Topology from differentiable viewpoint.
Exercises 3

21 marca 2014

Zad. 1 (BJ[1] 1.11.4 Generalized. Cf also MG[2] Oct.3)**.** Let M be a differentiable manifold and a discrete group G acts on M via diffemorphisms. Furthermore assume that the projection to the orbit space $p\colon M \to M/G$ is a covering space. Prove that M/G possesses a structure of a smooth manifold such that p is a local homemomorphism.

Zad. 2 (BJ 1.11.15)**.** Let $\mathbb{K} := \mathbb{R}, \mathbb{C}, \mathbb{H}$. The points of the projective space $\mathbb{K}P(n)$ are described by the homogeneous coordinates $x = [x_0; , \dots, x_n]$. Show that the mapping $f\colon \mathbb{K}P(n) \times \mathbb{K}P(m) \to \mathbb{K}P(mn + m + n)$, $f([x_0; , \dots, x_n], [y_0; , \dots, y_n]) := [x_0y_0; , x_0, y_1, \dots; x_iy_j, \dots; x_ny_n]$ is a smooth embedding. (It is known in algebraic geometry and the Veronese embedding).

Zad. 3 (BJ 1.11.16)**.** Let $\mathbf{V}, \mathbf{W}$ be finite dimensional vector spaces over a field $\mathbb{K} := \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathrm{Hom}_{\mathbb{K}}(\mathbf{V}, \mathbf{W})$ denote the space of homomorphisms. Prove that for a given natural number $r \leqslant \min\{\dim \mathbf{V}, \dim \mathbf{W}\}$ the subset $\mathrm{Hom}^r_{\mathbb{K}}(\mathbf{V}, \mathbf{W}) \subset \mathrm{Hom}_{\mathbb{K}}(\mathbf{V}, \mathbf{W})$ consisting of all operators of rank r is a submanifold. Compute its dimension. Note that if $r = \dim \mathbf{V}$ then $\mathrm{Hom}^r_{\mathbb{K}}(\mathbf{V}, \mathbf{W})$ is the non-compact Stiefel manifold.

Zad. 4 (BJ 3.23.2)**.** Let $\pi\colon E \to X$ be a vector bundle over a connected space and $f\colon E \to E$ a bundle homomorphism such that $f^2 = f$ (i.e. a projection on each fiber) Show that f has a constant rank. (Thus $\operatorname{im} f$ and $\ker f$ are sub bundles such that $E := \operatorname{im} f \oplus \ker f$.)

Zad. 5 (BJ 3.23.3)**.** Let $\pi\colon E \to X$ be a vector bundle and $f\colon E \to E$ a bundle homomorphism such that $f^2 = id$ (i.e. group $\mathbb{Z}_2$ acts on E). Show that restriction of p to the fixed point set $\mathrm{Fix}(f) := \{e \in E \mid f(e) = e\} \subset E$ is a subbundle of $p\colon E \to X$. (Generalize the statement to an arbitrary group action on $p\colon E \to X$.)

Zad. 6 (BJ 5.14.4)**.** Let A be a symmetric $(n \times n)$-matrix, and $0 \neq b \in \mathbb{R}$. Show that the quadric $M := \{x \in \mathbb{R}^n \mid {}^tXAx = b\}$ is an $n-1$-dimensional submaniold of $\mathbb{R}^n$.

Zad. 7 (BJ 5.14.5)**.** Brieskorn manifold

Zad. 8 (BJ 5.14.6)**.** Milnor manifolds

Zad. 9 (BJ 5.14.10)**.** Local form of a smooth retraction.

[1] Th. Bröcker, K. Jäinch "Introduction to differential topology".
[2] M. Gualtieri "Geometry and Topology 1300Y"