Topology from differentiable viewpoint.
Exercises 4

SJ

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Zad. 1 (1300F, Ass. 2)**.** Let K, K' be transverse submanifolds of codimension k, k' in the n-manifold M. Prove that each point $p \in K \cap K'$ has a neighbourhood $U \subset M$ and a diffeomorphism from U to a neighbourhood of the origin in $\mathbb{R}^n$ which takes K and K' to the coordinate planes $V(x_1, \ldots, x_k)$ and $V(x_{n-k'+1}, \ldots, x_n)$, respectively. It is useful to use the algebraic geometry notation $V(x^1, \ldots, x^k)$ to mean the "vanishing" subspace $x^1 = \cdots = x^k = 0$.

Zad. 2 (1300F, Ass. 3)**.** Let K, L be submanifolds of a manifold M, and suppose that their intersection $K \cap L$ is also a submanifold. Then K, L are said to have *clean* intersection when, for each $p \in K \cap L$, we have $T_p(K \cap L) = T_pK \cap T_pL$. Show that there are coordinates near $p \in K \cap L$ such that K, L, and $K \cap L$ are given by linear subspaces of $\mathbb{R}^n$ of the form $V(x^{i_1}, \ldots, x^{i_k})$ for some subset of the coordinates. Also, can the intersection of submanifolds be transverse but not clean? Can it be clean but not transverse? Give examples or proofs as necessary.

Zad. 3. For a smooth map $f\colon M \to N$ we define its graph $\Gamma(f) := \{(x, f(x)) \in M \times N \,|\, x \in M\}$. Prove that $\Gamma(f)$ is a submanifold and its tangent space $T\Gamma_{(x,f(x))} \subset T(M_x \times N_{f(x)}) = TM_x \times TN_{f(x)}$ coincides with the graph of the derivative Df_x.

Zad. 4. A smooth manifold is called *paralelizable* iff its tangent bundle is trivial. Prove that any Lie group is a paralelizable manifold.

Zad. 5. For any vector bundle $p\colon E \to B$ define its orientation bundle $p_{or}\colon E_{or} \to B$ such that $p_{or}^{-1}(b) = \{\text{orientations of}\, p^{-1}(b)\}$. Define a topology in E_{or} such that p_{or} is a double covering and its sections correspond to orientations of the bundle $p\colon E \to B$. (In particular p is orientable iff the covering p is trivial.)

Zad. 6. Prove that a smooth submanifold $M^{m-1} \subset \mathbb{R}^m$ of codimension 1 is orientable if and only if it admits a non vanishing normal vector field i.e. there exists a map $\mathfrak{n}\colon M \to \mathbb{R}^{n-1}$ such that for all $x \in M$, $\mathfrak{n}(x) \neq 0$ and $\mathfrak{n}(x) \perp TM_x$. Deduce that the (generalized) Möbius band, real projective space of even dimension, Klein bottle are not orientable. Identify their orientation bundles as covering spaces.

Zad. 7. Let $E(\gamma_n^1) \to \mathbb{R}P(n)$ will be the canonical bundle over the projective space. Prove that the one point compactification of its total space $E(\gamma_n^1)^+$ is homeomorphic to $\mathbb{R}P(n+1)$.

Zad. 8. For a smooth submanifold $M^m \subset \mathbb{R}^n$ define $E_\nu(M) := \{(x, \mathbf{v}) \in M \times \mathbb{R}^n \,|\, \mathbf{v} \perp TM_x\}$ and the map $p(x, \mathbf{v}) = x$. Prove that $p\colon E_\nu(M) \to M$ is a locally trivial $n-m$-dimensional vector bundle (called a normal bundle) and $TM \oplus E_\nu(M)$ is a trivial bundle.

Zad. 9. For a smooth submanifold $M^m \subset \mathbb{R}^n$ define a map $E_\nu(M) \ni (x, \mathbf{v}) \mapsto x + \mathbf{v} \in \mathbb{R}^n$. Prove that if M is compact then there exists $\delta > 0$ such that the map is a diffeomorphism for $||\mathbf{v}|| < \delta$. (Image of $E_\nu(M)_\delta$ is called a tubular neighborhood of M in $\mathbb{R}^n$.)

Zad. 10. If $(M, \partial M)$ is a submanifold with boundary of R^n then there exists a non-vanishing vector field normal to the boundary i.e. $\mathfrak{n}\colon \partial M \to \mathbb{R}^n$ such that for all $x \in \partial M$, $\mathfrak{n}(x) \neq 0$ and $\mathfrak{n}(x) \perp T\partial M_x$. Deduce that ∂M posseses a collar neighborhood i.e. diffeomorphic to $\partial M \times \mathbb{R}^n_{\geqslant 0}$.