Topology from differentiable viewpoint. Excersises 2

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Definition. *Let $f\colon S^r \longrightarrow M \setminus \partial M$ be an embedding of an r–dimensional sphere into an interior of a manifold with boundary $(M, \partial M)$ of dimension $n > r$ which extends to an embedding $f\colon S^r \times D^{n-r}_\epsilon \longrightarrow M$, where $D^k_\epsilon \subset \mathbb{R}^k$ is a k–dimensional closed disc of diameter $1+\epsilon$ where $\epsilon > 0$. We construct a manifold M_f in the following way:*

a) Consider $M' := M \setminus f(S^r \times \mathring{D}^{n-r})$,

b) The boundary of $M' := M \setminus f(S^r \times \mathring{D}^{n-r})$ is diffeomorphic via f to

$$\partial(S^r \times D^{n-r}) = S^r \times S^{n-r-1} = \partial(D^{r+1} \times S^{n-r-1}),$$

c) $M_f := (D^{r+1} \times S^{n-r-1}) \cup_f M'$.

Wa say that the manifold M_f is obtained from M by a surgery along f. (cf. Wiki Surgery Theory)

Zad. 1. Check that M_f is a manifold and $\partial M_f = \partial M$.

Zad. 2. Assume M is a manifold without boundary. Construct a bordism W_f between M_f a M (called *trace* of the surgery) which is homotopy equivalent to $M \cup_f D^{r+1}$. (Recall that a bordism between n-dimensional manifolds M_1, M_2 is an $n+1$–dimensional manifold with boundary $(W, \partial W)$ such that $\partial W \simeq M_1 \coprod M_2$.)

Zad. 3. Prove that M can be obtained via a surgery from M_f and (if $\partial M = \emptyset$) the trace of the surgery is W_f.

Zad. 4. Consider examples of surgery in low dimensions.

Zad. 5. Note that the connected sum of two n–dimensional manifolds $M_1 \# M_2$ can be defined as a result of a surgery on the disjoint sum $M_1 \sqcup M_2$, thus $M_1 \# M_2$ and $M_1 \sqcup M_2$ are bordant.

Zad. 6. Formulate and prove the Morse lemma. cf. J. Milnor "Morse Theory" and Brian Conrad – Stanford handout

Zad. 7. Solve problems 1,3, 5, 7 and 8 from 50 zadań z K-teorii (in English)

Zad. 8. Prove that the tangent bundle to the Grassmann manifold $G_k(\mathbb{R}^n)$ is isomorphic to the bundle $\mathrm{Hom}(\gamma, \gamma^\perp)$ where γ denotes the canonical bundle and $\gamma \oplus \gamma^\perp$ is a trivial bundle of dimension n.