Geometric bordism and cobordism.

A.Bojanowska, S.Jackowski

Contents

Chapter 1

Stable Homotopy Theory

We assume that the reader is familiar with § 1 - § 6 of G.W.Whitehead's paper [17]. We recall the basic definitions.

Definition 1.1 *A spectrum $E = (E_n, e_n)_{n\in\mathbb{Z}}$ consists of a sequence of based CW-complexes E_n and inclusions of subcomplexes $SE_n \hookrightarrow E_{n+1}$.*

Definition 1.2 *A map between spectra $f : E \to F$ consists of a sequence of maps $F_k : E_k \to F_k$ defined for $k > n_0$ such that the diagram:*

$$\begin{array}{ccc} SE_k & \hookrightarrow & E_{k+1} \\ Sf_k \downarrow & & \downarrow f_{k+1} \\ SF_k & \hookrightarrow & F_{k+1} \end{array}$$

commutes for $k > n_0$.
The category of spectra is denoted by $\mathcal{S}$.

We say that maps $f, f' : \mathbb{E} \to \mathbb{F}$ are equal iff $f_k = f'_k$ for k sufficiently large, and similarly they are homotopic iff $f_k \sim f'_k$ for k large enough. Let $\{\mathbb{E}, \mathbb{F}\}$ denote the set of homotopy classes of maps between $\mathbb{E}$ and $\mathbb{F}$, and $\mathcal{S}_h$ the homotopy category of spectra.

There is a natural functor: $\mathcal{C} \to \mathcal{S}_h$ from the category C of of CW-complexes to $\mathcal{S}_h$ associating with every complex X a spectrum $(S^kX, \mathrm{id})_{k\in\mathbb{N}}$.

The suspension functor $S : \mathcal{C} \to \mathcal{C}$ extends to $\mathcal{S}_h$ so that:

$$\begin{array}{ccc} \mathcal{S}_h & \xrightarrow{S} & \mathcal{S}_h \\ \uparrow & & \uparrow \\ \mathcal{C} & \xrightarrow{S} & \mathcal{C} \end{array}$$

commutes where $(SE)_k = E_{k+1}$.

Let us notice that $S : \mathcal{S}_h \to \mathcal{S}_h$ is an automorphism.

For every CW-complex
$\{S^n X, \mathbb{E}\} = \mathrm{colim}_k [S^{n+k} X, E_k]$.
For finite CW-complex the formulas:
$h^k(X; \mathbb{E}) := \{S^{-k} X, \mathbb{E}\}$
$h_k(X; \mathbb{E}) := \{S^k X, X \wedge \mathbb{E}\}$
define cohomology and homology theories associated with the spectrum $\mathbb{E}$. Here $X \wedge \mathbb{E}$ denotes the spectrum: $(X \wedge \mathbb{E})_k = X \wedge E_k$ with obvious inclusions.

A similiar "stabilization process" can be applied to vector bundles. We will assume that a map of bundles is an isomorphism on fibres.

Definition 1.3 *A spectrum of vector bundles $\xi = (\xi_n, \epsilon_n)$ consists of a sequence of bundles ξ_n over B_n (defined for $n > k_0$) and the inclusions:*

$$\begin{array}{ccc} \xi_n \oplus \theta^1 & \overset{\widetilde{\varepsilon}_n}{\hookrightarrow} & \xi_{n+1} \\ \downarrow & & \downarrow \\ B_n & \xrightarrow{\varepsilon_n} & B_{n+1} \end{array}$$

where e^1 denotes one dimensional trivial vector bundle.

The definitions of maps between spectra of vector bundles and their homotopy classes are analogous to those for spectra of complexes.

Let η , ξ be spectra of vector bundles. We denote by $\{\eta, \xi\}$ the set of homotopy classes of maps $\eta \to \xi$. We write $\mathcal{SV}_k$ for the homotopy category of spectra of vector bundles. There is a natural functor from the homotopy category of spectra of vector bundles to the homotopy category of spectra of vector bundles: $\mathcal{V}_h \to \mathcal{SV}_h$ associating with a bundle η a spectrum $\{\eta \oplus \theta^k, \mathrm{id}\}_{k \in \mathbb{N}}$. The dimension function can be extended to the objects of $\mathcal{SV}_h$: $\dim \xi =: \dim \xi_k - k$. It is clear that if $\dim \eta \neq \dim \xi$ then $\{\eta, \xi\} = \emptyset$.

We extend the functor $\theta : \mathcal{V} \to \mathcal{V}$, $\theta(\eta) = \eta \oplus \theta'$, $\theta(f) = f \oplus \mathrm{id}$ to the category $\mathcal{SV}_h$ putting:
$[\theta(\xi)]_k = \xi_{k+1}$. The diagram commutes:

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\theta} & \mathcal{V}_h \\ \downarrow & & \downarrow \\ \mathcal{SV}_h & \xmapsto{\theta} & \mathcal{SV}_h. \end{array}$$

If η is a vector bundle and ξ is a spectrum of vector bundles then $\{\theta_{-n}(\eta), \xi\} = \mathrm{colim}_k[\eta + \theta^{k-n}, \xi_k]$.

Let ξ be a spectrum of vector bundles and $\dim \xi = 0$.

Definition 1.4 *Let η be a n-dimensional vector bundle. A stable ξ-structure on η is an element of $\{\theta^{-n}(\eta), \xi\}$. We will also use the simpler notation $\theta(\xi) = \xi + 1$. In this notation a ξ-structure on η is a homotopy class of the map $\eta \to \xi + n$ of bundle spectra.*

Remark 1.5 *This definition of ξ-structure coincides with the definition given in Stong's book [16, ch.II].*

We describe in the above terms the stable reduction of the group of the vector bundle to the subgroup of the orthogonal group. Assume that the subgroups $G(n) < O(n)$ are defined in such way that the diagram:

$$\begin{tikzcd} G_n \arrow[r, hook] \arrow[d, hook] & O(n) \arrow[d, hook] \\ G_{n+1} \arrow[r, hook] & O(n+1) \end{tikzcd}$$

commutes.

This sequence defines the spectrum of vector bundles: $\gamma(G) = \{i_n^*\gamma_n, i^*\epsilon_n\}$, where $i_n : BG_n \to BO(n)$ and γ_n denotes a classyfying bundle over $BO(n)$. The stable reduction of the group of the n-dimensional bundle η to the group G is the $\gamma(G)$ structure on η.

Consider now the Thom space functor from the homotopy category of vector bundles to the homotopy category of based CW-complexes: $\mathcal{V}_n \to^T \mathcal{C}_n$. We use the notation $T(\xi) = B^\xi$ where ξ is a bundle over B.

Theorem 1.6 *There exists an extention of the Thom-space functor to the category of spectra i.e. there exists a commutative diagram:*

$$\begin{array}{ccc} \mathcal{V}_h & \xrightarrow{T} & \mathcal{C}_h \\ \downarrow & & \downarrow \\ \mathcal{S}\mathcal{V}_h & \xrightarrow{T} & \mathcal{S}_h \end{array}$$

Proof: We define $T(\xi)_k = T(\xi_k)$.
If $\epsilon_k : \xi_k + \theta' \to \xi_{k+1}$ then $T\epsilon_k : T(\xi_k \oplus \theta') = ST(\xi_k) = T\xi_{k+1}$.

Chapter 2

Differential Topology.

In this section we recall basic notions of differential topology which will be used in the sequel. Our material consists of three parts:

1. tubular neighborhoods.
2. imbeddings of manifolds.
3. transversality.

A smooth manifold is a C^∞ manifold and a smooth map is a C^∞ map. We will assume that all manifolds considered are paracompact.

2.1 Tubular Neighborhoods.

For a complete treatment of the subject we refer the reader to Lang [10].

Let V, W be manifolds and let $i : V \to W$ be an immersion. A differential $Di : TV \to TW$ induces then a monomorphism $Di : TV \to i^*TW$ over V.

Definition 2.1 *The quotient bundle $\nu(i) = i^*TW/TV$ is called a normal bundle to the immersion $i : V \to W$.*

If $i : V \hookrightarrow W$ is an inclusion of a submanifold then a normal bundle $\nu(i)$ is called a normal bundle of V in W and is denoted by $\nu_W(V)$. In this situation we define a tubular neighborhood of V in W.

Definition 2.2 *A tubular neighborhood of V in W consists of:*
a) a vector bundle $\pi : E \to V$ and an open neighborhood Z of its zero-section.
b) a diffeomorphism $h : Z \to U$ where U is an open neighborhood of V in W such that diagram:

$$\begin{array}{ccc} V & \hookrightarrow & U \\ s_0 \downarrow & h\nearrow & \\ Z & & \end{array}$$

commutes ($s_0 : V \to Z$ is the zero-section).

Example 2.3 *Let $\xi = (p : E \to M)$ be a smooth vector bundle over a manifold M. A normal bundle of the zero-section $s_0(M) \hookrightarrow E$ is canonically isomorphic to ξ. This follows from the exact sequence: $\theta \to p^*\xi \to TE \to^{Dp} p^*TM \to 0$ which has a canonical splitting Ds_0 over the zero-section. The tubular neighborhood is obviously the whole $E(\xi)$ with the icentity map.*

We shall formulate the existence theorem in general situation.

Theorem 2.4 *Let V be a closed submanifold of W. There exists a tubular neighborhood of V in W.*

Sketch of the proof: We choose a Riemannian metric on W. This metric defines the map $\nu(V) \to (TV)^{\perp}$ and thus the splitting of the sequence: $0 \to TV \to TW|_V \to \nu(V) \to 0$. The Riemannian metric defines also an exponential map: $TW \supset Z_0 \to^{\exp} W$. This exponential map being an identity on the zero-section is thus a diffeomorphism on an open subset $Z \subset Z_0 \cup (TV)^{\perp}$; $s_0(V) \subset Z$ (here we need the assumption that V is a closed submanifold).

The proof of theorem 10 gives us a canonical method of construction of tubular neighborhood after a Riemannian metric is chosen. It is important for our purposes to have a tubular neighborhood defined over the whole bundle and not only on a neighborhood of its zero-section.

Definition 2.5 *A vector bundle $p : E \to X$ is compressible iff for every open neighborhood Z of the zero-section $s_0(X) \subset E$ there exists an open neighborhood Z_1, $s_0(X) \subset Z_1 \subset Z \subset E$ and a homeomorphism $h : E \to Z_1$*

over X (i.e the diagram:

$$\begin{array}{ccc} E & \rightarrow & X \\ h\downarrow & \nearrow & \\ Z_1 & & \end{array}$$

commutes).

Lemma 2.6 *Any vector bundle over a manifold is compressible.*

For the proof see Lang [10;VII.4.]. It is important to notice that in this case the compressions are diffeomorphisms isotopic in E to the identity.

Thus haveing chosen a Riemannian metric we have the canonical construction of the total tubular neighborhood:

$$\nu(V) \to (TV)^{\perp} \xrightarrow{\text{compression}} Z \xrightarrow{\exp} W$$

We shall see that total tubular neighborhoods are unique up to the isomorphism of vector bundles.

Definition 2.7 *Let $p : E \to V$ be a vector bundle; $V \subset W$ and $Z \to W$ a tubular neughborhood of V in W. The isotopy $F : Z \times R \to W$ is called an isotopy of tubular neighborhoods iff each $F_t : Z \to W$ is a tubular neighborhood.*

Theorem 2.8 *Let $h : E \to W$ and $g : E' \to W$ be total tubular neighborhoods of V in W. Then there exists a vector bundle isomorphism $\lambda : E \to E'$ and an isotopy of tubular neighborhoods $f_t : E \to W$ such that $f_0 = h$ and $f_1 = g \circ \lambda$. Moreover, if E and E' are endowed with Riemannian metric, then λ can be chosen to be an isometry.*

Corollary 2.9 *If $E \to W$ is a tubular neighborhood of V in W, then E is isomorphic to $\nu(W)$.*

For the proof see Lang [10, IV.6].

Remark 2.10 *Any two canonical tubular neighborhoods are isotopic. This follows easily from the fact that any two Riemannian metrics are homotopic through Riemannian metrics (thus in this case λ is an identity). We will use thic fact later.*

2.2 Imbeddings of Manifolds.

We start with the definition:

Definition 2.11 *A smooth map $f : V \to W$ is called a proper imbedding iff the following conditions are satisfied:*

1. *Df is monomorphism,*
2. *f is injective,*
3. *f is proper (i.e. the inverse image of every compact set is compact).*

If $f : V \to W$ is a proper imbedding then $f(V) \subset W$ is a closed submanifold of W. From now on we assume that all imbeddings under consideration are proper.
We recall Whitney's imbedding theorem:

Theorem 2.12 *Let $\xi : V \to R$ be a smooth function such that for every $x \in V$, $\xi(x) > 0$. Let $f : V \to R^q$ be a proper map which is an imbedding of a neighborhood U of a closed set $A \subset V$. If $2 \cdot \dim V < q$ then there exists $g : V \to R^q$ such that $g|_A = f|_A$, g is an ε-approximation of f, and g is a proper imbedding.*

Remark 2.13 *R^q may be replased by a manifold N with $\dim N > 2 \cdot \dim V$.*

For the proof of the above theorem see Narasimhan [12, 2.15].
We will say that two proper imbeddings $f_0, f_1 : V \to W$ are properly isotopic iff there exists an isotopy $F : V \times R \to W$ such that F_t is proper for each t, $F_0 = f_0$ and $F_1 = f_1$. It is clear that $F : V \times R \to W$ is a proper isotopy iff a map $F \triangle \mathrm{pr}_2 : V \times R \to W \times R$, $(F \triangle \mathrm{pr}_2)(\nu, t) = (F(\nu, t), t)$ is a proper imbedding. From the Whitney theorem it follows:

Theorem 2.14 *If $q > 2 \cdot \dim V + 4$ then any two proper imbedding $i_0, i_1 : V \to R^q$ are properly isotopic. Any two such isotopies are themselves properly isotopic leaving the endpoints fixed.*

The last theorem enables us to identify the normal bundles to different imbeddings of the same manifold. Note first that an imbedding $i : V \to R^q$ defines a sequence of imbeddings $i_{q+k} : V \to R^{q+k}$. We can define a bundle spectrum $\{\nu_n, f_n\}$ which is determined for $n \geq q$ as follows: $\nu_n = \nu(i_n)$

(where $i = i_q$), and f_n are canonical isomorphisms $\nu(i_n) \oplus \theta = \nu(i_{n+1})$ determined by the standard orientation of the Euclidean space. We denote this spectrum by $\nu^S(i)$ and call it a stable normal bundle to an imbedding i. Note that $\dim \nu^S(i) = -\dim V$.

Recall now that any bundle η over $X \times I$ defines a map $\eta \to \eta|_{X\times\{0\}}$ which induces a canonical homotopy class of isomorphism $\eta|_{X\times\{0\}} to^{\simeq} \eta|_{X\times\{1\}}$ (see for instance Husemoller [7]).

Thus for any isotopy $F : V \times R \to R^q$ between proper imbeddings $i_0, i_1 : V \to R^q$ the normal bundle of $F : V \times R \to R^q \times R$, $\widetilde{F}(\nu, t) = (F\nu, t, t)$ defines a homotopy class of isomorphism $\nu(i_0) \simeq \nu(i_1)$. If q is sufficiently large (as in theorem 2.14) then any two isotopies between i_0 and i_1, being themselves isotopic through isotopies, define the same homotopy class of isomorphism $\nu(i_0) \simeq \nu(i_1)$.

Corollary 2.15 *Let $i : V \to R^q$ and $j : V \to R^q$ be imbeddings. Then there is a canonical homotopy class of isomorphisms $\nu^S(i) \to \nu^S(j)$.*

The class of all stable normal bundles of imbeddings of V in R^q will be denoted by $\tau(V)$.

Let ξ be a spectrum of vector bundles, $\dim \xi = 0$ and suppose $\dim V = n$.

Definition 2.16 *A ξ-structure on $\tau(V)$ is family of ξ-structures on stable normal bundles to all imbeddings of V, such that for every $\theta^n(\nu^S(i)) \to \xi$, $\theta^n(\nu^S(j)) \to \xi$, and the canonical homotopy class of isomorphisms $\nu^S(i) \to \nu^S(j)$ the following diagram commutes in $\mathcal{SV}_h$:*

$$\begin{array}{ccc} \theta^n(\nu^S(i)) & \to & \xi \\ \downarrow & \nearrow & \\ \theta^n(\nu^S(j)) & & \end{array}$$

Consider now a more general situation.

Let Y be a manifold. we will deal with imbeddings $i : V \to Y \times R^q$ (for Y = pt we have the previous situation). As before every such an imbedding determines a bundle spectrum $\{\nu_k, \varkappa_k\}$ defined for $k \geq q$ such that $\nu_k = \nu(i_k)$ where (as $k = q+1$) $i_k = i_{q+1} : V \to^i Y \times R^q \hookrightarrow Y \times R^{q+1}$ and $\varkappa_k$ is a canonical isomorphism $\nu(i_k) + \theta^1 = \nu(i_{k+1})$ determined by the standard orientation of the Euclidean space. The dimension of this spactrum is $\dim Y - \dim V$ and we denote it by $\nu^S(i)$.

Proposition 2.17 *Let $\pi : Y \times R^q \to Y$ be the projection. If $i_0, i_1 : V \to Y \times R^q$ satisfy $\pi i_0 = \pi i_1 = f$ then i_0 and i_1 are homotopic.*

Indeed: $H_t(\nu) = (1-t)i_0(\nu) + ti_1(\nu)$, defines this homotopy on $V \times I$, we can prolong it smoothly to $H : V \times R \to Y \times R^q$.

Proposition 2.18 *Let*

$$\begin{array}{ccc} V & \xrightarrow{i} & Y \times R^q \\ & f \searrow & \downarrow \pi \\ & & Y \end{array} \quad \textit{and} \quad \begin{array}{ccc} V & \xrightarrow{j} & Y \times R^q \\ & f \searrow & \downarrow \pi \\ & & Y \end{array}$$

be imbeddings lifting the same map f. Then there is a canonical homotopy class of isomorphism $\nu^S(i) \to \nu^S(j)$.

Outline of the proof: A homotopy from proposition refpro:2 gives us a map: $H : V \times R \to Y \times R^q \times R$. According to the Whitney theorem we can take an imbedding $\widetilde{H} : V \times R \to Y \times R^q \times R$ arbitrality close to H (and thus homotopic to H), such that $\widetilde{H}|_{V\times\{i\}} = H|_{V\times\{i\}}, i = 0, 1$. The normal bundle to $\widetilde{H}$ gives the required isomorphism determined by isotopies $H : V \times R \to Y \times R^q \times R$ for which the following diagram commutes homotopically:

$$\begin{array}{ccc} V \times R & \xrightarrow{H} & Y \times R^q \times R \\ & f \times \mathrm{id} \searrow & \downarrow \\ & & Y \times R \end{array}$$

To establish the uniqueness of the homotopy class of such isomorphism we just have to repeat the previous argument.

Remark 2.19 *For the geometric reasons we will be interested only in such imbeddings $i : V \to Y \times R^q$ that $f = \pi i : V \to Y$ is proper and in such homotopies $H : V \times R \to Y \times R^q \times R$ between them that $\pi H : V \times R \to Y \times R$ is proper. Notice, that when f is proper the homotopy from proposition refpro:2, as well as homotopies used in the proof of proposition 2.18 satisfy the above property. This fact will be used in the sequel.*

The class of all stable normal bundles of imbeddings lifting the map f will be denoted $\nu(f)$ and called a stable normal bundle to f.

Suppose $i : V \to R^l$ is an imbedding and $f : V \to Y$ a proper map. The diagonal $(f, i) : V \to Y \times R^l$ is then an imbedding "lifting" f whose normal bundle satisfies the equality:

$$\nu(i, f) \oplus \theta^l = f^*\tau(Y) \oplus \nu(i) \oplus \theta^l$$

For this reason the normal bundle $\nu(f)$ to the map f is also denoted by $f^*\tau(Y) - \tau(V)$.

The definition of ξ-structure on $\nu(f)$ is analogous to definition 2.16.

Definition 2.20 *ξ-orientation of f is a ξ-structure on $\nu(f)$.*

Remark 2.21 *Let $f : V \to Y$ and $g : V_1 \to Y$ be proper maps and suppose that there is a diffeomorphism $\varphi : V \to V_1$ for which the diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{f} & Y \\ \varphi\downarrow & \nearrow g & \\ V_1 & & \end{array}$$

The diffeomorphism φ induces a canonical isomorphism $\nu(f) \to \nu(g)$ (i.e. a family of isomorphisms compatible with the canonical isomorphisms defining $\nu(f)$ and $\nu(g)$).

Proof: Let $i : V \to Y \times R^q$, $\quad j : V \to Y \times R^q$ be imbeddings lifting f and g respectively. The canonical homotopy class of isomorphism $\nu^S(i) \to \nu^S(j)$ is the composition of the canonical homotopy class of isomorphism $\nu^S(i) \to \nu^S(j\varphi)$ with an isomorphism $\nu^S(j\varphi) \to \nu^S(j)$ (induces by φ in obyious way).

□

2.3 Transversality.

Definition 2.22 *Let $Z \subset X$ be a submanifold and $g : Y \to X$ a map of manifolds. We say that g is transversal to Z on a set $A \subset Y$ iff for every $a \in A$ such that $g(a) \in Z$ the following equality holds:*

$$T_{g(a)}Z + Dg(T_aY) = T_{g(a)}X.$$

($Dg : TY \to TX$ denotes the differential of g)
If $g : Y \to X$ is transversal to Z on Y we say simply that it is transversal to Z.

Definition 2.23 *Let $f : Z \to X$, $\quad g : Y \to X$ be maps. We say that f and g are transversal ($f \pitchfork g$) iff $f \times g : Z \times Y \to X \times X$ is transversal to the "diagonal submanifold" $\triangle(X) \hookrightarrow X \times X$.*

Theorem 2.24 *If $g : Y \to X$ is transversal to $Z \subset X$ then $g^{-1}(Z)$ is a submanifold and the differential $Dg : TY \to TX$ induces a canonical isomorphism of bundles:*

$$\nu_Y(g^{-1}(Z)) \to^{\approx} g^*\nu_X(Z)$$

This isomorphism is defined in the following way. Consider the composition:

$$\begin{tikzcd}
g^{-1}(Z) \arrow[d, "g"'] & T(Y)|_{g^{-1}(Z)} \arrow[l] \arrow[d, "Dg"] \\
Z & T(X)|_Z \arrow[l] \arrow[d, "\substack{canonical \\ projection}"] \\
& \nu_X(Z) \arrow[ul]
\end{tikzcd}$$

From the transversality condition the kernel of this composition is precisely $T(g^{-1}(Z))$ and thus we obtain the desired isomorphism which we denote by $D\overline{g}$.

Remark 2.25 *If f and g are transversal then $V = (f \times g)^{-1}(\triangle X)$ is a submanifold of $Z \times Y$ which is the pull-back of the following diagram:*

$$\begin{tikzcd}
V \arrow[r, dashed, "back"] \arrow[d, dashed] & Z \arrow[d, "f"] \\
Y \arrow[r, "g"] & X
\end{tikzcd}$$

We recall the Thom transversality theorem.

Theorem 2.26 *Let $g : Y \to X$ be a map which is transversal to $Z \subset X$ on a closed set $A \subset Y$. Let $\delta : Y \to R, \quad \delta > 0$ be a smooth function. Then there exists a δ-approximation $g' : Y \to X$ such that $g'|_A = g|_A$ and*

Notice that from this theorem it follows that for any map $g : Y \to X$ as above we can find a map $g' : Y \to X$ which is homotopic to g relative A and transversal to Z. (we use the theorem that δ is small enough g and g' must be homotopic.) The above theorem can be formulated in the following, slightly stronger form (Karoubi [9]).

Theorem 2.27 *Let $f : Z \to X$ and $g : Y \to X$ be maps, and $\delta : Y \to R, \quad \delta > 0$ a smooth function. Then there exists a δ-approximation $g' : Y \to X$ of g such that $g' \pitchfork f$.*

Chapter 3

The Thom construction.

Consider the following situation. Let η be a smooth vector bundle $p : E \to Y$ and let $i : X \to E$ be an imbedding such that the composition $\pi = f : X \to Y$ is a proper map.

$$\begin{array}{ccc} & & E \\ & i \nearrow & \downarrow p \\ X & \longrightarrow & Y \end{array}$$

The Thom construction associates with this imbedding a homotopy class of maps $Y^\eta \to X^{\nu(i)}$ where Y^η and $X^{\nu(i)}$ denote the Thom spaces of appropriate bundles.

We will formulate the description of tho Thom space in the form convenient for our puposes. Let $\eta = \{p : E \to Y\}$ be a vector bundle. $Y^\eta = E \cup \{*\}$. We topologize Y^η by taking as a base of neighborhoods of $\{*\}$ sets $Y^\eta \backslash A$ where $A \subset E$ is closed and $p|_A : A \to Y$ os proper. Note, that the Thom space of a bundle over paracompact space is paracompact. To describe the Thom construction let us fix a Riemannian metric on a manifold E. This gives us a tubular neighborhood

$$\begin{tikzcd}
\nu(i) \simeq T(X)^{\perp} \arrow[r, "\text{compression}"] \arrow[ddr, "h"'] & Z \arrow[d, "\exp"] \\
& E \arrow[d, hook] \\
& Y^{\eta}
\end{tikzcd}$$

with $h(\nu(i)) = U$. The Riemannian metric on E defines through isomorphism $TX^{\perp} \simeq \nu(i)$ a metric g on $\nu(i)$. We denote by $Dnu(i)$ and $Sg\nu(i)$ the disc bundle and the sphere bundle with respect to this metric.
Let $\dot{D}g\nu(i) =^{\text{def}} Dg\nu(i) - Sg\nu(i)$.
We have an identification map:

$$\begin{tikzcd}
Y^{\eta}/(Y^{\eta} - h(\mathrm{int} Dg\nu(i))) \arrow[r, phantom, "\simeq"] & Dg\nu(i)/Sg\nu(i) \arrow[d, equal] \\
& X^{\nu(i)} \\
Y^{\eta} \arrow[uu] \arrow[ur, "i^*"'] &
\end{tikzcd}$$

The homotopy class of $i^* : Y^{\eta} \to X^{\nu(i)}$ is independent of the choice of tubular neighborhood with respact to a given metric. It is independent also of the choice of a particular metric on E, as any two are homotopic through the Riemannian metrics. Note however that it is essential that we restrict ourselves to the tubular neighborhoods determined by some metric.

Remark 3.1 *The assumption that $pf : X \to Y$ is proper was necessary for the Thom construction. This assumption was used in establishing the continuity of*

$$h^{-1} : Y^{\eta}/Y^{\eta} \backslash h(\dot{D}g(\nu_i)) \longrightarrow Dg\nu(i)/Sg\nu(i).$$

Proposition 3.2 *Let imbeddings $i_0, i_1 : X \to E$ be isotopic through the isotopy $H : X \times R \to E$ such that if $\widetilde{H} : X \times R \to E \times R$ is the imbedding determined by H then the composition $X \times R \to^{\widetilde{H}} E \times R \to^{p\times\mathrm{id}} Y \times R$ is proper. The Thom maps $Y^\eta \to X^{\nu(i_0)}$ and $Y^\eta \to X^{\nu(i_1)}$ are then homotopic.*

Proof: Take the product metric on $E \times E$ and apply the Thom construction to $\widetilde{H}$ on each level. This gives the requied homotopy.

□

The next result is formulated for a compact manifold X.

Proposition 3.3 *If $i_0, i_1 : X \to E$ are imbeddings which are stable homotopic then maps $Y^\eta \to X^{\nu(i_0)}$ and $Y^\eta \to X^{\nu(i_1)}$ are also stable homotopic.*

Proof: In view of the provious proposition this one follows from Whitney imbedding theorem. The fact that the obtained isotopy is proper is guaranteed by the compactness of X.

□

Let us consider the following generalization of the situation described above. As before, consider imbedding $i : X \to E$ such that the composition

$$\begin{tikzcd} & E \arrow[d] \\ X \arrow[ur, "i"] \arrow[r, "f"'] & Y \end{tikzcd}$$

is proper and suppose $\alpha = (p : E' \to Y)$ is an arbitrary vector bundle.

Theorem 3.4 *The Thom construction defines a canonical homotopy class of map $Y^{\eta\oplus\alpha} \to X^{\nu(i)\oplus f^*\alpha}$ which makes the following diagram commutative:*

$$\begin{tikzcd} X^{\nu(i)\oplus f^*\alpha} \arrow[d, "\triangle"] & & Y^{\eta\oplus\alpha} \arrow[ll] \arrow[d, "\triangle"'] \\ X^{\nu(i)} \wedge X^{f^*\alpha} \arrow[dr, "1\wedge f"'] & & Y^\eta \wedge Y^\alpha \arrow[dl] \\ & X^\nu \wedge Y^\alpha & \end{tikzcd}$$

Proof: Without loss of generality we can assume that $\alpha : E' \to^{p'} Y$ is a smooth bundle. Let $p'' : E'' \to Y$ be the Whitney sum $\alpha \oplus \eta$.
With $i : X \to E$ and the vector bundle α we associate an imbedding $i' : X \to E''$ which is the composition

$$\begin{array}{ccccc} X & \stackrel{(i,f)}{\longrightarrow} & E \times Y & \hookrightarrow & E \times E' \\ & & \downarrow & & \| \\ & & E'' & \hookrightarrow & E \times E' \end{array}$$

The Thom construction applied to i' gives us $X^{nu(i')} \to Y^{\alpha \oplus \eta}$. There is a canonical isomorphism of $\nu(i')$ with $\nu(i) \oplus f^*\alpha$. To establish this recall first that $TE'' = p_{"}{}^*(TY \oplus (\alpha \oplus \eta))$.
$\nu(i') = i'^*p''^*(TY \oplus (\alpha \oplus \eta))_{/TX} = i'^*p''^*(TY \oplus \eta) \oplus i'^*p''^*\alpha_{/TX} = i^*p^*(TY \oplus \eta) \oplus f^*\alpha_{/TX} = i^*p^*(TY \oplus \eta)_{/TX} \oplus f^*\alpha = \nu(i) \oplus f^*\alpha$.
In view of this canonical isomorphism we obtain the desired map $X^{\nu(i) \oplus f^*\alpha} \to Y^{\eta \oplus \alpha}$. The commutativity of the diagram from theorem 3.4 follows immediately from the method of the Thom construction.

Chapter 4

The Pontriagin–Thom theorem.

Let $\xi : E' \to^{p'} E$ be a n-dimensional vector bundle. Let $\eta : E \to^{p} Y$ be a smooth k-dimensional vector bundle over a manifold $Y; \dim Y = 1$. We shall consider submanifolds $V \subset E$ of codimension n such that the restriction $p|_V : V \to Y$ is a prpoer map. A ξ-structure on such a submanifold is a map $\nu(i) \to \xi$, where $i : V \hookrightarrow E$ is the inclusion and a submanifold with a ξ-structure is called a ξ-submanifold of η.

Definition 4.1 *Two ξ-submanifolds of η*

$$\begin{array}{ccccccc} V_0 & \overset{i_0}{\hookrightarrow} & E & & V_1 & \overset{i_1}{\hookrightarrow} & E \\ & f_0 \searrow & \downarrow p & , & & f_1 \searrow & \downarrow p \\ & & Y & & & & Y \\ \nu(i_0) & \xrightarrow{\varphi_0} & \xi & , & \nu(i_1) & \xrightarrow{\varphi_1} & \xi \end{array}$$

are cobordant iff there exists a ξ-submanifold $j : W \hookrightarrow E \times R$ of codimension n, such that $(p \times \mathrm{id}) \circ j : W \to Y \times R$ is a proper map and W satisfies the following compatibility condition:

1. *W is transversal to $E \times \{k\}$, $\quad k = 0, 1$ and $W \cap (E \times \{k\}) = V_k$.*
2. *ξ-structure on W correspondents under the isomorphism $\nu(j)|_{W \cap (E \times \{k\})} = \nu(i_k)$ to $\varphi : \nu(i_k) \to \xi$.*

Cobordism is an equivalence relation between ξ-submanifolds of η. The set of equivalence classes of this relation is denoted by $L(\eta; \xi)$.

Remark 4.2 *1. If Y is compact then this definition of cobordism coincides with the definition of bordism of compact submanifolds of a manifold E (see Bröcker, tom Dieck [4]).*

2. Let $\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "f"'] & E \arrow[d, "p"] \\ & Y \end{tikzcd}$ *be as in the definition, and suppose $\varphi_o : \nu(i) \to \xi$ and $\varphi_1 : \nu(i) \to \xi$ are homotopic as vector bundle maps. Then $(i : V \to E, \quad \varphi_0 : \nu(i) \to \xi)$ and $(i : V \to E, \quad \varphi_1 : \nu(i) \to \xi)$ are cobordant in $L(\eta, \xi)$.*

3. It is possible to give the analogous definition and prove similiar theorems if $E \to Y$ is locally trivial smooth map with the fibre being a manifold.

For each ξ-submanifold of η the map obtained from the Thom construction composed with the map induced on the Thom spaces by a ξ-structure gives a homotopy class of maps $Y^\eta \to B^\xi$. It is easy to see that cobordant ξ-submanifolds yield the same homotopy class $Y^\eta \to B^\xi$. Thus we obtain a map

$$P : L(\eta, \xi) \longrightarrow [Y^\eta, B^\xi].$$

The Pontriagin–Thom theorem holds.

Theorem 4.3 *The map $P : L(\eta, \xi) \to [Y^\eta, B^\xi]$ is a bijection.*

First we will prove this theorem in the case when ξ is a smooth bundle.
Proof: (ξ smooth)
We will construct an inverse map $Q : [Y^\eta, B^\xi] \to L(\eta, \xi)$.
Let $f : Y^\eta \to B^\xi$ be an arbitrary map. We deform f within its homotopy class to a "good" map in three steps:
Step 1:
There is a map $f_1 \in [f]$ which is differentiable on $A = f^{-1}(E')$, transversal to the zero section $B \hookrightarrow B^\xi$ and $f^{-1}(E') = A$.
Proof of step 1:
Take on E' a metric d for which compact sets are exactly the closed sets bounded with respect to d (such a metric clearly exists). There is a homotopy $H : A \times I \to E'$ with $H_0 = f|_A$, H_1 differentiable and transversal to the zero section and moreover for every $a \in A, \quad t \in I, \quad d(H_t(a), f(a)) \leq 1$. Prolong H to $\widetilde{H} : Y^\eta \times I \to B^\xi$ putting $\widetilde{H}(x,t) = \infty$ for $x \notin A$. To demonstrate the continuity of $\widetilde{H}$ it remains to show that for $U \ni \infty$ being an open neighborhood of ∞ in $B^\xi, \quad H^{-1}(U)$ is open in $Y^\eta \times I$. Let $D = B^\xi \backslash U$.

$H^{-1}(D) = \widetilde{H}^{-1}(D) \subset f^{-1}\{x : \quad d(x,D) \leq 1\} \times I$. To show that $H^{-1}(D)$ is a complement of an open neighborhood of $\infty \times I$ in $Y^\eta \times I$ we must have:

1. $H^{-1}(D)$ is closed in E.
2. projection of $H^{-1}(D)$ onto $Y \times I$ (Y = zero cross section of Y^η is proper).

Point 1. follows from the continuity of H and 2. from the fact that if f is continous then $f^{-1}\{x : \quad d(x,D) \leq 1\}$ is a complement of ∞ in Y^η and thus its projection on Y is proper, $f_1 := \widetilde{H}_1$.
Step 2:
Let $N = f_1^{-1}(B)$. It is a submanifold of E and from continuity of $f_1, \quad p| : N \to Y$ is a proper map. Let U be a tubular neighborhood of N in E. There is a map $f_2 : Y^\eta \to B^\xi$, homotopic to f_1 such that

1. f_2 is transversal to the zero section, $f_2^{-1}(B) = N$.
2. $f_2(x) = \infty$ for $x \notin U$.

Proof of step 2:
Let $V \subseteq U$ be a tubular neighborhood and $\overline{V} \subseteq U$. Let $s : Y^\eta \to [0,1]$ be a smooth function such that $s^{-1}(0) = \overline{V}, \quad s^{-1}[0,1) = U$. Let

$$H_t(x) = \begin{cases} \frac{1}{1-tsx} \cdot f_1(x) & x \in A \text{ and } t < 1 \text{ or } x \in U \text{ and } t = 1 \\ \infty & \text{elsewhere} \end{cases}$$

and put $f_2 = H_1$.
Step 3:
There is a map $f_3 : Y^\eta \to B^\xi$, homotopic to f_2 such that:

1. $U = f_3^{-1}(E')$.
2. the composition

$$\begin{tikzcd} \nu(N) \arrow[r] \arrow[dr, dashed, "\varphi"'] & U \arrow[d, "f_3"] \\ & E' \end{tikzcd}$$

is a vector bundle map.

Proof of this theorem can be found in Bröcker, tom Dieck [4] (only step 1. needed a small modification).
We would like to put $Q([F]) = [N, \varphi], \quad Q : [Y^\eta, B^\xi] \to l(\eta, \xi)$. This definition is dependent neither on the choice of f from $[f]$ nor on choices in steps 1,2,3. Any two maps obtained in step 3. starting from $[f]$ are homotopic. The ξ-submanifold on η determined by them are cobordant because we can proceed with the homotopy as in steps 1,2,3.
It is clear that $Q \circ P = \mathrm{id}$ and $P \circ Q = \mathrm{id}$.

□

Remark 4.4 *The inverse map to $P : L(\eta, \xi) \to [Y^\eta, B^\xi]$ can also be constructed more directly. We start from the map $f_1 : Y^\eta \to B^\xi$ which is differentiable on $f_1^{-1}(E)$ and transversal to $B \hookrightarrow B^\xi$. We put $Q'[f_1] = [f_1^{-1}(B), Df_1]$ where $Df_1 : \nu(f_1^{-1}(B)) \to \nu B \simeq \xi$ (see). It is easy to see that $Q = Q'$.*

The proof of theorem 5.3 is valid for $\xi : E' \to^{p'} B$ being a smooth bundle. We will extend this result to a vector bundle over an arbitrary CW complex B. Suppose first that B is a finite dimensional, locally finite countable simplicial complex. Every such complex can be imbedded as a deformation retract of some neighborhood in Euclidean space $U \subset R^N$. Let $r : U \to B$ be the retraction. We have an induced bundle:

$$\begin{array}{ccc} r^*E' & \longrightarrow & E' \\ \downarrow & & \downarrow p' \\ U & \longrightarrow & B \end{array}$$

We may assume that $r^*E' \to U$ is a smooth bundle. On the other hand it is clear that $L(\eta, \xi) = L(\eta, r^*\xi)$ and $[Y^\eta, B^\xi] = [Y^\eta, U^{r^*\xi}]$, and thus $L(\eta, \xi) = [Y^\eta, B^\xi]$ in case B is a finite dimensional, locally finite countable simplicial complex.
Suppose now that B is an arbitrary simplicial complex. From the triangulation theorems for manifolds it follows that $L(\eta, \xi) = \mathrm{colim}_C L(\eta, \xi|_C)$ where C are finite dimensional, locally finite countable simplicial subcomplexex of B (since every structure $\nu(V) \to \xi$, where $V \hookrightarrow E$ is the submanifold considered, factors through $\xi|_C$). From the approximation theorem we have also $[Y^\eta, B^\xi] = \mathrm{colim}_C [Y^\eta, C^\xi]$ and the quality holds for arbitrary simplicial complex B. Since every CW complex has the homotopy type of some simplicial complex thus $L(\eta, \xi) = [Y^\eta, B^\xi]$ holds for a vector bundle $\xi : E' \to E$ over an arbitrary CW complex B.

The analogue of the Pontriagin–Thom theorem can be formulated in a relative form.

Definition 4.5 *A pair (Y, A) is relative n-dimensional manifold iff A is a closed subset of Y and $Y \backslash A$ is a n-dimensional manifold.*

Assume that $\eta : E \to^p Y$ is a vector bundle over Y such that its restriction $\eta|_{Y\backslash A}$ over $Y\backslash A : E(\eta|_{Y\backslash A}) \to^p Y\backslash A$ is a smooth bundle.

Definition 4.6 *A proper submanifold of E over a relative manifold (Y, A) is a submanifold of $E(\eta|_{Y\backslash A})$ such that the projection on the base is proper as a map into Y.*

Let $i : W \hookrightarrow E(\eta|_{Y\backslash A})$ be such a submanifold. As in the absolute case a ξ-structure on W is a bundle map $\nu(i) \to \xi$ where $\xi : E' \to^{p'} B$ is the vector bundle over CW complex B, $\quad \dim \xi = \mathrm{codim} W$.

Consider all proper ξ-submanifolds of E over a relative manifold (Y, A).

Definition 4.7 *Two ξ-submanifolds of E over a relative manifold (Y, A) are cobordant over (Y, A) iff they are cobordant as submanifolds of $E(\eta|_{Y\backslash A})$ and in addition the submanifold of $E(\eta|_{Y\backslash A}) \times R$ establishing the cobordism is a proper ξ-submanifold over a relative manifold $(Y \times R, A \times R)$.*

We denote by $L(\eta, \eta|_A; \xi)$ the equivalence classes of the above relation. The following theorem holds:

Theorem 4.8 *The Pontriagin–Thom construction establishes the isomorphism between $L(\eta, \eta|_A; \xi)$ and the set of homotopy classes $\left[Y^\eta / A^{\eta|_A}, B^\xi\right]$, where ξ is the vector bundle over an arbitrary CW complex B.*

Chapter 5

Stabilization. Bordism and Cobordism Groups.

We now stabilize the previous situation. Assume that we are given a bundle spactrum $\xi = (\xi_k, \varepsilon_k), \quad \dim \xi = 0$. Let Y be a fixed manifold. We will consider $(\dim Y - n)$-dimensional ξ_k-submanifolds of the succesive product bundles $Y \times R^{k-n} \to Y$ (k varies) whose projection on Y is aproper map (see chapter 4.). The inclusion $R^{k-n} \hookrightarrow R^{k-n+1}$ taking $(x_1, \ldots . x_{k-n})$ to $(x_1, \ldots, x_{k-n}, 0)$ induces a map $\varkappa_k : L(\vartheta_{k-n}, \xi_k) \to L(\vartheta_{k-n+1}, \xi_{k+1})$ in the following way: if $V \subset Y \times R^{k_n}$ is a ξ_k-submanifold of ϑ_{k-n} we can define ξ_{k+1}-structure on $V \subset Y \times R^{k-n+1}$ to be $\vartheta_{k+1}(V) \to^{\varkappa}_{\simeq} \nu_k(V) \oplus \theta^1 \to \xi_k \oplus \theta^1 \to^{\varepsilon_k} \xi_{k+1}$ where $\varkappa$ is the isomorphism determined by the standart orientation of R^{k-n+1}.

For simplicity of notation we will write also $L(Y \times R^{k-n}, \xi_k)$ for $L(\vartheta_{k-n}, \xi_k)$.

It is easy to verify the commutativity of the following diagram:

$$\begin{array}{ccc}
L(Y \times R^{k-n}, \xi_k) & \xrightarrow{p} & \left[S^{k-n}Y^0, B_k^{\xi_k}\right] \\
\varkappa_k \downarrow & & \downarrow \tau_k \\
L(Y \times R^{k-n+1}, \xi_{k+1}) & \xrightarrow{p} & \left[S^{k-n+1}Y^0, B_{k+1}^{\xi_{k+1}}\right]
\end{array}$$

where B_k is the base of the bundle ξ_k and τ_k is the composition:

$$\begin{tikzcd}
\left[S^{k-n}Y^0, B_k^{\xi_k}\right] \arrow[r, "S"] \arrow[ddd, dashed, no head] \arrow[dddr, dashed, "\tau_k"] & \left[S^{k-n+1}Y^0, SB_k^{\xi_k}\right] \arrow[d, "\simeq"] \\
& \left[S^{k-n+1}Y^0, B_k^{\xi_k \oplus \theta'}\right] \arrow[dd] \\
& \\
& \left[S^{k-n+1}Y^0, B_{k+1}^{\xi_{k+1}}\right]
\end{tikzcd}$$

Passing to the limit on both sides we obtain:

$$\text{colim}_k\left[S^{k-n}Y^0, B_k^{\xi_k}\right] = \text{colim}_k L(Y \times R^{k-n}, \xi_k)$$

If Y has the homotopy type of finite CW complex then $\text{colim}_k\left[S^{k-n}Y^0, B_k^{\xi_k}\right]$ is the n^{th} cohomology group of Y with coefficients in the spectrum $\{B_k^{\xi_k}, \varepsilon_k\}$. We denote this theory by $B^*(;\xi)$.

We have also a similiar formula for homology groups.

$$B_n(Y,\xi) = \text{colim}_k\left[S^{k+n}, Y^0 \wedge B_k^{\xi_k}\right] = \text{colim}_k\left[S^{k+n}, (Y \times B_k)^{\theta_Y^0 \times \xi_k}\right] = \text{colim}_k L(R^{k+n}, \xi \times \theta_Y^0),$$

where R^{k+n} denotes a $(n+k)$-dimensional vector bundle over a point and θ_Y^0 denotes a zero-dimensional vector bundle over Y. Notice that in the case of homology all considered submanifolds of R^{k+n} are compact.

Remark 5.1 *If Y is a finite CW complex then there exists an open manifold of the homotopy type of Y (e.g. an open neighborhood of Y in R^m for suitable m).*

The Whitney imbedding and isotopy theorems enable us to get rid of particular imbeddings and thus to interpret $\text{colim}_k L(Y \times R^{k-n}, \xi_k)$ in terms of abstract manifolds.

Let $f : V \to Y$ be a smooth map and $\nu(f); f^*\tau(Y) - \tau V$ a spectrum of normal bundle of f. We denote: $\dim f =: \dim \nu(f)$. We will introduce the equivalence relation in the set of n-dimensional proper maps $f : V \to Y$ edowed with ξ-orientations (V varies). For appropriate definition see chapter 1.

Definition 5.2 *We will say that the n-dimensional proper maps $(f_0 : V_0 \to Y, \alpha_0 : \nu(f_0) \to \xi \oplus n)$, $(f_1 : V_1 \to Y, \alpha_1 : \nu(f_1) \to \xi \oplus n)$ are cobordant iff there exists a ξ-oriented n-dimensional proper map $g : W \to Y \times R$, $\beta : \nu(g) \to \xi \oplus n$ such that:*

1. *$g : W \to Y \times R$ is transversal to $Y \times \{0\}$ and $Y \times \{1\}$,*

2. *there are diffeomorphisms:*

$$\varphi_0 : V_0 \to g^{-1}(Y \times \{0\})$$
$$\varphi_1 : V_1 \to g^{-1}(Y \times \{1\})$$

   for which the following diagrams commute:

$$\begin{array}{ccc} V_0 & \xrightarrow{\varphi_0} & g^{-1}(Y \times \{0\}) \\ & f_0 \searrow & \downarrow g \\ & & Y \end{array} \qquad \begin{array}{ccc} V_1 & \xrightarrow{\varphi_1} & g^{-1}(Y \times \{1\}) \\ & f_1 \searrow & \downarrow g \\ & & Y \end{array}$$

3. *The diffeomorphism φ_i $(i = 0, 1)$ induces the isomorphism of $f_i^*\tau(Y) - \tau(V_i)$ with $g^*|_{g^{-1}(Y \times \{i\})}\tau(Y) - \tau(\varphi_i(V_i))$ (see 2.21). The latter bundle is $\nu(g)|_{g^{-1}(Y \times \{i\})}$. We require that the diagram:*

$$\begin{array}{ccc} \nu(f_i) & \xrightarrow{\approx} & \nu(g)|_{g^{-1}(Y \times \{i\})} \\ & \alpha_i \searrow & \downarrow p|_{\nu(g)|_{g^{-1}(Y \times \{i\})}} \\ & & \xi + n \end{array}$$

   commutes for $i = 0, 1$.

The above relation is an equivalence and its classes are denoted by $\mathcal{B}^n(Y, \xi)$.

There exists a natural map:

$$\Phi : \mathrm{colim}_k L(Y \times R^{k-n}, \xi_k) \longrightarrow \mathcal{B}^n(Y, \xi)$$

which with every submanifold $V \subset Y \times R^{k-n}$ and $\nu(V) \to \xi_k$ associates a manifold V and a proper map $\pi|_V : V \to Y$ where $\pi : Y \times R^{k-n} \to Y$ is the projection. The map of bundle spectra $\nu(\pi) \to \xi + n$ is induced in the obvious way by $\nu(V) \to \xi_k$. It is clear that this map is well defined.

Theorem 5.3 *The map Φ is a bijection.*

Proposition 5.4 *Let $i_0, i_1 : V \to Y \times R^{k-n}$ be imbeddings lifting the same proper map f (e.g.* $\begin{array}{ccc} V & \overset{i_0,i_1}{\rightrightarrows} & Y \times R^{k-n} \\ & f \searrow & \downarrow \pi \\ & & Y \end{array}$ *) and let $\nu(i_0(V)) \to \xi_k, \quad \nu(i_1(V)) \to \xi_k$ be the ξ_k-structures. If there exists an isotopy for which the ξ_k-structures correspondend under the isomorphism $\nu(i_0(V)) \to \nu(i_1(V))$ defined by the isotopy then the ξ_k-submanifolds $(i_0(V), \nu(i_0(V)) \to \xi_k)$ and $(i_1(V), \nu(i_1(V)) \to \xi_k)$ are cobordant in $L(Y \times R^{k-n}, \xi_k)$.*

Proof: recall results os chapter 2. If $H : V \times R \to Y \times R^{k-n} \times R$ is the isotopy then $H(Y \times R) \hookrightarrow Y \times R^{k-n} \times R$ is the submanifold establishing the required cobordism, where the ξ_k-structure on $\nu(H(V \times R))$ arises from the composition

$$\nu(H(V \times R)) \to \nu(H(V \times \{0\})) \to \nu(i_0(V)) \to \xi_k.$$

□

Remark 5.5 *If the condition in the proposition 5.4 is fullfilled then the bundle maps $\nu(i_0(V)) \to \xi_k$ and $\nu(i_1(V)) \to \xi_k$ determine the same map of bundle spactra $\nu(f) \to \xi + n$. Also, if they determine the same map of bundle spectra then the thesis of proposition 5.4 is true for large k and thus $(i_0(V), \nu(i_0(V)) \to \xi_k)$ and $(i_1(V), \nu(i_1(V)) \to \xi_k)$ represent the same element in* $\operatorname{colim}_k L(Y \times R^{k-n}, \xi_k)$.

Proof of theorem 5.3:

It is clear that Φ is surjective. We will now prove that Φ is injective. Suppose that images of two ξ_k-submanifolds $i_0, i_1 : V_0, V_1 \hookrightarrow Y \times R^{k-n}; \quad \alpha_0 : \nu(V_0) \to \xi_k, \quad \alpha_1 : \nu(V_1) \to \xi_k$ are cobordant in the sense of definition 5.2. Thus there exists a manifold W and a proper ξ-oriented map $g : W \to Y \times R$ satisfying 1,2,3. Let $j : W \to R^{k-n}$ be an imbedding (we can assume k is large) and consider the diagonal:

$$\begin{array}{ccc} & & Y \times R^{k-n} \times R \\ \widetilde{g} = (g, j) & \nearrow & \downarrow \\ W & \xrightarrow{g} & Y \times R \end{array}$$

with the ξ_k-structure $\beta : \nu(\widetilde{g}(W)) \to \xi_k$ inducing the ξ-orientation of g. according to definition 4.1, the ξ-submanifold $\widetilde{g}(W)$ establishes cobordism in $L(Y \times R^{k-n}, \xi_k)$ between $U_0 := \widetilde{g}(g^{-1}(Y \times \{0\})), \quad \beta|_{U_0} : \nu(\widetilde{g}(W))|_{U_0} \to \xi_k$ and $U_1 := \widetilde{g}(g^{-1}(Y \times \{1\})), \quad \beta|_{U_1} : \nu(\widetilde{g}(W))|_{U_1} \to \xi_k$.

Consider imbeddings $i_0 : V_0 \hookrightarrow Y \times R^{k-n}$ and $V_0 \to^{\varphi_0} g^{-1}(Y \times \{0\}) \to^{\widetilde{g}} Y \times R^{k-n}$. From condition 2. we have $\pi i_0 = \pi \widetilde{g} \varphi_0$ and from 3. to see that ξ_k-structures on $\nu(i_o(V))$ and $\nu(\widetilde{g}\varphi_0 V) = \varphi_0^* \nu(\widetilde{g}(g^{-1}(Y \times \{0\})))$ define the same ξ-orientation $\nu(\pi i_0 \to \xi + n$. Analougously for V_1 and U_1. Thus injectivity of Φ follows from proposition 5.4.$\Box$

We will give now another definition of the cobordism relation. it is less natural then the previous one arising from stabilisation of $L(Y \times R^{k-n}, \xi_k)$ but it is similiar to the wellknown definition of geometric bordism.

Definition 5.6 *Two n-dimensional ξ-oriented proper maps $(f_i : V_i \to Y, \quad \alpha_i : \nu(f_i) \to \xi + n), \quad i = 0, 1$ are cobordant iff there exists a manifold with boundary W and a ξ-oriented proper map $g : W \to Y$ such that:*

1. *there is a diffeomorphism $\varphi : V_0 \cup V_1 \to \partial W$ such that the diagram*

$$\begin{array}{ccc} V_0 \cup V_1 & \xrightarrow{\varphi} & W \\ f_0 \cup f_1 & \searrow & \downarrow g \\ & & Y \end{array}$$

   commutes.

2. *the ξ-orientation of g agrees with those of $f_i, \quad i = 0, 1$ in the sense that the following diagram for f_0 and the analogous for f_1 commutes:*

$$\begin{array}{ccc} \nu(f_0) & \xrightarrow{\simeq} & \nu(g|_{\varphi(V_0)}) \\ \alpha_0 \downarrow & & \downarrow \delta_0 \\ \xi + n & & \nu(g)|_{\varphi(V_0} \oplus \theta^1 \\ & \nwarrow \quad \swarrow & \\ & \xi \oplus \theta^1 + n - 1 & \end{array}$$

   where the isomorphism $\simeq : \nu(f_0) \to \nu(g|_{\varphi(V_0)})$ is induced by φ and the map $\xi \oplus \theta' + n - 1 \to \xi + n$ is defined by id_ξ. The isomorphisms

$$\delta_i : \nu(g|_{\varphi(V_i)}) \longrightarrow \nu(g)|_{\varphi(V_i)} \oplus \theta^1$$

   require the choice of field of vectors normal to the boundary; we choose the inner normal field for V_0 and the outer normal field for V_1.

Theorem 5.7 *Definitions 5.2 and 5.6 are equivalent.*

Proof: It is clear that two maps cobordant in the sense of definition 5.2 are also cobordant in the sense of definition 5.6. Suppose $(f_0 : V_0 \to Y, \alpha_0 : \nu(f_0) \to \xi)$ and $(f_1 : V_1 \to Y, \alpha_1 : \nu(f_1) \to \xi)$ are equivalent in the sense of definition 5.6 and $g : W \to Y, \beta : \nu(g) \to \xi)$ establishes this equivalence: we will show that they are cobordant in the sense of definition **??**
Step 1:
Adding the Morse function on the last coordinate we obtain a proper map $h : W \to Y \times R$ such that:

1. $h(W) \subseteq Y \times [0,1]$,
2. $h^{-1}(Y \times \{i\}) = \varphi(V_i), \quad i = 0,1,$
3. $h \pitchfork Y \times \{i\}, \quad i = 0,1,$
4. $h|_{\varphi(V_i)} = g|_{\varphi(V_i)}, \quad i = 0,1.$

We can glue smoothly $\varphi(V_0) \times (-\infty, 0 >$ and $\varphi(V_1) \times < 1, +\infty)$ and prolong h to get a manifold without boundary $\widetilde{W}$ and $\widetilde{h} : \widetilde{W} \to Y \times R$ putting $\widetilde{h}(x,t) = (h(x), t)$ for $x \in \varphi(V_0 \cup V_1), \quad t \leq 0$ or $t \geq 1$ (see the figure below).
The orientation of $\widetilde{h} : \widetilde{W} \to Y \times R$:
The map $\beta : \nu(g) \to \xi$ determines the ξ-orientation $\nu(h) \to \xi$ because there is a natural map $\nu(h) \to \nu(g) + 1$ (any imbedding of W into $Y \times R \times R^{k-n}$ "over h" is an imbedding into $Y \times R^{k-n+1}$ "over g"). The latter ξ-orientation can be prolonged in a natural way to the ξ-orientation $\gamma : \nu(\widetilde{h}) \to \xi$.
We claim that $(\widetilde{h} : \widetilde{W} \to Y \times R, \gamma : \nu(\widetilde{h}) \to \xi)$ realizes the cobordism between $(f_0 : V_0 \to Y, \alpha_0 : \nu(f_0) \to \xi)$ and $(f_1 : V_1 \to Y, \alpha_1 : \nu(f_1) \to \xi)$ in the sense of definition **??**.
Step 2:
We see that $(\widetilde{h} : \widetilde{W} \to Y \times R, \gamma : \nu(\widetilde{h}) \to \xi)$ sets up the equivalence in the sense of definition **??** between $(i : V_i \to Y, \beta_i : \nu(f_i) \to \xi)$ for $i = 0,1$ where β_i is the orientation: $\nu(f_i) \to^{(\varphi_i)} \nu(\widetilde{h})|_{\varphi_i(V_i)} \to^{\gamma} \xi$.
Step 3:
It remains to show that the orientations $\alpha_0 : \nu(f_0) \to \xi$ and $\beta_0 : \nu(f_0) \to \xi$ are the same and analogously for $\nu(f_1)$.
Take any imbedding $j : \widetilde{W} \to (Y \times R) \times R^{k-n}$ "over h" with $\widetilde{\gamma} : \nu(\widetilde{W} \to \xi_k$ being the representant of $\gamma : \nu(\widetilde{h}) \to \xi$ as on the diagram:

$$\begin{array}{ccc} & & (Y \times R) \times R^{k-n} \\ & j \nearrow & \downarrow \\ \widetilde{W} & \xrightarrow{\widetilde{h}} & Y \times R \end{array}$$

V_0 is then imbedded into $Y \times \{0\} \times R^{k-n}$, $\quad j\varphi(V_0) \hookrightarrow Y \times \{0\} \times R^{k-n}$ where $\nu_k(j\varphi(V_0))$ is the normal bundle. Then $\widetilde{\gamma}|_{\varphi(V_0)} : \nu_k(j\varphi(V_0)) = \nu(\widetilde{W})|_{\varphi(V_0)} \to \xi_k$ represents $\beta_0 : \nu(f_0) \to \xi$ and let $\widetilde{\alpha}_0 : \nu_k(j\varphi(V_0)) \to \xi_k$ represents $\alpha_0 : \nu(f_0) \to \xi$.
When we consider $j\varphi(V_0)$ as a submanifold of $Y \times R^{k-n+1}$ with the normal bundle $\nu_{k+1}(j\varphi(V_0))$ the above structures are respectively:

1. $\nu_{k+1}(j\varphi(V_0)) \overset{\varkappa,\simeq}{\to} \nu_k(j\varphi(V_0)) \oplus \theta \overset{\widetilde{\gamma}_1 \oplus \mathrm{id}}{\to} \xi_k \oplus \theta \overset{\varepsilon}{\to} \xi_{k+1}$
2. $\nu_{k+1}(j\varphi(V_0)) \overset{\varkappa,\simeq}{\to} \nu_k(j\varphi(V_0)) \oplus \theta \overset{\widetilde{\alpha}_0 \oplus \mathrm{id}}{\to} \xi_k \oplus \theta \overset{\varepsilon_k}{\to} \xi_{k+1}$

where $\varkappa$ denotes the canonical isomorphism determined by the orientation of the Euclidean space. The isomorphism $\delta_0 : g^*|_{\varphi(V_0))}\tau(Y) - \tau(\varphi(V_0)) \simeq g^*\tau(Y) - \tau(W)|_{\varphi(V_0)} \oplus \theta$ from definition 5.6.2 gives an isomorphism of bundles: $\widetilde{\delta}_0 : \nu_{k+1}(j\varphi(V_0)) \to \nu(j(\widetilde{W}))|_{\varphi(V_0))} \oplus \theta = \nu_k(j\varphi(V_0)) \oplus \theta$.
From condition 1 in definition 5.6 we have the homotopic commutativity of the diagram

$$\begin{tikzcd}
& \nu_{k+1}(j\varphi(V_0)) \arrow[dl, "\varkappa"'] \arrow[dr, "\widetilde{\delta}_0"] & \\
\nu_k(j\varphi(V_0)) \oplus \theta \arrow[d, "\widetilde{\alpha}_0 \oplus \mathrm{id}"'] & & \nu_k(j\varphi(V_0)) \oplus \theta \arrow[d, "\widetilde{\gamma}| \oplus \mathrm{id}"] \\
\xi_k \oplus \theta \arrow[dr, "\varepsilon_k"'] & & \xi_k \oplus \theta \arrow[dl, "\varepsilon_k"] \\
& \xi_{k+1} &
\end{tikzcd}$$

Because the choice of the inner field of vectors normal to the boundary we have $\widetilde{\delta}_0 = \varkappa$ (see the figure below) and therefore the structures 1 i 2 are homotopic.
Analogous reasoning holds for V_1. This ends the proof.

□

When Y has the homotopy type of a finite CW complex then we have a geometric interpretation of cohomology groups $B^*(Y, \xi)$.

$$\mathcal{B}^n(Y;\xi) = \mathrm{colim}_k \left[S^{k-n} Y^0, B_k^{\xi_k} \right]$$

This equality is established by a map
$\mathcal{B}^n(Y;\xi) \to^{\Phi^{-1}} \mathrm{colim}_k L(Y \times R^{k-n}, \xi_k) \to^p B^n(Y;\xi)$ which will be also denoted by P and called the Pontriagin–Thom isomorphism.

To obtain the interpretation of homology groups

$$\mathcal{B}_n(X;\xi) = \mathrm{colim}_k \left[S^{k+n}, X^0 \wedge B_k^{\xi_k} \right] = \mathcal{B}^{-n}(\mathrm{pt}, \xi \times \theta_X^0)$$

it remains to put $Y = \mathrm{pt}$. In this case the proper map into a point with $\xi \times \theta_X^0$ orientation is simply $-\tau(M) \to \xi \times \theta_X^0 - n$ where M is a compact manifold. this orientation consists of a ξ-structure $-\tau(M) \to \xi - n$ and a map $f : M \to X$. Thus we can reformulate definition 5.6.

Definition 5.8 *A singular n-dimensional ξ-manifold of X is a compact n-dimensional manifold H with a structure $-\tau(M) \to \xi - n$ and a map $f : H \to X$. The singular manifolds $(V_i, f_i : V_i \to X, -\tau(V_i) \to \xi - n), \quad i = 0, 1$ are bordant iff there exists a singular ξ-manifold with boundary $g : W \to X$ such that $\partial W \simeq V_0 \cup V_1, \quad g|_{V_i} = f_i, \quad i = 0, 1$ and ξ-structure on W extends those on V_0 and V_1 in the sense of definition 5.6.*

Geometric interpretation of $B^n(Y, A; \xi)$.

If Y is a finite CW complex and A its subcomplex then we can find a pair of the same homotopy type such that (Y, A) is a relative manifold.
We can now give the definition of $\mathcal{B}^n(Y, A; \xi)$.

Definition 5.9 *Let $(f_i : V_i \to Y \backslash A, \nu(f_i) \to \xi), \quad i = 0, 1$ be proper ξ-oriented maps into (Y, A) (i.e. the composition $f_i : V_i \to Y \backslash A \hookrightarrow Y$ is proper for $i = 0, 1$). We say that they are equivalent iff there is a proper ξ-oriented map $(g : W \to Y \backslash A, \nu(g) \to \xi)$ into (Y, A) which establishes the cobordism between them in $\mathcal{B}^n(Y \backslash A, \xi)$. The equivalence classes of this relation are elements of $\mathcal{B}^n(Y, A; \xi)$.*

Theorem 5.10 *$\mathcal{B}^*(Y, A; \xi)$ is isomorphic to $B^*(Y, A; \xi)$.*

The interpretation of relative groups in bordism is well known (Conner, Floyd [5]).

Chapter 6

Functoriality and Group Structure.

The set of homotopy classes of maps $[Y^\eta, B^\xi]$ has functorial properties with respect to maps of vector bundles (also with respect to their homotopy classes): a map $\varphi : \eta \to \eta'$ induces $\varphi^* : [Y^{\eta'}, B^\xi] \to [Y^\eta, B^\xi]$ and $\psi : \xi \to \xi'$ induces $\psi_* : [Y^\eta, B^\xi] \to [Y^\eta, B^{\xi'}]$. We will give a geometric description of induced maps $\varphi^\# : L(\eta, \xi) \to L(\eta, \xi)$ and $\psi_\# : L(\eta, \xi) \to L(\eta, \xi)$ corresponding to the previous ones under the Pontriagin–Thom isomorphism.

Definition 6.1 *Let V be a ξ-submanifold of η with a normal bundle $\nu(V)$. We define $\psi_\# : L(\eta, \xi) \to L(\eta, \xi')$ putting*

$$\psi_\#(V, \nu(V) \to \xi) = (V, \nu(V) \to \xi \to^\psi \xi').$$

Theorem 6.2 *The following diagram is commutative:*

$$\begin{array}{ccc} L(\eta, \xi) & \xrightarrow{\psi_\#} & L(\eta, \xi) \\ P \downarrow & & \downarrow P \\ [Y^\eta, B^\xi] & \xrightarrow{\psi_*} & [Y^\eta, B^{\xi'}] \end{array}$$

The proof is evident and we omit it.

The description of $\varphi^\# : L(\eta, \xi) \to L(\eta, \xi)$ is more delicate. We need first the followiing fact.

Let $\begin{array}{ccc} E(\eta) & \xrightarrow{\overline{\varphi}} & E(\eta') \\ \pi \downarrow & & \downarrow \pi' \\ Y & \xrightarrow{\varphi} & Y' \end{array}$ be a smooth map between vector bundles η and η'.

Let $V \subset E(\eta')$ be a submanifold. Then there exists a vector bundle map

$(\overline{\varphi}_1, \varphi_1)$ homotopic to $(\overline{\varphi}, \varphi)$ and such that $\overline{\varphi}_1$ is transversal to V.
This fact follows from the Thom transversality theorem for maps and the Homotopy Covering Property, when we notice that $\overline{\varphi}_1 \pitchfork V$ iff $\varphi_1 \pitchfork \pi'|_V$.

Definition 6.3 *Let V be a ξ-submanifold of η'. we define $\varphi^{\#} : L(\eta,\xi) \to L(\eta,\xi)$ putting $\varphi^{\#}(V, \nu(V) \to \xi) = (\varphi_1^{-1}(V), \overline{D\varphi_1}\nu(\varphi_1^{-1}(V)) \to \nu(V) \to \xi)$ where φ_1 is homotopic to φ and $\overline{\varphi}_1 \pitchfork V$ (thus $\overline{\varphi}_1^{-1}(V)$ is a submanifold of η). For definition of $\overline{D\varphi_1}$ see 2.24. Notice that $\overline{\varphi}_1^{-1}(V)$ is the pull-back of the diagram*

$$\begin{tikzcd}
\varphi_1^{-1}(V) \arrow[r, dashed, "back"] \arrow[d, dashed] & V \arrow[d, "\pi'|_V"] \\
Y \arrow[r, "\varphi_1"'] & Y
\end{tikzcd}$$

Independence of the particular choice of φ_1 follows from the Thom transversality theorem. it can also be deduced from the following theorem.

Theorem 6.4 *The diagram*

$$\begin{array}{ccc}
L(\eta',\xi) & \xrightarrow{\varphi^{\#}} & L(\eta,\xi) \\
P\downarrow & & \downarrow P \\
[Y^{\eta'}, B^{\xi}] & \xrightarrow{\varphi^{*}} & [Y^{\eta}, B^{\xi}]
\end{array}$$

is commutative.

Proof: It will be more convenient to verify the commutativity of the diagram

$$\begin{array}{ccc}
L(\eta,\xi) & \xrightarrow{\varphi^{\#}} & L(\eta,\xi) \\
Q'\uparrow & & \uparrow Q' \\
[Y^{\eta'}, B^{\xi}] & \xrightarrow{\varphi^{*}} & [Y^{\eta}, B^{\xi}]
\end{array}$$

where Q' is the inverse of P described in remark 4.4.
Let $f : Y^{\eta'} \to B^{\xi}$ be transversal to the zero section. We have $Q'([f]) = (f^{-1}(B), \overline{Df} : \nu(f^{-1}(B)) \to \xi)$. We assume $\varphi \pitchfork f^{-1}(B)$.
Then $\varphi^{\#}(f^{-1}(B), \overline{Df} : \nu(f^{-1}(B)) \to \xi) = ((f \circ \varphi)^{-1}(B), \overline{D\varphi} \circ \overline{Df} : \nu((f \circ \varphi)^{-1}(B)) \to \xi)$.
It is easy to verify that $\overline{D\varphi} \circ \overline{Df} = D(f \circ \varphi)$ and therefore to see that the right hand side of the above equation is $Q'\varphi^*$.

□

From this interpretation of induced maps we can obtain easily an interpretation of induced maps in bordism and cobordism theories.

Definition 6.5 *Let $f : X \to Y$ be a continous map. We define $f_{\#} : \mathcal{B}_*(X,\xi) \to \mathcal{B}_*(Y,\xi)$ putting:*

$$f_{\#}(M, g : M \to X, -\tau(M) \to \xi) = (M, f \circ g : M \to Y, -\tau(M) \to \xi).$$

To give the analogous definition for cobordism assume that X, Y are manifolds and $f : X \to Y$ is a smooth map (we can always do that within the same homotopy type).

Definition 6.6 *We define $f^{\#} : \mathcal{B}^*(Y,\xi) \to \mathcal{B}^*(X,\xi)$ as follows:*

$$f^{\#}(V, g : V \to Y, \nu(g) \to \xi) = (f^*(V), \overline{g} : f^*V \to X, \overline{Df} : \nu(g) \to \nu(g) \to \xi)$$

assuming $f \pitchfork g$ and f^*V is the pull back:

$$\begin{array}{ccc} f^*V & \xrightarrow{\overline{f}} & V \\ \overline{g}\downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

The map $\overline{Df} : \nu\overline{g} \to \nu(g)$ is induced by corresponding maps on unstable levelas in definition 6.3.

It is easy to verify that all conditions allowing the passage to the limit with the diagrams from theorems 6.2 and 6.4 are fulfilled. Therefore we obtain:

Theorem 6.7 *Let $f : X \to Y$ and let*

$$\begin{array}{c} f_* : \mathbb{B}_*(X,\xi) \to \mathbb{B}_*(Y,\xi); f_{\#} : \mathcal{B}_*(X,\xi) \to \mathcal{B}_*(Y,\xi) \\ f^* : \mathbb{B}^*(X,\xi) \to \mathbb{B}^*(Y,\xi); f^{\#} : \mathcal{B}^*(X,\xi) \to \mathcal{B}^*(Y,\xi) \end{array}$$

denote the induced maps in "homotopical" and "geometrical" bordism and cobordism theories respectively. The maps $f_*, f_{\#}$ and $f^*.f^{\#}$ correspond under the Thom–Pontriagin isomorphism.

Unlike the functorial properties, the group structure can be defined in the stable range only. The track-addition in $B^q(X,\xi) = \mathrm{colim}_k \left[S^{k-n}X, B_k^{\xi_k}\right]$ corresponds to disjoint sum of manifolds in "geometric" cobordism; exactly the same holds for bordism.

We can also describe the "inverse element" operation in $\mathcal{B}^*(X, xi)$. Let $[V, f : V \to X, \varphi : \nu(f) \to \xi]$ be an element of $\mathcal{B}^*(X,\xi)$. Its inverse element $-[V, f : V \to X, \varphi : \nu(f) \to \xi]$ is represented by $[V, f : V \to X, \varphi' : \nu(f) \to \xi]$ where φ' is the composition: $\varphi \oplus \theta : \nu(f) \oplus \theta \to \xi \oplus \theta \to \xi$.

Chapter 7

Products.

The pairing of spectra (see Whitehead [17]) defines the products between the associated homology and cohomology theories. The purpose of this chapter is to give the geometric interpretation of those products in the case of bordism and cobordism theories.
Let $(\xi'_n, \theta'_n), (\xi"_n, \theta"_n), (\xi_n, \theta_n)$ be the bundle spectra such that for each p, q there is a bundle map:

$$\begin{array}{ccc} \xi'_p \times \xi"_q & \longrightarrow & B'_p \times B"_q \\ m_{p,q} \downarrow & & \downarrow \\ \xi_{p+q} & \longrightarrow & B_{p+q} \end{array}$$

Passing to Thom spaces of these bundles we obtain maps:

$$B'^{\xi'_p}_p \wedge B"^{\xi"_q}_q \overset{T(m_{p,q})}{\longrightarrow} B^{\xi_{p+q}}_{p+q}.$$

Assume that the bundle maps above satisfy the properties analogous to those which maps $T(m_{p,q})$ have to satisfy in order to define the pairing of spectra. We shall call maps $m_{p,q}$ the pairing of the bundle spectra.
Similarly a pairing $m_{pq} : \xi_p \times \xi_q \to \xi_{p+q}$ and a unit map $\mu : \theta \to \xi$ where θ denotes a trivial spectrum defines a bundles ring spectrum.

Remark 7.1 *For simplicity we will write $m : \xi' \times \xi' \to \xi$ to denote the pairing of bundle spectra. This notation is justified because it is possible to define the smash-product in the homotopy category of bundle spactra. Its construction is similiar to that of the smash-product in the category of spectra (see Adams [1]).*
The pairing of the bundle spectra $m_{p,q} : \xi'_p \times \xi"_q \to \xi_{p+q}$ defines then a map $m : \xi' \times \xi" \to \xi$.

Example 7.2 *For each bundle spectrum ξ there is a pairing with the trivial spectrum $m : \theta \times \xi \to \xi$. This pairing defines canonical pairing $m^* : S \wedge B^\xi \to B^\xi$ where S denotes the sphere spectrum and B^ξ the Thom spectrum of ξ.*

Example 7.3 *Let $\{G_n\}$ be a sequence of groups such that $G_n \subset O(n)$ and that the diagram*

$$\begin{tikzcd} G_n \arrow[r, phantom, "\subset"] \arrow[d, hook] & G_{n+1} \arrow[d, hook] \\ O(n) \arrow[r, hook] & O(n+1) \end{tikzcd}$$

commutes for each n.
We assume that $G_n \times G_m \hookrightarrow G_{n+m}$. Such a sequence defines a bundle ring spectrum $\gamma(G) = (\gamma(G_n), \theta_n)$ where $\gamma(G_n)$ denotes the classyfying G_n bundle over BG_n.

Let $m : \xi' \times \xi" \to \xi$ be the pairing of bundle spectra. We consider four external products determined by the associated pairing of Thom spectra.

$$\text{Cross products} \begin{cases} \underline{\wedge} : B_p(X;\xi') \otimes B_q(Y;\xi") \longrightarrow B_{p+q}(X \times Y;\xi) \\ \overline{\wedge} : B^p(X;\xi') \otimes B^q(Y;\xi") \longrightarrow B^{p+q}(X \times Y;\xi) \end{cases}$$

$$\text{Slant products} \begin{cases} / : B^p(X \times Y;\xi') \otimes B_q(Y;\xi") \longrightarrow B^{p-q}(X;\xi) \\ \backslash : B^p(X,\xi') \otimes B_q(X \times Y;\xi") \longrightarrow B_{q-p}(Y;\xi) \end{cases}$$

These products can also be defined using a geometric description of bordism and cobordism.

Definition 7.4 *Cross product in bordism*

$$\underline{\wedge} : \mathcal{B}_p(X;\xi') \otimes \mathcal{B}_q(Y;\xi") \longrightarrow \mathcal{B}_{p+q}(X \times Y;\xi)$$

$[M, f : M \to X, -\tau(M) \to \xi]\underline{\wedge}[N, g : N \to Y, -\tau(N) \to \xi"] := [M \times N, f \times g : M \times N \to X \times Y, -\tau(M \times N) \to \xi]$ where $-\tau(M \times N) \to \xi$ is the compositon: $\substack{\text{canonical}\\ \text{isomorphism}} : -\tau(M) \times -\tau(N) \to \xi' \times \xi" \to^m \xi$.

Definition 7.5 *Cross product in cobordism*

$$\overline{\wedge} : \mathcal{B}^p(X;\xi') \otimes \mathcal{B}^q(Y;\xi") \longrightarrow \mathcal{B}^{p+q}(X \times Y;\xi)$$

$[Z, f : Z \to X, \nu(f) \to \xi'] \,\underline{\vee}\, [W, g : W \to Y, \nu(g) \to \xi"] := [Z \times W, f \times g : Z \times W \to X \times Y, \nu(f \times g) \to \xi]$ where $\nu(f \times g) \to \xi$ is the composition: $\substack{\text{canonical}\\ \text{isomorphism}} : \nu(f \times g) \to \nu(f) \times \nu(g) \to \xi' \times \xi" \to \xi$.

Definition 7.6 */-slant product*

$$/ : \mathcal{B}^p(X \times Y : \xi') \otimes \mathcal{B}_q(Y;\xi'') \longrightarrow \mathcal{B}^{p-q}(X;\xi)$$

$[Z, f : Z \to X \times Y, \nu(f) \to \xi]/[M, g : M \to Y, -\tau(M) \to \xi''] := [W, h : W \to X, \nu(h) \to \xi]$ *where* $h : W \to X$ *is defined by the following pull back diagram (we choose* $f \pitchfork 1 \times g$*):*

$$\begin{tikzcd}
W \arrow[r, "\overline{f}"] \arrow[d, "\overline{(\mathrm{id} \times g)}"'] \arrow[dr, phantom, "back"] \arrow[ddr, bend right, "h"'] & X \times M \arrow[d, "\mathrm{id} \times g"] \\
Z \arrow[r, "f"'] & X \times Y \arrow[d] \\
 & X
\end{tikzcd}$$

$h := \pi \circ f \circ \overline{(\mathrm{id} \times g)} : W \to X$ *is a proper map.*

Definition 7.7 *$\backslash$-slant product*

$$\backslash : \mathcal{B}^p(X;\xi') \otimes \mathcal{B}_q(X \times Y;\xi'') \longrightarrow \mathcal{B}_{q-p}(Y;\xi)$$

$[Z, f : Z \to X, \nu(f) \to \xi']\backslash[M, g : M \to X \times Y, -\tau(M) \to \xi''] := [N, h : N \to Y, -\tau(N) \to \xi]$ *where* $h : N \to Y$ *is defined by the pull back (assuming* $g \pitchfork f \times \mathrm{id}$*):*

$$\begin{tikzcd}
N \arrow[r, "\overline{g}"] \arrow[d, "\overline{(f \times \mathrm{id})}"'] \arrow[ddr, bend right] & Z \times Y \arrow[d, "f \times \mathrm{id}"] \\
M \arrow[r, "g"'] & \overline{X} \times Y \arrow[d, "\pi"] \\
 & Y
\end{tikzcd}$$

N is a compact manifold.
$h := \pi(f \times \mathrm{id})\overline{g} = \pi g\overline{(f \times \mathrm{id})} : N \to Y$.

Remark 7.8 *The definitions of products can be extended to relative groups.*

All homotopy products are defined on the level of maps. From the properties of the pairing of spectra we are able to pass to the limit and thus to define the products on homology and cohomology classes. In the geometric description we will proceed similarly – define products and then pass to the limit. Verifications showing that the geometric definitions of products correspond to the homotopic ones the Thom–Pontriagin isomorphism will also be done on an unstable level.
We shall give proofs for the cross-product in cobordism and /-slant product.

Theorem 7.9 *The diagram*

$$\begin{array}{ccc} \mathcal{B}^p(X;\xi') \otimes \mathcal{B}^q(Y;\xi") & \xrightarrow{\bar{\wedge}} & \mathcal{B}^{p+q}(X\times Y:\xi) \\ P\otimes P\downarrow & & \downarrow P \\ B^p(X,\xi')\otimes B^q(Y,\xi") & \xrightarrow{\bar{\wedge}} & B^{p+q}(X\times Y,\xi) \end{array}$$

commutes.

Proof: We recall that $\mathcal{B}^p(X;\xi') = \mathrm{colim}_k L(X\times R^{k-q},\xi_k)$. From the remarks proceeding the theorem we see that it is sufficient to prove the commutativity of the diagram:

$$\begin{array}{ccc} L(X\times R^{k-p};\xi'_k)\times L(Y\times R^{l-q};\xi"_\theta & \xrightarrow{\bar{\wedge}} & L(X\times Y\times R^{k+l-p-q};\xi_{l+k} \\ P\otimes P\downarrow & & \downarrow P \\ \left[S^{k-p}X^0, B_k'^{\xi'_k}\right]\times\left[S^{l-p}Y^0, B"_l^{\xi"_l}\right] & \xrightarrow{\bar{\wedge}} & \left[S^{k+l-p-q}X^0\wedge Y^0, B_{l+k}^{\xi_{l+k}}\right] \end{array}$$

where the upper line is the "geometric" cross on unstable level which is defined as follows. Let $(V\subset X\times R^{k-p},\nu(V)\to\xi'_k)\in L(X\times R^{k-p},\xi_k)$ and $(W\subset Y\times R^{l-q},\nu(W)\to\xi"_l)\in L(Y\times R^{l-q},\xi"_\theta$. Then $V\times W\subset X\times R^{k-p}\times Y\times R^{l-q}$. After the change of coordinates: $\varphi: X\times R^{k-p}\times Y\times R^{l-q}\to X\times Y\times R^{k-p}\times R^{l-q}$ the image $\varphi(V\times W)=Z$ is a submanifold of $X\times Y\times R^{k-p}\times R^{l-q}$ and the projection on $X\times Y$ is a proper map. We define ξ_{k+l}-structure on W to be:
$\nu(Z)\to^D_\simeq \nu(V\times W)\simeq\nu(V)\times\nu(W)\to\xi'_k\times\xi"_l\to^{m_{k,l}}\xi_{k+l}$ where isomorphism $D\varphi:\nu(Z)\to\nu(V\times W)$ is induced by φ. Passing to the limit we obtain the stable cross-product as in definition 5.6.
Recall also the homotopic definition.

Let $x\in\left[S^{k-p}X^0, B_k'^{\xi'_k}\right]$
Then $x\,\bar{\wedge}\,y$ is the composition:

$$\begin{tikzcd}
S^{k-p+l-q}X^0 \wedge Y^0 \arrow[r, "\simeq"] \arrow[ddr, dashed, "x \bar{\wedge} y"'] & S^{k-p}X^0 \wedge S^{l-q}Y^0 \arrow[d, "x \wedge y"] \\
& B_k^{'\xi'_k} \wedge B"_l^{\xi"_l} \arrow[d, "m^*_{k,l}"] \\
& B_{k+l}^{\xi_{k+l}}
\end{tikzcd}$$

It remains to check commutativity with the Thom isomorphisms.

Having chosen the metrics d' and $d"$ for $X \times R^{k-p}$ and $Y \times R^{l-q}$ choose on $X \times Y \times R^{k-p} \times^{l-q}$ the metric d induced from $d' \times d"$ on $X \times R^{k-p} \times Y \times R^{l-q}$ by the diffeomorphism φ. It is clear that if $h_1 : E(\nu(V)) \to X \times R^{k-p}$ and $h_2 : E(\nu(W)) \to Y \times R^{l-q}$ are tubular neighborhoods with respect to d and d' then
$\varphi(h_1 \times h_2)D\varphi : E(\nu(Z)) \to E(\nu(V \times W)) \to X \times R^{k-p} \times Y \times R^{l-q} \to X \times Y \times R^{k+l-p-q}$ is a tubular neighborhood of Z with respect to the metric d.
The Thom–Pontriagin construction gives the map:
$S^{k+l-p-q}(X \times Y)^0 \to Z^{\nu(Z)} \to B_k^{'\xi'_k} \wedge B_l^{''\xi"_l} \to B_{k+l}^{\xi_{k+l}}$.
From the definition of the structure on $\nu(Z)$ and the chosen forms of tubular neighborhoods the following diagram commutes:

$$\begin{tikzcd}
S^{k-p}X^0 \wedge S^{l-q}Y^0 \arrow[rr] \arrow[d] & & S^{k+l-p-q}(X \times Y)^0 \arrow[d] \\
V^{\nu(V)} \wedge M^{\nu(M)} \arrow[d] & (V \times W)^{\nu(V \times W)} \arrow[l, "\approx"'] & Z^{\nu(Z)} \arrow[l] \arrow[d] \\
B_k^{'\xi'_k} \wedge B_l^{''\xi"_l} \arrow[dr] & & (B'_k \times B"_\theta)^{\xi'_k \times \xi"_\theta} \arrow[dl] \\
& B_{k+l}^{\xi_{k+l}} &
\end{tikzcd}$$

The lower arrows give clearly the "homotopic" cross-product of maps obtained by applying the Thom–Pontriagin construction to V and W and maps to X and Y. □
The theorem can also be proved by using the inverse p^{-1} as described in remark 4.4.

Theorem 7.10 *The diagram*

$$\begin{array}{ccc} \mathcal{B}^p(X,\xi') \otimes \mathcal{B}_q(X\times Y,\xi") & \stackrel{\backslash}{\longrightarrow} & \mathcal{B}_{q-p}(Y\xi) \\ P\otimes P\downarrow & & \downarrow P \\ \mathbb{B}^p(X;\xi)\otimes \mathbb{B}_q(X\times Y;\xi") & \stackrel{\backslash}{\longrightarrow} & \mathbb{B}_{q-p}(Y;\xi) \end{array}$$

commutes.

Proof: It is sufficient to prove the commutativity of the diagram:

$$\begin{array}{ccc} L(X\times R^{k-p};\xi'_k)\times L(R^{q+l};\theta^0_X\times\theta^0_Y\times\xi"_l) & \stackrel{\backslash}{\longrightarrow} & L(R^{k+l-p+q},\theta^0_Y\times\xi_{k+l}) \\ P\times P\downarrow & & \downarrow P \\ \left[S^{k-p}X^0, B'^{\xi'_k}_k\right]\times\left[S^{q+l},X^0\wedge Y^0\wedge B"^{\xi"l}_l\right] & \stackrel{\backslash}{\longrightarrow} & \left[S^{k+l-p+q},Y^0\wedge B^{\xi_{k+l}}_{k+l}\right] \end{array}$$

where the geometric slant is obtained by the following construction.
Let $V\subset X\times R^{k-p}, \nu(V)\to\xi'_k$ and $M\subset R^{q+l},\nu(M)\to\xi"_l, g: M\to X\times Y$ be elements of $L(X\times R^{k-p},\xi'_k)$ and $L(R^{q+l},\theta^0_X\times\theta^0_Y\times\xi"_l)$ respectively. The product $V\times Y$ is a submanifold of $X\times Y\times R^{k-p}$. Consider $g\times\mathrm{id}: M\times R^{k-p}\to X\times Y\times R^{k-p}$ and the inverse image $V'=(g\times\mathrm{id})^{-1}(V\times Y)$. From the compactness of M it follows that V' is compact and can be regarded as a submanifold $j: V'\hookrightarrow R^{q+l}\times R^{k-p}$.
To describe the ξ-structure on V' we have to find the normal bundle $\nu(V')$ to V' in $R^{q+l}\times R^{k-p}$.
The normal bundle to $i: V'\hookrightarrow M\times R^{k-p}$ is canonically isomorphic to $(g\times\mathrm{id})^*\nu(V\times Y$. The normal bundle to the imbedding $j: M\times R^{k-p}\hookrightarrow R^{l+q}\times R^{k-p}$ is $\pi^*_M\nu(M)$ where $\pi_M: M\times R^{k-p}\to M$.
There is a canonical isomorphism:
$\nu(V')/i^*\pi^*_M\nu(M)\simeq\nu(i)$ where $\nu(i)$ denotes the normal bundle to V' in $M\times R^{k-p}$.
Definition of an isomorphism between $\nu(V')$ and $\nu(i)\oplus i^*\pi^*_M\nu(M)$ require the choice of a complement of $\nu(i)$ in $\nu(V')$. We choose an orthogonal complement of $\nu(i)$ with respect to some metric on $\nu(V')$; since any two metric are homotopic the homotopy class of the needed isomorphism is unique.
Now the ξ-structure on $\nu(V')$ is:

$$\begin{array}{ccc}
\nu(V') & \longrightarrow & \nu(i) \oplus i^*\pi_M^*(M) \\
\vdots & & \downarrow \\
\vdots & & \nu(V \times Y) \times \nu(M) \\
\vdots & & \downarrow \\
\vdots & & \nu(V) \times \nu(M) \\
\vdots & & \downarrow \\
\vdots & & \xi'_k \times \xi"_l \\
\vdots & & \downarrow \\
\llcorner & \dashrightarrow & \xi_{k+l}
\end{array}$$

The map $V' \to Y$ is composition:
$V' \overset{g\times \mathrm{id}}{\to} X \times Y \times R^{k-p} \overset{\pi_Y}{\to} Y$.
After passing to the limit we get the stable \-product as in definition 7.7. We have to verify the commutativity with the Pontriagin–Thom isomorphism. Let $x \in \left[S^{k-p}X^0, B'^{\xi'_k}_k\right]$ and $y \in \left[S^{q+l}, X^0 \wedge Y^0 \wedge B"^{\xi"_l}_l\right]$ be homotopic representatives of $\left[V, \nu(V) \to \xi_k\right]$ and $\left[M, g : M \to X \times Y, \nu(M) \to \xi"_l\right]$.
Recall that the "homotopic" slant is defined by the diagram:

$$\begin{array}{ccc}
S^{k-p} \wedge S^{q+l} & \xrightarrow{1 \wedge y} & S^{k-p} \wedge X^0 \wedge Y^0 \wedge B"^{\xi"_l}_l \\
\vdots & & \downarrow x \wedge \mathrm{id} \wedge \mathrm{id} \\
x \backslash y \ \vdots & & B'^{\xi'_k}_k \wedge Y^0 \wedge B"^{\xi"_l}_l \\
\vdots & & \downarrow \tau \wedge 1 \\
Y^0 \wedge B^{\xi_{k+l}}_{k+l} & \xleftarrow{1 \wedge \overline{m}_{k,l}} & Y^0 \wedge B'^{\xi'_k}_k \wedge B"^{\xi"_l}_l
\end{array}$$

The Thom construction for V' in $R^{q+l} \times R^{k-p}$ can be divided into two steps.
The first step:
consists of Thom construction for $R^{k-p} \times M$ in $R^{k-p} \times R^{q+l}$. Choosing appropriately the tubular neighborhood the obtained map is exactly the smash product of $\mathrm{id}_{S^{k-p}}$ with the Thom map for M in R^{q+l}:

$S^{k-p+q+l} \simeq S^{k-p} \wedge S^{q+l} \rightarrow^{\mathrm{id}\wedge} S^{k-p} \wedge M^{\nu(M)}$.

Second step:

The manifold V' is in natural way, a submanifold of $R^{k-p} \times E\nu(M)$ with the normal bundle $\nu(V')$; therefore it is a submanifold of $E(\nu(M)) \oplus \theta^{k-p}$ over M and as such it is the pull back of the diagram:

$$\begin{tikzcd}
\mathbb{R}^{k-p} \times E\nu(M) \arrow[r, "\widetilde{g}"] \arrow[dd, bend right] & (X \times Y \times \mathbb{R}^{k-p}) \times E\nu(M) \arrow[dd, bend left] \\
V \arrow[u, hook] \arrow[r, "g", "\text{back}"'] \arrow[d] & X \times Y \times M \arrow[u, hook] \arrow[d] \\
M \arrow[r, "g \times \mathrm{id}_M"'] & X \times Y \times M
\end{tikzcd}$$

We have $V' = \widetilde{g}^{-1}(V \times Y \times M)$ where $V \times Y \times M$ is the product of $V \times Y \times R^{k-p}$ and $M \subset E\nu(M)$ (as the zero section). The Thom construction thus yields a map:
$M^{\nu(M)\oplus\theta^{k-p}} = S^{k-p} \wedge M^{\nu(M)} \rightarrow V'^{\nu(V')}$. It is obvious that the tubular neighborhoods of V' in $R^{k-p} \times E\nu(M)$ and of V' in $R^{k-p} \times R^{q+l}$ can be chosen so that the following diagram commutes:

$$\begin{tikzcd}
S^{k-p} \wedge S^{q+l} \arrow[rr, "\text{Thom construction for}", "V' \text{ in } \mathbb{R}^{k-p} \times \mathbb{R}^{q+l}"'] \arrow[dr, "\text{first step}"'] & & V'^{\nu(V')} \arrow[dl, "\text{second step}"] \\
& S^{k-p} \wedge M^{\nu(M)} &
\end{tikzcd}$$

The properties of the pull back imply the homotopically commutative diagram:

$$\begin{tikzcd}
S^{k-p} \wedge M^{\nu(M)} \arrow[r, "\text{step 2.}"] \arrow[d, "\text{induced on Thom spaces by } g"] & V'^{\nu(V)} \arrow[dddl, "\text{the map induced by a differential of } \widetilde{g}"] \\
S^{k-p}(X \times Y) \wedge M^{\nu(M)} \arrow[d, "\simeq"] & \\
S^{k-p} \wedge X^0 \wedge Y^0 \wedge M^{\nu(M)} \arrow[d, "\text{Thom construction for } V \times Y \times M \text{ in } X \times Y \times \mathbb{R}^{k-p} \times E\nu(M)"] & \\
(V \times Y)^{\nu(V \times Y)} \wedge M^{\nu(M)} \arrow[d, "\simeq"] & \\
V^{\nu(V)} \wedge Y^0 \wedge M^{\nu(M)} &
\end{tikzcd}$$

The Thom map for $V \times Y \times M$ in $X \times Y \times R^{k-p} \times E\nu(M)$ can be assumed to be the Thom map for $V \times Y$ in $X \times Y \times R^{k-p}$ smashed with the identity map on $M^{\nu(M)}$.

We glue the diagrams 1. and 2. and add the "structure maps" on $\nu(V)$ and $\nu(M)$ to obtain the "giant" diagram:

$$\begin{tikzcd}
S^{k-p} \wedge S^{q+l} \arrow[r] \arrow[d, "\text{first step}"] & V'^{\nu(V')} \arrow[ddddl] \\
S^{k-p} \wedge M^{\nu(M)} \arrow[ur] \arrow[d] & \\
S^{k-p}(X \times Y) \wedge M^{\nu(M)} \arrow[d, "\text{second step}", "\simeq"'] & \\
S^{k-p} \wedge X^0 \wedge Y^0 \wedge M^{\nu(M)} \arrow[d] & \\
(V \times Y)^{\nu(V \times Y)} \wedge M^{\nu(M)} \arrow[d, "\simeq"'] & \\
V^{\nu(V)} \wedge Y^0 \wedge M^{\nu(M)} \arrow[d] & \\
B'^{\xi'_k}_k \wedge Y^0 \wedge B"^{\xi"_l}_l \arrow[d] & \\
Y^0 \wedge B^{\xi_{k+l}}_{k+l} &
\end{tikzcd}$$

From the construction the left column is the "homotopic" slant $x\backslash y$.
The verification of the commutativity will be completed if we will show that the structure on $\nu(V')$ induced by the pull back diagram 2. is the same as for the geometric slant product. To show thus it is sufficient to check that bundle maps:

$$\begin{array}{cc}
\nu(V') \to \nu(i) \oplus i^*\pi^*\nu(M) \to & \nu(i) \times i^*\pi^*\nu(M) \\
& \downarrow \\
& \nu(V \times Y) \times \nu(M)
\end{array}$$

from the definition of "geometric slant" and
$\nu(V') \to \nu(V \times Y \times M) = \nu(V \times Y) \times \nu(M)$
induced by the pull back are the same. This we leave to the reader.

□

External products lead to internal products.

Cup product:

$$\cup : \mathcal{B}^p(X;\xi') \otimes \mathcal{B}^q(Y;\xi") \longrightarrow \mathcal{B}^{p+q}(X;\xi)$$

Cap product:

$$\cap : \mathcal{B}^p(X;\xi') \otimes \mathcal{B}_q(X;\xi") \longrightarrow \mathcal{B}_{q-p}(X;\xi)$$

$\cup y = \triangle^*(\cdot\backslash y)$,
$\cap\omega = \cdot\backslash\triangle_*\omega$,
where $\triangle : X \to X \times X$.

Geometric description of these products follow from those for cross and slant products an the induced map.

It is important to notice that the cap-product is given by the pull back. Let $= [V, f : V \to X, \nu(f) \to \xi'] \in \mathcal{B}^p(X;\xi')$ and $W = [M, g : M \to X, -\tau(M) \to \xi"] \in \mathcal{B}_q(X;\xi")$. Then (assuming $f \pitchfork g$) $\cap\omega$ is defined by the pull back:

$$\begin{array}{ccc} f^*M & \xrightarrow{\overline{f}} & M \\ \overline{g}\downarrow & & \downarrow g \\ V & \xrightarrow{f} & X \end{array}$$

where $\cap w = [f^*M, f \circ \overline{g} : f^*M \to X, -\tau(F^*M) \to \xi]$ with the induced ξ-structure.

If ξ is a bundle ring spectrum with a unit $\theta \to \xi$ then there is a unit for internal products: $1 \in \mathcal{B}^0(X)$. This unit is represented by the identity $X \to X$ with the ξ-orientation defined by the unit $\theta \to \xi$. It is also clear that this unit corresponds to the homotopic one under the Pontriagin–Thom isomorphism.

Chapter 8

The Thom isomorphism theorem.

Let α, β be vector bundles over space X. We denote by X^α, Y^β their Thom spaces. In every multiplicative cohomology theory h^* there are products:

$$\begin{array}{l} \cup : \widetilde{h}^p(X^\alpha) \otimes \widetilde{h}^q(X^\beta) \longrightarrow \widetilde{h}^{p+q}(X^{\alpha+\beta}) \\ \cap : \widetilde{h}^p(X^\alpha) \otimes \widetilde{h}_q(X^{\alpha+\beta}) \longrightarrow \widetilde{h}_{q-p}(X^\beta) \end{array}$$

These products are defined by corresponding external products and the diagonal $\triangle : X^{\alpha+\beta} \to X^\alpha \wedge X^\beta$.

Definition 8.1 *The h^*-orientation of the n-dimensional real vector bundle α over X is an element $U \in \widetilde{h}^n(X^\alpha)$ such that for each point $x \in B$ the restriction $U|_X \in \widetilde{h}^n(x^\alpha) \simeq \widetilde{h}^n(S^n)$ is equal $\pm i^n$ where i^n denotes n-suspension of the unit $1 \in \widetilde{h}^0(S^0)$.*

Lemma 8.2 *Let $u \in \widetilde{h}^n(X^\alpha), \quad v \in \widetilde{h}^m(X^\beta)$ be orientations of the vector bundles α, β respectively. Then $u \cup v \in \widetilde{h}^{m+n}(X^{\alpha+\beta})$ is h^*-orientation of $\alpha + \beta$.*

Let $u \in \widetilde{h}^*(X^\alpha), \quad w \in \widetilde{h}^*(X^{\alpha+\beta})$ be h^*-orientations of α and $\alpha + \beta$ respectively. There then exists h^*-orientation of $\beta, \quad v \in \widetilde{h}^*(X^\beta)$ such that $u \cup v = w$.

Every element $U \in \widetilde{h}^n(X^\alpha)$ defines two homomorphisms of the degree n and $-n$ respectively:

$$\begin{array}{ll} \Phi^U : \widetilde{h}^*(X^\beta) \longrightarrow \widetilde{h}^*(X^{\alpha+\beta}), & \Phi^U(z) = U \cup z \\ \Phi_U : \widetilde{h}_*(X^{\alpha+\beta}) \longrightarrow \widetilde{h}_*(X^\beta), & \Phi_U(z) = U \cap z \end{array}$$

We recall Thom-Dold isomorphism theorem:

Theorem 8.3 *If $U \in \widetilde{h}^n(X^\alpha)$ is the h^*-orientation of α then Φ^U and Φ_U are isomorphisms.*

The Thom-Dold theorem holds also for relative groups.

The homomorphisms Φ^U and Φ_U are natural with respect to the Thom-Pontriagin construction. Let ξ be a vector bundle over Y; $f : X \to Y$ a smooth proper map between manifolds with a factorization by imbedding $\widetilde{f}$:

$$\begin{array}{ccc} X & \overset{\widetilde{f}}{\dashrightarrow} & E \\ & f \searrow & \downarrow \xi \\ & & Y \end{array}$$

For the given vector bundle α over Y the Thom-Pontriagin construction defines identification map $T(f) : Y^{\xi+\alpha} \to X^{\nu+f^*\alpha}$ where $\nu = \nu(f^*)$.

Lemma 8.4 *For every element $U \in \widetilde{h}^*(X^\alpha)$ the diagram*

$$\begin{array}{ccc} h^*(X^{\nu+f^*\alpha}) & \xrightarrow{T(f)^*} & \widetilde{h}^*(Y^{\xi+\alpha}) \\ \Phi^{f^*U} \uparrow & & \uparrow \Phi^U \\ \widetilde{h}^*(X^\nu) & \xrightarrow{T(f)^*} & \widetilde{h}^*(Y^\xi) \end{array}$$

commutes. The similar diagram holds for homology.

It is clear that homomorphisms Φ^U, Φ_U are natural with respect to the maps of vector bundles. For the proofs of 8.1– **??** see Dyer [6] and Boardman [3].

We shall give a geometric interpretation of Thom isomorphism in bordism and cobordism theories. We fix a bundle spectrum ξ. Suppose the vector bundle α over manifold X has a fixed stable ξ-structure.

Proposition 8.5 *The vector bundle α has a canonical $\mathcal{B}^*(;\xi)$-orientation represented by the zero-section $s_0 : X \hookrightarrow E(\alpha)$.*

Proof: The zero-section s_0 is a proper map. There exists a ξ-structure on s_0 because the normal bundle to s_0 is α. Then s_0 determines an element $[s_0] \in \mathcal{B}^n(X^\alpha, \mathrm{pt}; \xi)$. Note that (X^α, pt) is a relative manifold. Restrictions of $[s_0]$ to fibres are represented by inclusions $x \to 0 \in R^n \subset S^n$ which are the canonical generators of $\mathcal{B}^*(S^n, \mathrm{pt}; \xi)$.

□

The classification of orientable bundles in a given cohomology theory seems to be an important problem. For cobordism theories only partial results are known. We list them in the following table:

cobordism theory	orientable bundles
unoriented cobordism	all
oriented cobordism	SO-bundles
complex cobordism	?
spin cobordism	?
framed cobordism (stable homotopy)	bundles belonging to $\ker\{J : K(X) \to J(X)\}$

The case of framed cobordism shows that a stable ξ-structure on the bundle α is not a necessery condition for orientability in $\mathcal{B}^{ast}(;\xi)$.

From proposition **??** and the interpretation of products given in chapter 7. It follows:

Theorem 8.6 *Let α be a vector bundle with stable ξ-structure over manifold X. Under the Thom-Pontriagin isomorphism the Thom homomorphism for α correspond to maps:*

$$\begin{array}{ll} \Psi^\alpha : \mathcal{B}^*(X;\xi) \longrightarrow \mathcal{B}^*(X^\alpha, \mathrm{pt};\xi) & \Psi^\alpha([f]) = [s_0 \circ f] \\ \Psi_\alpha : \mathcal{B}_*(X^\alpha, \mathrm{pt};\xi) \longrightarrow \mathcal{B}_*(X;\xi) & \Psi_\alpha([f]) = [f : f^{-1}(s_0 X) \to X] \end{array}$$

where $f \pitchfork s_0$.

We shall return to the Thom isomorphism in chapter 11. Where we shall give a geometric interpretation of inverse homomorphisms.

Chapter 9

Spanier-Whitehead duality.

In this section we describe a geometric interpretation of the Spanier-Whitehead duality in the bordism and cobordism theories.

First we recall some basic definitions and facts from the Spanier-Whitehead theory.

Definition 9.1 *Let X, X^* be pointed CW complexes. A stable homotopy class $u \in \{S^n, X^* \wedge X\}$ will be called the n-duality if the homomorphism*

$$\circ \backslash u_*[S^n] : \widetilde{H}^q(X^* : Z) \longrightarrow \widetilde{H}_{n-q}(X : Z)$$

is an isomorphism. We call X^* n-dual to X.

Let K be a subcomplex of n-dimensional sphere S^n and let $K^- \subset S^n \backslash K$ be a strong deformation retract of $S^n \backslash K$. There exists a $(n-1)$-duality $u \in \{S^{n-1}, K \wedge K^-\}$. Let $x \in K$ and $x^- \in K^-$. A diagonal map $S^n \to S^n \wedge S^n$ induces a map $S^n \to (S^n \backslash x^-)/K \wedge (S^n \backslash K)/x$. Since $S^n \backslash x^-$ is contractible we have $(S^n \backslash x^-)/K = S(K)$. Then we obtain a map $u : S^n \to S(K) \wedge K^- = S(K \wedge K^-)$ which defines the $(n-1)$-duality $u \in \{S^{n-1}, K \wedge K^-\}$.

The duality $u \in \{S^n, X^* \wedge X\}$ gives a natural isomorphisms of groups of stable homotopy classes. Let X, Y be finite CW complexes.

Theorem 9.2 *For arbitrary CW complex Z the maps*

$$\begin{array}{ll} \varphi_u : \{X \wedge Y, Z\} \longrightarrow \{S^n Y, X^* \wedge Z\} & \varphi_u(\{f\}) = \{(1 \wedge f) \circ (u \wedge 1)\} \\ \psi_u : \{Y \wedge X^*, Z\} \longrightarrow \{Y \wedge S^n, Z \wedge X\} & \psi(\{u\}) = \{(g \wedge 1) \circ (1 \wedge u)\} \end{array}$$

are isomorphisms.

If X^*, Y^* are n-duals to X, Y and $u \in \{S^n, X^* \wedge X\}, v \in \{S^n, Y^* \wedge Y\}$ are corresponding dualities then there exists an isomorphism $D(u, v) : \{X, Y\} \to \{X^*, Y^*\}$ defined by the composition

$$\begin{array}{ccc} \{X, Y\} & \overset{D(u,v)}{\dashrightarrow} & \{X^*, Y^*\} \\ & \varphi_u \searrow & \uparrow \psi_u \\ & & \{S^n, X^* \wedge Y\} \end{array}$$

It follows from this definition that $D(u, v)(f) = g$ iff the diagram

$$\begin{array}{ccc} S^n & \xrightarrow{u} & X^* \wedge X \\ v \downarrow & & \downarrow 1 \wedge f \\ Y^* \wedge Y & \xrightarrow{g \wedge 1} & X^* \wedge Y \end{array}$$

commutes in the stable homotopy category.
Assume that $u \in \{X \wedge X^*, S^n\}$ is a n-duality in the sense of Spanier [14]. Then an element $Du \in \{S^n, X^* \wedge X\}$ is a n-duality in the sense described above. The inverse statement is also true.

Let h^* be a cohomology theory associated with a spectrum $\mathbb{E}$. This spectrum determines also a homology theory $h_* = h_*(;\mathbb{E})$. WE have a slant product

$$\backslash : \widetilde{h}^p(X) \otimes \widetilde{\pi}^S_q(X \wedge Y) \longrightarrow \widetilde{h}_{q-p}(Y)$$

induced by the canonical pairing of spectra $\mathbb{S} \wedge \mathbb{E} \to \mathbb{E}$ (for details see Whitehead [17]). If an element $u \in \{S^n, X^* \wedge X\} = \widetilde{\pi}^S_n(X^* \wedge X)$ is n-duality then a map induced by slant product

$$\backslash^u : \widetilde{h}^p(\{u\}(X^*) \longrightarrow \widetilde{h}_{n-p}(X)$$

is an isomorphism. Theorem 9.2 is the special case of the last statement. In general form this can be proved by considering multiplicative structure in Stiyah-Hirzebruch spectral sequence for theory h^*.

The duality isomorphism $D(u, v)$ is natural in the following sense:
Let $f \in \{X, Y\}$ and let $u : S^n \to X^* \wedge X, v : S^n \to Y^* \wedge Y$ be dualities. Then the diagram

$$\begin{array}{ccc} \widetilde{h}^p(X^* & \xrightarrow{\cdot\backslash^u} & \widetilde{h}_{n-p}(X) \\ (D(u,v)f)^* \downarrow & & \downarrow f_* \\ \widetilde{h}^p(Y^*) & \xrightarrow{\cdot\backslash^v} & \widetilde{h}_{n-p}(Y) \end{array}$$

commutes.

We recall the Milnor-Spanier duality theorem for manifolds. Let M be a compact manifold with boundary ∂M. Let M be imbedded in the cube I^N

such that $\partial M \subset J^{N-1}$ and M is transversal to the normal bundle of ∂M in I^{N-1}.

Theorem 9.3 *The Thom space M^ν in N-dual to $M/\partial M$. The quotient space $M^\nu/\partial M^{\nu'}$ is N-dual to $M^0 = M + \mathrm{pt}$.*

For the proof see Atiyah [2] or Stong [16]. We remark only that the duality maps are defined by a diagonal $\triangle : M \to M \times M$;

$$S^N \overset{\approx}{\to} I^N/\mathrm{int}I^N \to M^\nu/\partial M^{\nu'} \overset{\triangle}{\to} (M \times M)^{\nu\times 0}/(M \times M)^{\nu'\times 0} \simeq M^\nu \wedge (M/\partial M)$$
$$S^N \overset{\approx}{\to} I^N/\mathrm{int}I^N \to M^\nu/\partial M^{\nu'} \overset{\triangle}{\to} (M \times M)^{\nu\times 0}/(M \times M)^{\nu'\times 0} \approx (M^\nu/\partial M^{\nu'}) \wedge M^0$$

We describe now a geometric interpretation of the Spanier-Whitehead dual to the map $f : M \to N$ between compact manifolds M, N. We assume for simplicity $\partial M = \partial N = 0$. We shall use Thom-Pontriagin construction. Let $i : M \subset R^q$. We have a commutative diagram

$$\begin{array}{ccc} M & \overset{\widetilde{f}=(f,\mathrm{id})}{\dashrightarrow} & N \times \mathbb{R}^q \\ & f \searrow & \downarrow \\ & & N \end{array}$$

For any vector bundle α over N the Thom-Pontriagin construction gives a map $T(f) : N^{q+\alpha} \to M^{\nu(\widetilde{f})+f^*\alpha}$. Let $\alpha = \nu(j)$ where $j : N \subset R^{q'}$. We obtain a map

$$T(f) : N^{q+\nu(j)} \longrightarrow M^{q'+\nu(i)}.$$

We can describe this map more explicitly. Consider the composition

$$M \overset{(f,i)}{\to} N \times R^q \hookrightarrow R^{q'} \times R^q \subset S^{q+q'}.$$

We choose tubular neighborhoods $P \supset M$ and $Q \supset N$ in $S^{q+q'}$ such that $P \subset Q$. $T(f)$ is an identification map $Q/\partial Q \to P/\partial P$. It is easy to see that the diagram

$$\begin{array}{ccc} S^{q+q'} & \overset{v}{\longrightarrow} & N^0 \wedge N^{\nu(j)} \\ u \downarrow & & \downarrow 1 \wedge T(f) \\ M^0 \wedge M^{\nu(i)} & \overset{f\wedge 1}{\longrightarrow} & N^0 \wedge M^{\nu(i)} \end{array}$$

where u, v denote canonical dualities, commutes up to homotopy.

The diagonal $\triangle$ gives the cap-products

$$\cap : \widetilde{h}^p(M^\nu) \otimes \widetilde{\pi}_q^S(M^\nu/\partial M^{\nu'}) \longrightarrow h_{q-p}(M, \partial M)$$
$$\cap : h^p(M^\nu, \partial M^{\nu'}) \otimes \pi_q^S(M^\nu, \partial M^{\nu'}) \longrightarrow \widetilde{h}_{q-p}(M^0)$$

The Spanier-Whitehead duality is induced by the cap-product with the element $\{S^N \to M^\nu/\partial M^{\nu'}\} \in \widetilde{\pi}_q^S(M^\nu/\partial M^{\nu'})$.

The manifold M is orientable in cohomology theory h^* if its tangent (normal) bundle is h^*-orientable. Similarly as in ordinary cohomology the h^*-orientability for manifolds can be defined in terms of a fundamental class in its h^*-homology groups. Let $(M, \partial M)$ denote a compact n-dimensional manifold with a boundary. For each point $x \in M\backslash\partial M$ we have the inclusion $i_x : (M, \partial M) \hookrightarrow (M, M\backslash x)$.

Definition 9.4 *The fundamental h_*-class of the manifold $(M, \partial M)$ is an element $z \in h_n(M, \partial M)$ such that for each point $x \in M\backslash\partial M$, $(i_x)_*(z) \in h_n(M, M\backslash x) = h_n(D^n, S^{n-1})$ is up to sign the canonical generator of $h_n(D^n, S^{n-1})$.*

This definition is equivalent to the Whitehead definition [17].

Theorem 9.5 *For each manifold $(M, \partial M)$ the Spanier-Whitehead duality gives a bijection between h^*-orientations of the normal bundle to M and the fundamental h_*-classes of $(M, \partial M)$.*

Proof: Let ν be a normal bundle to M, $\dim \nu = k$. We consider the Spanier-Whitehead duality

$$\approx : \widetilde{h}^k(M^\nu) \longrightarrow h_n(M, \partial M).$$

Let $p_x : M/\partial M \to S^n$ be the identification map determined by inclusion $(M, \partial M) \hookrightarrow (M, M\backslash D_x) \hookrightarrow (M, W\backslash x)$. Consider a canonical duality $S^{n+k} \to (M^\nu \wedge M)/\partial M$. For each duality $S^{n+k} \to S^k \wedge S^n$ we obtain the following commutative diagram:

$$\begin{array}{ccc} \widetilde{h}^k(M^\nu) & \xrightarrow{\approx} & \widetilde{h}_n(M/\partial M) \\ (Dp_x)^* \downarrow & & \downarrow (p_x)_* \\ \widetilde{h}^k(S^k) & \xrightarrow{\approx} & \widetilde{h}_n(S^n) \end{array}$$

Thus it remains to show that $Dp_x : S^k \to M^\nu$ is equal to an inclusion $S^k \hookrightarrow M^\nu$ onto fiber over x. For this purpose we choose duality $S^{n+k} \to S^n \wedge S^k$ in a special way as shown on the figure below. It is clear that the diagram

$$\begin{tikzcd}
S^{n+k} \arrow[rr] \arrow[d] & & M^\nu/\partial M^{\nu'} \arrow[d] \\
S^k \wedge S^n \arrow[dr, "j_x \wedge 1"'] & & M^\nu \wedge M/\partial M \arrow[dl] \\
& M^\nu \wedge S^n &
\end{tikzcd}$$

commutes up to homotopy.

□

We shall give a geometric interpretation of the Spanier-Whitehead duality for manifolds in the case of bordism and cobordism theories. First we recall a geometric interpretation of the cap-product in relative groups. If $(M, \partial M)$is a manifold with boundary then there exists an open manifold $\overline{M}$ and an inclusion $M \hookrightarrow \overline{M}$ which is the homotopy equivalence. The construction of M follows easily from the existence of open collaring of the boundary. If α is a vactor bundle over M then it extends over $\overline{M}$. The inclusion $M^\alpha \hookrightarrow \overline{M}^\alpha$ is also a homotopy equivalence. We shall consider two cap-products:

$$\cap : \mathcal{B}^p(\overline{M}, \mathrm{pt}; \xi) \otimes \mathcal{B}_q(M^\nu, \partial M^{\nu'}; \xi) \longrightarrow \mathcal{B}_{q-p}(M, \partial M; \xi)$$
$$\cap : \mathcal{B}^p(M^\nu, \partial M^{\nu'}; \xi) \otimes \mathcal{B}_q(M^\nu, \partial M^{\nu'}; \xi) \longrightarrow \mathcal{B}_{q-p}(M; \xi)$$

We observe that $(M^\nu, \partial M^{\nu'})$ is a relative manifold. θ denotes the spectrum determined by trivial bundle over point; $\mathcal{B}_*(;\theta) \simeq \pi_*^S$. Let $g : V \to \overline{M}^\nu$ be a ξ-oriented proper map such that $[g] \in \mathcal{B}^p(M^\nu, *; \xi)$. We can assume that $g(V) \subset \overline{M}$. Let $W, \partial W)$ be a framed manifold and $f : (W, \partial W) \to (M^\nu, \partial M^\nu)$ be a singular manifold : $[f] \in \mathcal{B}_q(M^\nu, \partial M^{nu'}; \theta)$. We assume $f \pitchfork g$. The pull back

$$\begin{tikzcd}
W \times_{\overline{M}} V \arrow[r] \arrow[d] \arrow[dr, dashed, "f \cap g"] & V \arrow[d, "g"] \\
W \arrow[r, "f"] & \overline{M}^\nu
\end{tikzcd}$$

defines $[f] \cap [g] = [f \cap g]$. $W \times_{\overline{M}} V$ is a compact ξ-manifold with boundary $f \cap g : W \times \overline{M} V \to M, \quad (f \cap g)(\partial(W \times_{\overline{M}})) \subset \partial M$.
In a similiar way we obtain an interpretation of the second cap-product. It is obvious that a canonical generator of $\mathcal{B}_N(S^N, \mathrm{pt}; \theta)$ is represented by an identification map $(D^N, S^{N-1}) \to (S^N, \mathrm{pt})$. Then its image under the induced homomorphism

$$\mathcal{B}_N(S^N, \mathrm{pt}; \theta) \to \mathcal{B}_N(M^\nu / \partial M^{\nu'}, \mathrm{pt}; \theta) = \mathcal{B}_N(M^\nu, \partial M^{\nu'}; \theta)$$

is represented by the identification $(I^N, \mathrm{int} I^N) \to (M^\nu, \partial M^{\nu'})$.
We define the duality maps:

$$\begin{array}{l} \Psi_1 : \mathcal{B}^p(\overline{M}^\nu, \mathrm{pt}; \xi) \longrightarrow \mathcal{B}_{N-p}(M, \partial M; \xi) \\ \Psi_1([g : V \to \overline{M}^\nu]) = [g : g^{-1}(M) \to M] \end{array}$$

where $g(V) \subset \overline{M}$ and $g \pitchfork \partial M$ as a map into $\overline{M}$. The inverse image $g^{-1}(M)$ is a compact ξ-manifold with boundary and $[g] \in \mathcal{B}_{N-p}(M, \partial M; \xi)$.

$$\begin{array}{l} \Psi_2 : \mathcal{B}^p(M^\nu, \partial M^{\nu'}; \xi) \longrightarrow \mathcal{B}_{N-p}(M; \xi) \\ \Psi_2([g : V \to M^\nu]) = [g : V \to M] \end{array}$$

where $g(V) \subset M$. The manifold V is compact ξ-manifold and $[g] \in \mathcal{B}_{N-p}(M; \xi)$.

Theorem 9.6 *The maps Ψ_1, Ψ_2 correspond under Thom-Pontriagin isomorphism to the Spanier-Whitehead duality between homotopy theoretic bordism and cobordism.*

The proof follows from the interpretation given above.

Corollary 9.7 *Let $(M, \partial M)$ be a compact ξ-manifold of dimension n. Then a canonical $\mathcal{B}^*(;\xi)$-orientation of the normal bundle of M corresponds under Spanier-Whitehead duality to the identity map $(M, \partial M) \to (M, \partial M) \in \mathcal{B}_n(M, \partial M; \xi)$.*

Chapter 10

Duality theorems for differentiable manifolds.

If M is amanifold oriented in a given multiplicative cohomology theory h^* then all well-known duality theorems hold. The machinery developed in Spanier's book can be applied here to obtain their proofs. Since we deal with differentiable manifolds we can apply the Spanier-Whitehead duality and the Thom isomorphism to obtain the duality theorems. We start from the Poincaré duality theorem.

Theorem 10.1 *For a compact h^*-oriented n-dimensional manifold with boundary $(M, \partial M)$ the following isomorphism hold:*

$$\begin{aligned} \cap U : h^q(M) &\xrightarrow{\approx} h_{n-q}(M, \partial M) \\ \cap U : h^q(M, \partial M) &\xrightarrow{\approx} h_{n-q}(M) \end{aligned}$$

where $U \in \widetilde{h}^k(M^\nu)$ is an orientation of the normal bundle of M.

Proof: The homomorphisms described in the theorem are compositions of the Spanier-Whitehead duality isomorphisms and the Thom isomorphisms:

$$\begin{aligned} h^q(M) &\longrightarrow \widetilde{h}^{q+k}(M^\nu) \longrightarrow h_{n-q}(M, \partial M) \\ h^q(M, \partial M) &\longrightarrow h^{q+k}(M^\nu, \partial M^{\nu'}) \longrightarrow h_{n-q}(M). \end{aligned}$$

It is easy to verify that the inverse homomorphisms are given by a cap-product with a corresponding fundamental class.

□

From the Poincaré duality we can deduce the Lefschetz duality theorem.

First we recall some facts about continuity of the homology and cohomology. Let CW complex X be a direct limit of its subcomplexes $X = \mathrm{colim}_i X_i, \quad X_i \subset X_{i+1}$. For every additive homology theory h_* the isomorphism $h_*(X) = \mathrm{colim}_i h_*(X_i)$ hold (see Milnor [11]). For cohomology we have only an epimorphism $h^*(X) \to \lim h^*(X_i)$. Let (A, B) be the pair of subspaces in a topological space X. We define cohomology groups of (A, B) in X:

$$\overline{h}^*(A, B) = \mathrm{colim} h^*(\overline{U}, \overline{V})$$

where $(\overline{U}, \overline{V}) \supset (A, B)$ varies over closed neighborhoods of (A, B) in X. The restriction maps define a homomorphism $i : \overline{h}^*(A, B) \to h^*(A, B)$. We say that (A, B) is taut with respect to h^* in X if i is an isomorphism. We can formulate the Lefschetz duality theorem.

Theorem 10.2 *Let (M, A) be a compact n-dimensional relative manifold such that $M \backslash A$ is h^*-oriented. There exists an isomorphism*

$$h_q(M \backslash A) = \overline{h}^{n-q}(M, A).$$

Proof: Let $\{N\}$ be a family of closed neighborhoods of A such that $M \backslash \mathrm{int} N, \partial N)$ is a manifold with boundary. The Lefschetz isomorphism is defined by the following composition:

$$h_q(M \backslash \mathrm{int} N) \cong h^{n-q}(M \backslash \mathrm{int} N, \partial N) = h^{n-q}(M, N).$$

By passing to the limit over the family $\{N\}$ we obtain the isomorphism

$$h_q(M \backslash A) = \overline{h}^{n-q}(M, A).$$

□

We restrict ourselves to the bordism and cobordism theories. Let $(M, \partial M)$ be a compact ξ-manifold. It follows easily from our geometric interpretations of products that the Poincaré duality induced by the canonical $\mathcal{B}^*(;\xi)$-orientation is an identity map. The Lefschetz duality is also an identity.

Chapter 11

Transfer homomorphisms.

There are important but not functorial homomorphism between homology (cohomology) groups induced by special kinds of maps.

We start from an axiomatic description. As usually, let h^* be a multiplicative cohomology theory.

Let $\mathcal{F}$ be a subclass of the class of all maps for which h^* is defined. We say that h admits transfers for maps $f \in \mathcal{F}$ if for $f : X \to Y, \quad f \in \mathcal{F}$ the homomorphisms

$$f_\natural : h^*(X) \longrightarrow h^*(Y), \quad f^\natural : h_*(Y) \longrightarrow h_*(X)$$

are defined. Both $f_\natural$ and $f^\natural$ have the same generally non-zero degree depending on f. The homomorphisms $f_\natural$ and $f^\natural$ satisfy the following conditions:

1. $\mathrm{id}_\natural = \mathrm{id}, \mathrm{id}^\natural = \mathrm{id}, \quad (g \circ f)_\natural = g_\natural \circ f_\natural, \quad (g \circ f)^\natural = (-1)^{mn} f^\natural \circ g^\natural$ where $m = \deg(f^\natural), n = \deg(g^\natural)$,
2. $f_\natural(\alpha \cup f^*\beta) = f_\natural\alpha \cup \beta$,
3. $f^\natural(x \cap \alpha) = f^\natural x \cap f^*\alpha$,
4. $f_*(f^\natural x \cap \alpha) = (-1)^{m\cdot|x|} x \cap f_\natural\alpha$.

We shall define transfers for two classes of maps (Boardman [3]).

The Grothendieck transfer of an oriented map.

Let $f : X \to Y$ be a proper h^*-oriented (i.e. the normal bundle $\nu(f)$ is h^*-oriented) smooth map between manifolds $X, Y, \quad \dim X = m, \dim Y = n$. We shall define a transfer of degree $n-m$. Let $i : X \hookrightarrow R^N$ be an imbedding. The Thom construction gives us the map $T(f) : S^N \wedge Y^0 \to X^{\nu(f)}$. Let

$U \in \widetilde{h}^{N+n-m}(X^{\nu(f)})$ be an orientation of the normal bundle of f. The homomorphism $f_\natural$ is defined by the following diagram

$$\begin{array}{ccc} h^q(X) & \stackrel{f}{\dashrightarrow} & h^{q+n-m}(Y) \\ \Phi\downarrow & & \downarrow\approx \\ h^{q+N+n-m}(X^{\nu(f)}) & \stackrel{T(f)^*}{\longrightarrow} & h^{q+N+n-m}(S^N \wedge Y^0) \end{array}$$

and the analogous one for the homology defines $f^\natural$. The definition of $f_\natural$ and $f^\natural$ depends only on the orientation of the normal bundle $\nu(f)$.

The Grothendieck bundle transfer.

Let $\pi : E \to B$ be a fiber bundle whose typical fiber F is a smooth compact manifold and whose structure group G is a Lie group acting smoothly on F.

We shall define the bundle $\tau_F(E)$ of tangents along the fibres. Let $\{U\}$ be a trivialisation of the bundle π. We define bundles $U \times \tau(F) \to U \times F$ on sets $\pi^{-1}(U) = U \times F$. Let $g_{ij} : U_i \cap U_j \to G$ be the transition function defined by π. Then we have identification maps $(U_i \cap U_j) \times \tau(F) \to (U_i \cap U_j) \times \tau(F)$, $\quad (x, \omega) \mapsto (x, d(g_{ij}(x))(\omega))$ where $x \in U_i \cap U_j$ and $\omega \in \tau(F)$. After this identification we obtain a vector bundle $\tau_F(E) \to E$. Its restriction to the fibres gives a tangent bundle $\tau(F) \to F$. We need the folloeing lemma.

Lemma 11.1 *Let F be a smooth compact manifold and G a compact Lie group acting smoothly on F. Then there exists a finite dimensional representation V for G and a smooth G-equivariant imbedding $F \hookrightarrow V$.*

For the proof see Janich [8].

A choice of a representation as in lemma 11.1 gives us an imbedding

$$\begin{array}{ccc} E & \stackrel{j}{\hookrightarrow} & E(\eta) \\ \pi\downarrow & \swarrow p & \\ B & & \end{array}$$

where η is a vector bundle determined by the tsransition functions of π and the representation V. We define a normal bundle to E in $E(\eta) : \nu_F^q(E) = j!\tau_V(E(\eta))/\tau_F(E)$. The inclusion $\tau_F(E) \hookrightarrow j!\tau_V(E(\eta))$ is defined by the following diagram

$$\begin{array}{ccc} \tau_F(E) & \stackrel{dj}{\longrightarrow} & \tau_V(E(\eta)) = p!\eta \\ \downarrow & & \downarrow \\ E & \stackrel{j}{\longrightarrow} & E(\eta) \end{array}$$

Observe that the restriction $\nu_F^\eta(E)|_F$ is the normal bundle of F in V. We have the following generalization "over B" of the tubular neighborhood theorem.

Lemma 11.2 *There exists an imbedding $t : \nu_F(E) \to E(\eta)$ onto an open subset such hat $t(s_0(E)) = E$.*

Proof: Choose an equivariant metric on V and let U be a metric tubular neighborhood of F in $V : \nu(F) \to U \subset V$. Then U gives rise to an associated subbundle of η with he fiber U. We denote its total space by $N \subset E(\eta)$. There exists an obvious imbedding onto $N : \nu_F(E) \to N \subset E(\eta)$.

□

Without loss of generality we can assume that N is contained in the unit disc bundle of η. The Thom-Pontriagin construction gives a map $B^\eta \to E^{\nu_F(E)}$ and hence $B^{\eta+\alpha} \to E^{\nu_F(E)+\pi\alpha}$ for any vector bundle α over B. We choose $\alpha = -\eta$; $\quad \eta + (-\eta) = \theta$. Since $\pi : \eta = \tau_F(E) + \nu_F(E)$ we obtain a map $T(\pi) : S^N \wedge B^0 \to E^{-\tau_F(E)}$ where $\tau_F(E) + (-\tau_F(E)) = \theta^N$. We define the transfer for $\pi : E \to B$ in the case when a bundle $\tau_F(E)$ is h^*-oriented.

$$\begin{array}{ccc} h^q(E) & \dashrightarrow & h^{q-n}(B) \\ \Phi\downarrow & & \downarrow\approx \\ h^{q+N-n}(E^{-\tau_F(E)}) & \xrightarrow{T(\pi)^*} & h^{q+N-n}(S^N \wedge B^0) \end{array}$$

where Φ is defined by the orientation of $\tau_F(E)$ and $n = \dim F$. Analogously for the homology theory. Definition of the Grothendieck bundle transfer is independent from the choice of a vector bundle and imbedding $E \subset E(\eta)$. This follows from the generalization of the standard isotopy arguments. The Grothendieck transfer depends only on the orientation of $\tau_F(E)$.

Both transfers described above satisfy multiplicative conditions 1.-4. This follows from suitable multiplicative properties of the Thom isomorphism which hold for an arbitrary cohomology and homology theory as well as for ordinary theory (see Spanier [15]).

Theorem 11.3 *Let $\pi : E \to B$ be a map for which the both Grothendieck transfers are defined. Then, after suitable choice of orientations these transfers agree.*

Proof: We choose an imbedding $i : E \subset R^N$. It defines the commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{(\pi,i)} & B \times R^N \\ \downarrow & \swarrow & \\ B & & \end{array}$$

It is easy to see that $\nu_F^N(E) = \nu(\pi)$. The theorem follows.

□

Let $\alpha : E \to^p B$ be a n-dimensional h^*-oriented vector bundle. Then the projection $p : S(\alpha + \theta') \to B$ admits bundle transfer $\tau_{S^n}(S(\alpha + 1)) + \theta^1 = \alpha + \theta^1$. If B is a manifold then we have a smooth zero section $s_0 : B \to S(\alpha + \theta)$, $\nu(s_0) = \alpha$ which also admits transfer. We recall that $B^\alpha = S(\alpha + 1)/B$ where $s_\infty : B \subset S(\alpha + \theta^1)$.

Theorem 11.4 *Let α be a n-dimensional h^*-oriented vector bundle over finite dimensional CW complex B. Then the following diagrams commute up to sign:*

$$\begin{array}{ccc} h_*(S(\alpha+\theta')) & \xrightarrow{j_*} & \widetilde{h}_*(B) \\ p^\natural \uparrow & \swarrow \Phi & \\ h_*(B) & & \end{array} \quad , \quad \begin{array}{ccc} j^* : \widetilde{h}^*(B^\alpha) & \longrightarrow & h^*(S(\alpha+\theta^1)) \\ & \Phi \nwarrow & \downarrow p_\natural \\ & & h^*(B) \end{array}$$

where Φ is the Thom isomorphism and $j : S(\alpha + \theta') \hookrightarrow (S(\alpha + \theta'), B)$.

Proof: For each finite CW complex B there exists an open manifold $\overline{B}$ and an inclusion $B \hookrightarrow \overline{B}$ which is homotopy equivalence. Therefore it is enough to prove the theorem in the case when B is a manifold. The the transfer $s^\natural$ is defined and the formula $s^\natural \circ p^\natural = (-1)^n (ps)^\natural = (-1)^n$ holds. Straightforward verification shows that $s^\natural = \Phi \circ j_*$. Analogously for cohomology.

□

At the end of this section we discuss a connection between transfers and the Poincaré duality. Let $f : X \to Y$ be a smooth map between compact h^*-oriented manifolds without boundary. Orientations of X and Y gives us a canonical orientation of the normal bundle $\nu(f)$.

Theorem 11.5 *The following diagrams commutes:*

$$\begin{array}{ccc} h^*(X) & \stackrel{f_\natural}{\to} & h^*(Y) \\ D \downarrow & & \downarrow D \\ h_*(X) & \stackrel{F_*}{\to} & h_*(Y) \end{array} \quad , \quad \begin{array}{ccc} h^*(X) & \stackrel{f^*}{\leftarrow} & h^*(Y) \\ D \downarrow & & \downarrow D \\ h_*(X) & \stackrel{f^\natural}{\leftarrow} & h_*(Y) \end{array}$$

where D denotes the Poincaré duality.

Proof: We prove the commutativity of the first diagram. From the definition of duality and the transfer we have

$$\begin{array}{ccc}
\widetilde{h}^*(X^{\nu(f)}) & \xrightarrow{T(f)^*} & \widetilde{h}^*(S^N \wedge Y^0) \\
\Phi \uparrow & & \uparrow \Phi \\
h^*(X) & & h^*(Y) \\
\Phi \downarrow & & \downarrow \Phi \\
\widetilde{h}^*(X^{\nu(X)}) & \xrightarrow{(Df)^*} & \widetilde{h}^*(Y^{\nu(Y)}) \\
SW \downarrow & & \downarrow SW \\
h_*(X) & \xrightarrow{F_*} & h_*(Y)
\end{array}$$

The map Df is the Spanier-Whitehead dual of f. By naturality of Spanier-Whitehead duality it is sufficient to prove the commutativity of the following diagram:

$$\begin{array}{ccc}
\widetilde{h}^*(X^{\nu(f)}) & \xrightarrow{T(f)^*} & \widetilde{h}^*(S^N \wedge Y^0) \\
\Phi^{f^*_\tau(Y)} \uparrow & & \uparrow \Phi^{\tau(Y)} \\
\widetilde{h}^*(X^{\nu(X)}) & \xrightarrow{Df} & \widetilde{h}^*(Y^{\nu(Y)})
\end{array}$$

From the geometric interpretation of Df it follows that Df is obtained by the Thom construction applied to $T(f)$. Then the theorem follows from lemma 8.4.

□

Chapter 12

Transfers in bordism and cobordism.

We shall give a geometric interpretation of the transfers defined in chapter 11. In the bordism and cobordism theories.

Let $f : X \to Y$ be a proper ξ-oriented map between smooth manifolds X and Y. It was shown that ξ-orientation defines a canonical $\mathcal{B}^*(;\xi)$-orientation of $\nu(f)$ (proposition 8.5). Thus we obtain transfer homomorphisms

$$\begin{array}{c} f_\natural : B^q(X;\xi) \longrightarrow B^{q+d}(Y;\xi) \\ f^\natural : B_q(Y;\xi) \longrightarrow B_{q-d}(X;\xi) \end{array}$$

where $d = \dim Y - \dim X$.

Theorem 12.1 *The transfers $f_\natural$ and $f^\natural$ defined in the homotopy theoretic bordism and cobordism*

$$f_\natural : \mathcal{B}^*(X;\xi) \longrightarrow \mathcal{B}^*(Y;\xi), \quad f_\natural([g]) = [f \circ g]$$

where ξ-orientation of $f \circ g$ is given by the isomorphism $\nu(f \circ g) = g!\nu(f) + \nu(g)$.

$$f^\natural : \mathcal{B}_*(Y;\xi) \longrightarrow \mathcal{B}_*(X;\xi), \quad f^\natural([g]) = [\widetilde{g}]$$

where $f \pitchfork g$ and g is defined by the pull back

$$\begin{array}{ccc} g!X & \xrightarrow{\widetilde{g}} & X \\ \downarrow & & \downarrow f \\ M & \xrightarrow{g} & Y \end{array}$$

with canonical ξ-orientation on $g!X$.

Proof: The proof follows from the interpretation of the Thom isomorphism and induced homomorphisms.

We shall give an interpretation of the Grothendieck bundle transfer in the bordism theory only because the geometric interpretation of cobordism is known for manifolds. In the case when $\pi : E \to B$ is a smooth bundle the theorem 11.3 states that the both transfers agree. Let $\pi : E \to X$ be a fiber bundle whose fiber F is a smooth compact manifold and whose structure group G is a compact Lie group acting smoothly on F. We assume that the ξ-orientation of the bundle $-\tau_F(E)$ is given. We have the Grothendieck bundle transfer

$$\pi^\natural : B_*(X;\xi) \longrightarrow B_*(E;\xi)$$

defined by the canonical $B^*(;\xi)$-orientation of $-\tau_F(E)$.

Theorem 12.2 *The transfer $\pi^\natural$ defined in homotopy theoretic bordism corresponds under the Thom-Pontriagin isomorphism to a map*

$$\pi^\natural : \mathcal{B}_*(X;\xi) \longrightarrow \mathcal{B}_*(E;\xi), \quad \pi^\natural([g]) = [\widetilde{g}]$$

where $\widetilde{g}$ is defined by the pull back

$$\begin{array}{ccc} g!E & \xrightarrow{\widetilde{g}} & E \\ \downarrow & & \downarrow \pi \\ M & \xrightarrow{g} & X \end{array}$$

Proof: The map $\pi^\natural$ is well defined: there exists a canonical smooth structure on $g!E$ (for the proof see Appendix). Standard arguments give a ξ-orientation on a manifold $g!E$. The map $\pi^\natural$ is natural with respect to maps of bundles which are diffeomorphisms on fibres. As in the proof of theorem 11.4 we can assume that X is a manifold. Then the theorem follows from theorem 12.1.

$\square$

Remark 12.3 *In a particular case of the bordism theory for a geometric definition of the transfer $\pi^\natural$ the compactness of G is not required.*

It is important to note that the Grothendieck bundle transfer gives a geometric interpretation of the inverse map to the Thom isomorphism in bordism theory for vector bundles arbitrary complexes. Let $\alpha : E \to^p X$ be a n-dimensional ξ-oriented vector bundle. A sphere bundle $S(\alpha+1) \to X$ admits a bundle transfer

$$p^\natural : B_*(X;\xi) \longrightarrow B_*(S(\alpha+1);\xi).$$

Theorem 12.4 *Under the assumptions described above the map*

$$\Psi : \mathcal{B}_q(X;\xi) \longrightarrow \mathcal{B}_{q+n}(D\alpha, S\alpha;\xi)$$

defined by the pull back

$$\begin{array}{ccc} (D(f!\alpha), S(f!\alpha)) & \xrightarrow{\widetilde{g}} & (D(\alpha), S(\alpha)) \\ \downarrow & & \downarrow p \\ M & \xrightarrow{g} & X \end{array}$$

is an inverse to the Thom isomorphism.

Proof: The theorem follows from the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{B}(S(\alpha+\theta');\xi) & \xrightarrow{j_*} & \mathcal{B}(S(\alpha+\theta'), X;\xi) \\ p^\natural \downarrow & & \downarrow \approx \\ \mathcal{B}_*(X;\xi) & \xrightarrow{\Psi} & \mathcal{B}_*(D\alpha, S\alpha;\xi) \end{array}$$

□

The last theorem is very important. We have defined the Thom isomorphism without use of transversality. In this form the Thom homomorphism is defined in an equivariant (geometric) bordism theory.

At the end of this section we describe, following Quillen, the ξ-cobordism theory as an universal contravariant functor on the category of smooth manifolds enowed with the transfers for ξ-oriented maps.

Let h be a contavariant functor from the homotopy category of smooth manifolds to the category of sets. Suppose that for each proper ξ-oriented map $f : Z \to X$ the map $f_\natural : h(Z) \to h(X)$ is given such that the following conditions are satisfied:

1. If f, g are proper ξ-oriented maps then $(g \circ f)_\natural = g_\natural \circ f_\natural$ where $g \circ f$ is endowed with canonical ξ-structure.

2. Assume that

$$\begin{array}{ccc} g!Z & \xrightarrow{\widetilde{g}} & Z \\ \widetilde{f} \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & X \end{array}$$

   is the pull back of manifolds where $f \pitchfork g$. If f is endowed with pull back ξ-structure then $g^* f_\natural = \widetilde{f}_\natural \widetilde{g}^*$.

Theorem 12.5 *Given is an element $a \in h(\mathrm{pt})$. Then there exists a unique natural transformation of functors $\theta : \mathcal{B}_*(;\xi) \to h$ commuting with transfers, such that $\theta(1) = a$ where $1 \in \mathcal{B}^*(\mathrm{pt}, \xi)$ is a class of identity map.*

Proof: An element $a \in h(\mathrm{pt})$ determines $a_X \in h(X)$ for all X by he formula $a_X = \pi_X^*(a)\pi_X : X \to \mathrm{pt}$. Furtheremore if $x = [f] \in \mathcal{B}^*(X;\xi)$ then $x = f_\S, \quad \pi_Z^*(1)$. Hence θ on the class x must be $\theta(x) = f_\natural \pi_Z^*(a)$ which proves the uniqueness of θ. For the existence of θ it is necessary to show that the right hand side of the definition $\theta(x) = f_\natural \pi_Z^*(a)$ where $x = [f]$ depends only on the cobordism class of f. Let f_0, f_1 be cobordant and $u : W \to X \times R$ be the cobordism between them. Then we have $(f_0)_\natural \pi_{Z_0}^*(a) = \varepsilon_0^* u_\natural \pi_W^*(a) = \varepsilon_1 u_\natural \pi_W^*(a) = (f_1)_\natural \pi_{Z_1}^*(a)$ where $\varepsilon_i : X \subset X \times 1$. It is obvious that θ is a natural transformation commuting with transfers.

$\square$

The above theorem gives us a description of $\mathcal{B}^*(;\xi)$ as a functor to the category of sets. It is possible to characterize the group structure (or the ring structure in the case of multiplicative bundle slectrum ξ) of $\mathcal{B}^*(;\xi)$ by a similar universal property. In this case we assume

1. the functor h takes values in the category of abelian groups and f are group homomorphisms,
2. conditions 1. and 2. stated above are satisfied.

From the general construction of transfers follows that the cobordism theory $\mathcal{B}^*(;\xi)$ is "universal" among cohomology theories in which bundles with ξ-structure are canonically oriented.

For example let $k^* = K^*$ be the complex K-theory. Let $\xi = \{\gamma_n^C\}$ be the spectrum determining the complex cobordism. There exists a unique multiplicative natural transformation $\theta : U^* \to K^*$ such that $\theta(1) = 1$. We obtain the following corollary.

Corollary 12.6 *If the vector bundle α is U^*-orientable then it is K^*-orientable.*

We recall that α is K^*-orientable iff $w_1(\alpha) = 0$ and $w_2(\alpha) = (c \mod 2)$ where $c \in H^2(B; Z)$. In other words there is a spin-structure on α. Complete characterization of U^*-orientability is unknown.

Chapter 13

Appendix.

Suppose $E \to B$ is a fibre bundle whose fibre F is a smooth compact n-dimensional manifold and whose structure group is aLie group acting smoothly on F. In this appendix we prove the following theorem:

Theorem 13.1 *Suppose the assumptions described above are fulfilled and moreover B is a smooth manifold. The space E admits then a smooth structure such that $pi : E \to B$ is a smooth bundle.*

We need first the following lemma.

Lemma 13.2 *Let $X \dashrightarrow X/G$ be a smooth principial bundle and F a smooth G-manifold. The associated bundle $X \times_G F \to X/G$ is then a smooth bundle.*

Lemma 13.3 *Assume that G is a compact Lie group. For each $k \in N$ there exists a k-universal smooth G-bundle $p : E_G^k \to B_G^k$, which we denote γ_G^k.*

Proof: Choose an imbedding of G into the orthogonal group $G \subset O(n)$. The principal bundle

$$O(N+k+1)/O(k+1) \longrightarrow O(N+k+1)/G \times O(k+1)$$

is the desired k-universal bundle.

$\square$

Lemma 13.4 *For each map $f : M \to N$ between manifolds there exists a smooth map $g : M \to N$ which is homotopic to f.*

The proof of the theorem in case G is a compact Lie group follows from the above lemmas:
Lemma 13.2 reduces the general case to the case of principal bundle. Since a manifold B has the homotopy type of a finite CW complex, there is a classifying map $f : B \to B_G^k$ for sufficiently large $k \in N$. Let g be a smooth map homotopic to f, then $g^j \gamma_G^k$ is a smooth bundle isomorphic to $\pi : E \to B$.

The proof of the general case is based on the following "folk theorem" formulated by Steenrod (N.E. Steenrod "Topology of fibre bundles" Princeton Univ. Press 1951).

Theorem 13.5 *A bundle whose group is a connected Lie group is equivalent in its group to a bundle whose group is a compact subgroup.*

Lemma 13.6 *Let M be a manifold and $p : \widetilde{M} \to M$ be a covering space over M. There exists a smooth structure on $\widetilde{M}$ such that p is a local isomorphism.*

We can prove now the theorem 13.1 in the general case. Let $G_\theta \hookrightarrow G$ be the component of the identity $\theta \in G$. Notice, that there exists a commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & X/G \\ \downarrow & \swarrow p & \\ X/G_\theta & & \end{array}$$

where $X \to X/G_\theta$ is a principal bundle whose structure group is a connected Lie group and $p : X/G_\theta \to X/G$ is a covering space. Theorem 13.1 follows then from its analogue for a compact group, theorem 13.5 and lemma 13.6.

$\square$

References:

[1] J.F.Adams, *Stable homotopy and generalized homology*, Lecture Notes, University of Chicago, 1971

[2] M.F.Atiyah, *Thom complexes*, Proc. London Math. Soc. 11 (1961), 291–310

[3] J.M.Boardman, *Stable homotopy theory*, chapter V–*Duality and Thom spectra*, University of Warwick, Warwick, 1966

[4] Th. Bröcker, T. tom Dieck, *Kobordismentheorie*, Lecture Notes in Mathematics 178, Berlin, Springer, 1970

[5] P.E.Conner, E.E. Floyd, *Differentiable periodic maps*, Berlin, Springer, 1970

[6] E.Dyer, *Cohomology theories*, New York, Benjamin, 1969

[7] D.Husemoller, *Fibre bundles*, New York, McGraw–Hill, 1966

[8] K.Jänich, *Differenzierbare G-Mannigfaltigkeiten*, Lecture Notes in Mathematics 59, Berlin, Springer, 1968

[9] M.Karoubi, *Cobordisme et groupes formals*, Sem. Bourbaki, 408, 1971–72

[10] S.Lang, *Differential manifolds*, Reading Mass., Addison Wesley, 1972

[11] J.W.Milnor, *On axiomatic homology theory*, Pacific J. Mathematics, 12 (1962), 337–341

[12] R.Narasimhan, *Analysis on real and complex manifolds*, Amsterdam, North Holland, 1968

[13] D.Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math. 7 (1971), 29–56

[14] E.H.Spanier, *Function spaces and duality*, Ann. Math. 70 (1959), 338–378

[15] E.H.Spanier, *Algebraic topology*, New York, McGraw–Hill, 1966

[16] R.Stong, *Notes on cobordism theory*, Princeton N.J., Princeton University Press, 1968

[17] G.W.Whitehead, *Generalised homology theories*, Trans. Amer. Math. Soc. 102 (1962), 227–283