Topology from differentiable viewpoint
Exercises 1.

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Zad. 1. If $X_i \supset A_i \xrightarrow{f_i} Y_i$ for $i = 1, 2$ are two continous maps, $h\colon (X_1, A_1) \to (X_2, A_2)$ and $g\colon Y_1 \to Y_2$ two homeomorphisms, such that $f_2 = gf_1h^{-1}$ then the quotient spaces $X_1 \cup_{f_1} Y_1$ oraz $X_2 \cup_{f_2} Y_2$ are homeomorphic.

Zad. 2. Let M, N are topological manifolds and $h\colon U \to V$ a homeomorphism between their open subsets i $U \subset M$, $V \subset N$. The quotient space $M \cup_h N$ is a topological manifold if and only if it is Hausdorff. Analogous assertions holds for differentiable manifolds and a diffeomorphism.

Zad. 3. Prove that if K is a submanifold of L and L is a submanifold of M, then K is a submanifold of M. Define a submanifold with boundary.

Zad. 4. Let $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_n \geqslant 0\}$, $\mathbb{R}^n_- := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_n \leqslant 0\}$, $\mathbb{R}^{n-1} := \mathbb{R}^n_+ \cap \mathbb{R}^n_-$ and $h\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be a diffeomorphism. On the quotient space $\mathbb{R}^n_+ \cup_h \mathbb{R}^n_-$ consider two charts: identity on the subset $\mathbb{R}^n_{>0} \cup \mathbb{R}^n_{<0} \subset \mathbb{R}^n_+ \cup_h \mathbb{R}^n_-$ and $\phi\colon \mathbb{R}^n_+ \cup_h \mathbb{R}^n_- \to \mathbb{R}^n$ given by the formula:

$$\phi([x_1, \ldots, x_n]) := \begin{cases} (x_1, \ldots, x_n) \text{ for } x_n \geqslant 0 \\ (h \times id)^{-1}(x_1, \ldots, x_n) \text{ for } x_n \leqslant 0 \end{cases}$$

Check that the charts form a smooth atlas on $\mathbb{R}^n_+ \cup_h \mathbb{R}^n_-$ and the resulting manifold is diffeomorphic to $\mathbb{R}^n$ (with identity atlas).

Zad. 5. Let M be a one-dimensional manifold and $M = U_1 \cup U_2$, where two proper open subsets $U_i \subset M$ are homeomorphic to $\mathbb{R}$. Prove that:

a) $U_1 \cap U_2$ has at most two connected components.

b) If i $U_1 \cap U_2$ is connected then M is homeomorphic to $\mathbb{R}$.

c) If If $U_1 \cap U_2$ has two components , then M is homeomorphic to the circle S^1.

d) If $M = \bigcup U_i$, where $U_1 \subset U_2 \subset \ldots$ and all sets U_i are homeomorphic to $\mathbb{R}$, then $M \simeq \mathbb{R}$.

If $M$is a smooth manifolds the above assertions hold when homeomorphism is replaced with a diffeomorphism.
Hint. See J. Milnor "Topology from differentiable viewpoint". Appendix, or G. Granja "The Classification of 1–dimensional manifolds.", or D.B. Fuks, V.A.Rokhlin "Beginner's Course in Topology". Ch.3 §1.

Zad. 6.

1. If $(W, \partial W)$ i $(V, \partial V)$ are topological manifolds with boundary, then $W \times V$ is a topological manifold with boundary $\partial(W \times V) = W \times \partial V \cup \partial W \times V$.
2. If $(W, \partial W)$ is a smooth manifold with boundary and M is a smooth manifold (without boundary), then $W \times M$ is a smooth manifold with boundary such that $\partial(W \times M) = \partial W \times M$. Thus cartesian product of a manifold which bounds and an arbitrary manifold is a boundary of a manifold and cartesian product is well defined on the bordism classes of manifolds.

Zad. 7. Prove that the set of orthogonal matrices $O(n) := \{A \in M_{\mathbb{R}}(n, n) \,|\, AA^T = Id\} \subset M(n, n)$ is a compact submanifold. What's its dimension? Identify its connected components. Note that $O(n)$ is a Lie group. Prove an analogous theorem for unitary and symplectic group.

Zad. 8. Let $f \in \mathbb{C}[z_1, \dots, z_n]$ be a (homogeneous) polynomial such that $f'(z) = 0$ only for $z = 0$. Then $L = \{z \in S^{2n-1} \,|\, f(z) = 0\} \subset \mathbb{C}^n$ where $S^{2n-1} \subset \mathbb{C}^n$ is a unit sphere is a submanifold. The manifold L bounds.

Zad. 9. Prove that n–dimensional projective spaces over fields $\mathbb{R}, \mathbb{C}, \mathbb{H}$ have a smooth manifold structure. What's their dimension as smooth manifolds?

Zad. 10 (MAT1300HF)**.** Consider the following spaces:

1. $S(T\mathbb{R}^3) := \{(x, \mathbf{v}) \in S^2 \times \mathbb{R}^3 \,|\, ||\mathbf{v}|| = 1,\, \langle \mathbf{v}, x \rangle = 0\}$ – unit tangent vectors to the sphere S^2.
2. Intersection of the sphere $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$ in $\mathbb{C}^3$ with the complex cone $z_1^2 + z_2^2 + z_3^2 = 0$.

Are any of the above manifolds diffeomorphic to the projective space $\mathbb{R}P(3)$? Show that $\mathbb{R}P(3)$ bounds.