Topology from differentiable viewpoint.
Problems for the oral exam

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1 Local structure of smooth manifolds and maps

Zad. 1. Prove that if K is a submanifold of L and L is a submanifold of M, then K is a submanifold of M. Give a proof for submanifolds with boundary.

Zad. 2 (1300F, Ass. 3)**.** Let K, L be submanifolds of a manifold M, and suppose that their intersection $K \cap L$ is also a submanifold. Then K, L are said to have *clean* intersection when, for each $p \in K \cap L$, we have $T_p(K \cap L) = T_pK \cap T_pL$. Show that there are coordinates near $p \in K \cap L$ such that K, L, and $K \cap L$ are given by linear subspaces of $\mathbb{R}^n$ of the form $V(x^{i_1}, \dots, x^{i_k})$ for some subset of the coordinates. Also, can the intersection of submanifolds be transverse but not clean? Can it be clean but not transverse? Give examples or proofs as necessary. (We use the algebraic geometry notation $V(x^{i_1}, \dots, x^{i_k})$ to mean the "vanishing" subspace $x^{i_1} = \cdots = x^{i_k} = 0$.)

Zad. 3. Formulate and prove the constant rank theorem and its stronger form for a smooth retraction [BJ 5.14.10].

Zad. 4. Let $p\colon E \to M$ be a proper (i.e. inverse images of the compact sets are compact) submersion. Prove that p is smoothly locally trivial. If M is connected then for any two points $x_1, x_2 \in M$ the fibers $f^{-1}(x_1)$ and $f^{-1}(x_2)$ are diffeomorphic.

Zad. 5 (BJ 6.7.1)**.** Show that, if $M^n \subset \mathbb{R}^p$ is a smooth submanifold, then there exists a $p-1$ dimensional subspace in $\mathbb{R}^p$ which intersects M^n transversally.

2 Important examples

Zad. 6. Prove that any connected 1-dimensional topological (resp. smooth) manifold is homeomorphic (resp. diffeomorphic) to the real $\mathbb{R}$ line or circle S^1.

Zad. 7. Prove that the following smooth manifolds are diffeomorphic:

1. Special orthogonal group $SO(3)$.
2. Real projective space $\mathbb{R}P(3)$.
3. The orbit space $S^3/\mathbb{Z}_2$ of the antipodal action of the group $\mathbb{Z}_2$ on the sphere S^3. (Define a smooth atlas on $S^3/\mathbb{Z}_2$!).
4. The unit sphere bundle of the tangent bundle $S(TS^2) := \{(x, \mathbf{v}) \in S^2 \times \mathbb{R}^3 \,|\, ||\mathbf{v}|| = 1,\ \langle \mathbf{v}, x\rangle = 0\}$.
5. Intersection of the sphere $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$ in $\mathbb{C}^3$ with the complex cone $z_1^2 + z_2^2 + z_3^2 = 0$.

Show that each of the above manifold bounds. Prove that they are parallelizable. Note that the group $SO(3)$ acts on itself by translations. Define natural $SO(3)$–actions on manifolds 2)–5) such that the diffeomorphisms are $SO(3)$–equivariant (i.e. commute with $SO(3)$–actions.)

Zad. 8. Let $f \in \mathbb{C}[z_1, \dots, z_n]$ be a (homogeneous) polynomial such that $f'(z) = 0$ only for $z = 0$. Then $L = \{z \in S^{2n-1} \,|\, f(z) = 0\} \subset \mathbb{C}^n$ where $S^{2n-1} \subset \mathbb{C}^n$ is a unit sphere is a submanifold. The manifold L bounds.

Zad. 9 (BJ 1.11.16)**.** Let $\mathbf{V}, \mathbf{W}$ be finite dimensional vector spaces over a field $\mathbb{K} := \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathrm{Hom}_{\mathbb{K}}(\mathbf{V}, \mathbf{W})$ denote the space of homomorphisms. Prove that for a given natural number $r \leqslant \min\{\dim \mathbf{V}, \dim \mathbf{W}\}$ the subset $\mathrm{Hom}^r_{\mathbb{K}}(\mathbf{V}, \mathbf{W}) \subset \mathrm{Hom}_{\mathbb{K}}(\mathbf{V}, \mathbf{W})$ consisting of all operators of rank r is a submanifold. Compute its dimension. Note that if $r = \dim \mathbf{V}$ then $\mathrm{Hom}^r_{\mathbb{K}}(\mathbf{V}, \mathbf{W})$ is the non-compact Stiefel manifold. How $\mathrm{Hom}^r_{\mathbb{K}}(\mathbf{V}, \mathbf{W})$ if $r = \dim \mathbf{W}$ is related to the Grassmann manifold ?

Zad. 10. Let $k \leqslant n$ be natural numbers. Prove that the set

$$V_k(\mathbb{R}^n) = \{(v, \dots, v_k) \mid < v_i, v_j >= \delta_{ij}\} \subset S^{n-1} \times \cdots \times S^{n-1}$$

is a smooth submanifold (Stiefel manifold). What's its dimension? Is its normal bundle trivial? Identify its connected components.

3 Surgery and cobordism

Definition. *Let $f\colon S^r \longrightarrow M \setminus \partial M$ be an embedding of an r–dimensional sphere into an interior of a manifold with boundary $(M, \partial M)$ of dimension $n > r$ which extends to an embedding $f\colon S^r \times D^{n-r}_\epsilon \longrightarrow M$, where $D^k_\epsilon \subset \mathbb{R}^k$ is a k–dimensional closed disc of diameter $1 + \epsilon$ where $\epsilon > 0$. We construct a manifold M_f in the following way:*

a) Consider $M' := M \setminus f(S^r \times \mathring{D}^{n-r})$,

b) The boundary of $M' := M \setminus f(S^r \times \mathring{D}^{n-r})$ is diffeomorphic via f to

$$\partial(S^r \times D^{n-r}) = S^r \times S^{n-r-1} = \partial(D^{r+1} \times S^{n-r-1}),$$

c) $M_f := (D^{r+1} \times S^{n-r-1}) \cup_f M'$.

We say that the manifold M_f is obtained from M by a surgery along f. (cf. Wiki Surgery Theory)

Zad. 11. Check that M_f is a manifold and $\partial M_f = \partial M$. Prove that M can be obtained via a surgery from M_f (with the same trace – cf. next problem.)

Zad. 12. Assume M is a manifold without boundary. Construct a bordism W_f between M_f a M (called *trace* of the surgery) which is homotopy equivalent to $M \cup_f D^{r+1}$. (Recall that a bordism between n-dimensional manifolds M_1, M_2 is an $n+1$–dimensional manifold with boundary $(W, \partial W)$ such that $\partial W \simeq M_1 \coprod M_2$.)

Zad. 13. Note that the connected sum of two n–dimensional manifolds $M_1 \# M_2$ can be defined as a result of a surgery on the disjoint sum $M_1 \sqcup M_2$, thus $M_1 \# M_2$ and $M_1 \sqcup M_2$ are bordant.

Zad. 14. Present torus $T = S^1 \times S^1$ and the Klein bottle K as results on surgeries on 2-dimensional sphere S^2. Note that K is a connected sum of two projective planes $\mathbb{R}P(2)$.

4 Vector bundles and their morphism

Zad. 15. Prove that any (smooth) vector bundle over the real line $\mathbb{R}$ (or more generally $\mathbb{R}^n$) is (smoothly) trivial.

Zad. 16. For a smooth submanifold $M^m \subset \mathbb{R}^n$ define $E_\nu(M) := \{(x, \mathbf{v}) \in M \times \mathbb{R}^n \mid \mathbf{v} \perp TM_x\}$ and the map $p(x, \mathbf{v}) = x$. Prove that $p\colon E_\nu(M) \to M$ is a locally trivial $n - m$-dimensional vector bundle (called a normal bundle) and $TM \oplus E_\nu(M)$ is a trivial bundle.

Zad. 17 (BJ 3.23.2)**.** Let $\pi\colon E \to X$ be a vector bundle over a connected space and $f\colon E \to E$ a bundle homomorphism such that $f^2 = f$ (i.e. a projection on each fiber) Show that f has a constant rank. (Thus $\mathrm{im}\, f$ and $\ker f$ are sub bundles such that $E := \mathrm{im}\, f \oplus \ker f$.)

Zad. 18 (BJ 3.23.3)**.** Let $\pi\colon E \to X$ be a vector bundle and $f\colon E \to E$ a bundle homomorphism such that $f^2 = id$ (i.e. group $\mathbb{Z}_2$ acts on E). Show that restriction of p to the fixed point set $\mathrm{Fix}(f) := \{e \in E \mid f(e) = e\} \subset E$ is a subbundle of $p\colon E \to X$. (Generalize the statement to an arbitrary group action on $p\colon E \to X$.)

Zad. 19 (BJ 11.7.1,2)**.** Let $p\colon E \to M$ be a smooth vector bundle and $Dp\colon TE \to TM$ be its derivative. We identify $M = s_0(M)$. Prove that $(Dp)|M\colon (TE)|M \to TM$ is an epimorphism of vector bundles and $\ker(Dp)|M$ is isomorphic to the bundle $p\colon E \to M$. Prove that $TE \simeq p^*(E \oplus TM)$.

Zad. 20. Prove that the tangent bundle to the Grassmann manifold $G_k(\mathbb{R}^n)$ is isomorphic to the bundle $\mathrm{Hom}(\gamma, \gamma^\perp)$ where γ denotes the canonical bundle and $\gamma \oplus \gamma^\perp$ is a trivial bundle of dimension n.

5 Orientability of manifolds

Zad. 21. Let M^m be an orientable smooth manifold. Prove that a submanifold $N^{m-1} \subset M$ of codimension 1 is orientable if and only if it admits a non vanishing normal vector field i.e. there exists a map $\mathfrak{n}\colon N \to TM$ such that for all $x \in N$, $\mathfrak{n}(x) \notin TN_x$.

Zad. 22. If $(M, \partial M)$ is a submanifold with boundary of $\mathbb{R}^n$ then there exists a non-vanishing vector field normal to the boundary i.e. $\mathfrak{n}\colon \partial M \to \mathbb{R}^n$ such that for all $x \in \partial M$, $\mathfrak{n}(x) \neq 0$ and $\mathfrak{n}(x) \perp T\partial M_x$. Deduce that ∂M possesses a collar neighborhood i.e. diffeomorphic to $\partial M \times \mathbb{R}^n_{\geqslant 0}$ and boundary of an orientable manifold is orientable.

Zad. 23. Prove that a smooth manifold is orientable if and only if the normal bundle to any immersion (embedding) $f\colon S^1 \to M$ is trivial.

6 Diffeomorphisms and smooth functions

Zad. 24 (BJ 9.6.1)**.** Let M be a connected manifold with $\dim(M) > 1$. Let $\{x_1, \dots x_k\}$ be distinct points of M, and let $\{y_1, \dots y_k\}$ also be distinct points of M. Show that there exists a diffeomorphism (with compact support) homotopic (even isotopic) to identity $\phi\colon M \to M$ such that $\phi(x_i) = y_i$ for $i = 1, \dots k$.

Zad. 25 (BJ 9.6.2)**.** Let $M \subset N$ be a submanifold of the connected manifold N such that $\dim N - \dim M \geqslant 2$ and $p, q \in N \backslash M$. Show that there exists a diffeomorphism $h\colon N \to N =$ such that $h(p) = q$ and $h|N = id$.

Zad. 26 (BJ 10.11.1)**.** Let M be an (oriented) connected manifold, $p, q \in M$ and $\phi\colon TM_p \to TM_q$ an (orientation preserving) isomorphism. Show that there exists a diffeomorphism $f\colon M \to M$ such that $f(p) = q$ and $Df_p = \phi$. (*Remark.* There exist orientable manifolds which do not admit an orientation reversing diffeomorphisms!)

Zad. 27. For an arbitrary smooth manifold M construct a proper (i.e. inverse images of the compact sets are compact) smooth function $f\colon M \to \mathbb{R}$. For an arbitrary compact manifold with boundary $(M, \partial M)$ such that $\partial M = \partial_0 M \cup \partial_1 M$ is a disjoint sum of two submanifolds there exists a smooth function $f\colon M \to [0,1]$ such that $\partial_0 M = f^{-1}(0)$ and $\partial_1 M = f^{-1}(1)$. Note that in both cases if f is a submersion then f is a trivial bundle with the fiber $f^{-1}(0)$. (*Remark.* The function f can be choosen such that all their critical points are non-degenerate i.e. the Morse function.)

Zad. 28 (BJ 7.13.11)**.** Show that there exists a closed embedding of the real line $\mathbb{R}$ into an arbitrary connected non-compact smooth manifold M. Moreover, if $D \subset M$ is any closed connected countable discrete subset, then an embedding $j\colon \mathbb{R} \to M$ can be chosen such that $D \subset j(M)$.

7 Framed cobordism and the Pontryagin construction

Zad. 29. Read the Chapter 7 of J. Milnor *Topology from Differentiable Viewpoint.* and complete details of the proofs.

Zad. 30. Complete the last section of Chapter 7 (The Hopf Theorem) proving that for a non-compact connected manifold M^m every map $M^m \to S^m$ is contractible.