Progressive Algorithms for Domination and Independence*

Grzegorz Fabiański† Michał Pilipczuk‡ Sebastian Siebertz§ Szymon Toruńczyk¶

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Abstract

We consider a generic algorithmic paradigm that we call *progressive exploration*, which can be used to develop simple and efficient parameterized graph algorithms. We identify two model-theoretic properties that lead to efficient progressive algorithms, namely variants of the *Helly property* and *stability*. We demonstrate our approach by giving linear-time fixed-parameter algorithms for the Distance-r Dominating Set problem (parameterized by the solution size) in a wide variety of restricted graph classes, such as powers of nowhere dense classes, map graphs, and (for $r = 1$) biclique-free graphs. Similarly, for the Distance-r Independent Set problem the technique can be used to give a linear-time fixed-parameter algorithm on any nowhere dense class. Despite the simplicity of the method, in several cases our results extend known boundaries of tractability for the considered problems and improve the best known running times.

1 Introduction

It is widely believed that many important algorithmic graph problems cannot be solved efficiently on general graphs. Consequently, a natural question is to identify the most general classes of graphs on which a given problem can be solved efficiently. Structural graph theory offers a wealth of concepts that can be used to design efficient algorithms for generally intractable problems on restricted graph classes. An important result in this area states that every property of graphs expressible in monadic second-order logic can be tested in linear time on every class of bounded treewidth [4]. Similarly, every property expressible in first-order logic can be tested in almost linear time on every nowhere dense graph class [11].

Nowhere denseness is an abstract notion of uniform sparseness in graphs, which is the foundational definition of the theory of sparse graphs; see the monograph of Nešetřil and Ossona de Mendez [14] for an overview. Formally, a graph class $\mathscr{C}$ is *nowhere dense* if for every $r \in \mathbb{N}$, one cannot obtain arbitrary large cliques by contracting disjoint connected subgraphs of radius at most r in graphs from $\mathscr{C}$. Many well-studied classes of sparse graphs are nowhere dense, for instance the class of planar graphs, any class of graphs with a fixed bound on the maximum degree, or any class of graphs excluding a fixed (topological) minor, are nowhere dense. Furthermore, under certain closure conditions, nowhere denseness

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†Faculty of Mathematics, Informatics, and Mechanics, University of Warsaw, Poland, grzegorz.fabianski@students.mimuw.edu.pl

‡Institute of Informatics, University of Warsaw, Poland, michal.pilipczuk@mimuw.edu.pl.

§Institut für Informatik Humboldt-Universität zu Berlin siebertz@informatik.hu-berlin.de.

¶Institute of Informatics, University of Warsaw, Poland szymtor@mimuw.edu.pl

constitutes the frontier of parameterized tractability for natural classes of problems. For instance, while the first-order model-checking problem is fixed-parameter tractable on every nowhere dense class $\mathscr{C}$ [11], on every subgraph-closed class $\mathscr{D}$ that is not nowhere dense, it is as hard as on the class of all graphs [6, 9]. Similar lower bounds are known for many individual problems, e.g. for the Distance-r Independent Set problem and the Distance-r Dominating Set problem, on subgraph-closed classes which are not nowhere dense [8, 15].

Towards the goal of extending the border of algorithmic tractability for the above mentioned problems beyond graph classes that are closed under taking subgraphs, we study a very simple and generic algorithmic paradigm that we call *progressive exploration*.

The idea of progressive exploration can be applied to a *parameterized subset problem*: given a graph G and parameter k, we look for a vertex subset S of size k that has some property. A progressive exploration algorithm works in rounds $i = 1, 2, \ldots$, where each round i finishes with constructing a *candidate solution* S_i, short, a *candidate*, and, in case S_i is not a solution, a *witness* W_i for this fact. More precisely, in round i, we first attempt to find a candidate S_i of size k which *agrees* with all the previously found witnesses $W_1, W_2, \ldots, W_{i-1}$, in the sense that none of them witnesses that S_i is not a feasible solution. Obviously, if no such S_i exists, then we can terminate and return that there is no solution. On the other hand, if the found candidate S_i is in fact a feasible solution, then we can also terminate and return it. Otherwise, we find another witness W_i, which witnesses that all the candidates found so far – $S_1, \ldots, S_{i-1}, S_i$ – are not feasible solutions, and we proceed to the next round. In this way, we progressively explore the whole solution space, while constructing more and more problematic witnesses that the future candidates must agree with, until we either find a solution or enough witnesses to certify that no solution exists.

Progressive graph exploration is a generic approach to solving graph problems, which so far rather resembles a wishful-thinking heuristic than a viable algorithmic methodology. Such algorithms can be applied to any input graph, however, a priori there are multiple problem-dependent details to be filled. First, in order to implement the iteration, we need to efficiently compute candidates S_i and small witnesses W_i in every round. Second, to analyze the running time we need to give an upper bound on the number of rounds in which the algorithm terminates. If we can guarantee that each round can be implemented efficiently and that the number of rounds is bounded in the parameter k, we immediately obtain a fixed-parameter algorithm for the problem under consideration.

In this work we study properties of algorithmic problems and graphs that ensure these desired features. We consider problems for which the property that a candidate S agrees with a witness W is, in a way to be made precise, definable by a first-order formula. We identify model-theoretic properties of formulas that lead to efficient progressive exploration algorithms. The properties that guarantee the existence of small witness sets are variants of the *Helly property*, called *nfcp* in model theory. The property that guarantees that the progressive exploration algorithms stop after a bounded number of rounds is the model-theoretic notion of *stability*. Under these conditions, for problems formulated using short distances in the graph, we are able to implement progressive exploration efficiently, yielding fast and simple fixed-parameter algorithms. See the caption in Figure 1 for an overview of the paper.

We demonstrate our approach by applying it to the Distance-r Dominating Set problem and the Distance-r Independent Set problem on a variety of restricted graph classes. Precisely, we prove that:

- For every $r \in \mathbb{N}$ and graph class $\mathscr{C}$ that is either nowhere dense, or is a power of a nowhere dense class, or is the class of map graphs, the Distance-r Dominating Set problem on any graph $G \in \mathscr{C}$ can be solved in time $2^{\mathcal{O}(k \log k)} \cdot \|G\|$. Here and throughout the paper, $\|G\|$ denotes the total number of vertices and edges in a graph G, while $|G|$ denotes the number of vertices in G.

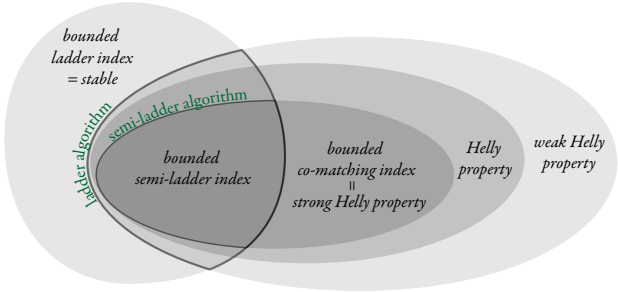

Figure 1: The figure depicts various properties of classes of bipartite graphs, which are introduced in Section 2. Domination- and independence-type problems studied in Section 3 reduce to the problem of determining whether the right part of a given bipartite graph has a common neighbor. In Section 4 two algorithms for the latter problem are devised, and their domains of applicability are marked above. The *ladder algorithm* has a larger domain, but requires a more powerful access oracle and has higher running time. Finally, we apply these algorithms to specific graph classes, yielding new fixed-parameter tractability results for domination- and independence-type problems.

- For every $t \in \mathbb{N}$, the Dominating Set problem on any $K_{t,t}$-free graph G can be solved in time $2^{\mathcal{O}(k \log k)} \cdot \|G\|$; here, a graph is $K_{t,t}$-free if it does not contain the complete bipartite graph $K_{t,t}$ as a subgraph.
- For every $r \in \mathbb{N}$ and nowhere dense graph class $\mathscr{C}$, the Distance-r Independent Set problem on any graph $G \in \mathscr{C}$ can be solved in time $f(k) \cdot \|G\|$, for some function f.

Actually, for the last result, we also give a different algorithm with running time $2^{\mathcal{O}(k \log k)} \cdot \|G\|$, which however does not rely on the concept of progressive exploration and uses some black-boxes from the theory of nowhere dense graphs.

Our results extend the limits of tractability for Distance-r Dominating Set and Distance-r Independent Set and, in some cases, improve the best known running times. We include a comprehensive comparison with the existing literature at the end of Section 4. However, let us stress here a key point: all our algorithms are derived in a generic way using the idea of progressive exploration, hence they are very easy to implement and they do not use any algorithmic black-boxes that depend on the class from which the input is drawn. In fact, properties of the considered classes are used only when analyzing the running time.

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2 Complexity-measures for bipartite graphs

In this section, we define the basic notions used in this paper, related to various complexity measures associated with bipartite graphs.

A *bipartite graph* is a triple $G = (L, R, E)$, where L and R are two sets of vertices, called the *left part* and *right part*, respectively, and $E \subseteq L \times R$ is a binary relation whose elements are called *edges*. Hence, bipartite graphs with parts L, R correspond bijectively to binary relations with domain L and codomain R. Note that each bipartite graph has a uniquely determined left and right part. Also, those parts are not necessarily disjoint.

Ladders, semi-ladders, and co-matchings. We now define various complexity measures for bipartite graphs, based on the size of a largest "obstruction" found in a given bipartite graph. There are several types of obstructions, leading to different complexity measures. We start with defining the various types of obstructions. Let $G = (L, R, E)$ be a bipartite graph. Two sequences, $a_1, \ldots, a_n \in L$ and $b_1, \ldots, b_n \in R$, form:

- a *co-matching* of order n in G if we have $(a_i, b_j) \in E \iff i \neq j$, for all $i, j \in \{1, \ldots, n\}$;
- a *ladder* of order n in G if we have $(a_i, b_j) \in E \iff i > j$, for all $i, j \in \{1, \ldots, n\}$; and
- a *semi-ladder* of order n in G if $(a_i, b_j) \in E$ for all $i, j \in \{1, \ldots, n\}$ with $i > j$, and $(a_i, b_i) \notin E$ for all $i \in \{1, \ldots, n\}$.

Note that in case of a semi-ladder we do not impose any condition for $i < j$. Observe that any ladder of order n and any co-matching of order n are also semi-ladders of order n.

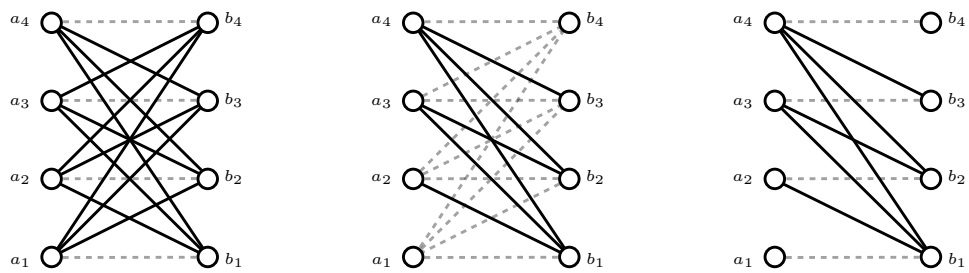

Figure 2: A co-matching, a ladder, and a semi-ladder of order 4, respectively. Dashed gray lines represent non-edges.

The *co-matching index* of a bipartite graph is the maximum order of a co-matching that it contains. A class of bipartite graphs has *bounded* co-matching index if the supremum of the co-matching indices of its members is finite. We define analogous notions for the *ladder index* and the *semi-ladder index*, in the expected way.

In this paper, we will often not care about the precise bounds on the indices of graphs, and it will only matter whether the respective index is bounded in a given class. Classes of bipartite graphs with bounded ladder index are also known as *stable* classes. We will later relate the property of having a bounded co-matching index to a variant of the *Helly property*. However, using a simple Ramsey argument, we now observe that boundedness of the semi-ladder index is equivalent to boundedness of both the co-matching and the ladder index. Let us first state Ramsey's theorem in the form used in this paper.

Theorem 1 (Ramsey's theorem). *For all $c, \ell \in \mathbb{N}$ there exists a number $R^c(\ell)$ with the following property. If the edges of a complete graph on $R^c(\ell)$ vertices are colored using c colors, then there is a set of ℓ vertices which is* monochromatic*, that is, all edges with both endpoints in this set are of the same color.*

The standard proof of Ramsey's theorem yields an upper bound of $R^c(\ell) \leqslant c^{c\ell-1}$ for $c \geqslant 2$. We include a proof for completeness. However, we defer the proof, and other proofs which are merely technical, to Section 8 in order to not disturb the flow of presentation.

From now on, we adopt the notation $R^c(\ell)$ for the multicolored Ramsey numbers as described in Theorem 1. Also the proof of the following lemma is relegated to Section 8.

Lemma 2. *A class of bipartite graphs has finite semi-ladder index if and only if both its ladder index and its co-matching index are finite.*

Helly property and its variants. Let $p \in \mathbb{N}$ and let $G = (L, R, E)$ be a bipartite graph. We say that a subset $B \subseteq R$ is *covered* by a subset $A \subseteq L$ if there exists a vertex $a \in A$ which is adjacent to all the vertices of B. Then subsets A and B have the *p-Helly property* if either B is covered by A, or B contains a subset of size at most p that is not covered by A. We shall say that G has:

- the *weak p-Helly property* if L and R have the p-Helly property;
- the *p-Helly property* if L and B have the p-Helly property, for all $B \subseteq R$; and
- the *strong p-Helly property* if all $A \subseteq L$ and $B \subseteq R$ have the p-Helly property.

We say that a class $\mathscr{C}$ of bipartite graphs has the (weak/strong) Helly property if there is some $p \in \mathbb{N}$ such that all graphs in $\mathscr{C}$ have the (weak/strong) p-Helly property. The Helly property is called *nfcp* in model theory [19, Chapter II.4].

It turns out that the strong p-Helly property corresponds precisely to having co-matching index at most p. A proof of the lemma is presented in Section 8.

Lemma 3. *Let $p \in \mathbb{N}$ and G be a bipartite graph. Then G has the strong p-Helly property if and only if it has co-matching index at most p.*

In the following paragraphs we will see specific examples of classes of bipartite graphs satisfying variants of the (weak/strong) Helly property.

Bipartite graphs defined by formulas. We construct bipartite graphs using logical formulas. In principle, we could consider formulas of any logic, but in this paper we only consider first-order logic in the vocabulary of graphs, i.e., using the binary relation symbol E representing edges, the binary relation symbol $=$ representing equality, and logical constructs $\vee, \wedge, \neg, \rightarrow, \forall, \exists$. E.g, the property $\mathrm{dist}(x, y) \leqslant 5$, expressing that x and y are at distance at most 5 in a graph G, can be expressed by a first-order formula using four existential quantifiers.

We write $\overline{x}$ to represent a non-repeating tuple of variables. If V is a set, then $V^{\overline{x}}$ denotes the set of all assignments mapping variables in $\overline{x}$ to V. Let $\varphi(\overline{x}; \overline{y})$ be a formula with free variables partitioned into two disjoint tuples, $\overline{x}$ and $\overline{y}$. Given any graph G with vertex set V, the formula φ induces a bipartite graph $\varphi(G)$ with left part $V^{\overline{x}}$, right part $V^{\overline{y}}$, and where $\bar{a} \in V^{\overline{x}}$ and $\bar{b} \in V^{\overline{y}}$ are adjacent if and only if $\varphi(\bar{a}; \bar{b})$ holds in G. If $\mathscr{C}$ is a class of graphs and $\varphi(\overline{x}; \overline{y})$ is a formula, then by $\varphi(\mathscr{C})$ we denote the class of all bipartite graphs $\varphi(G)$, for $G \in \mathscr{C}$. We say that φ has *bounded ladder index* on a class $\mathscr{C}$ if the class $\varphi(\mathscr{C})$ has bounded ladder index; similarly for the co-matching and the semi-ladder index. The same definitions apply if instead of graphs we consider logical structures over some fixed signature, and $\varphi(\overline{x}; \overline{y})$ is a formula over that signature. For simplicity, we consider only graphs in this paper.

We note that the various indices are preserved by adding spurious free variables to formulas. Precisely, let $\varphi(\overline{x}; \overline{y})$ be a first-order formula and let $\varphi'(\overline{x}'; \overline{y}')$ be the same formula, but having extra free variables, i.e., $\overline{x}$ is a subtuple of $\overline{x}'$ and $\overline{y}$ is a subtuple of $\overline{y}'$. Then, for any graph G, the ladder index of $\varphi(G)$ is equal to the ladder index of $\varphi'(G)$, although the bipartite graphs $\varphi(G)$ and $\varphi'(G)$ may differ. The same holds for all other properties studied in this paper: co-matching index, semi-ladder index, (weak/strong) Helly property.

The next lemma shows that a positive boolean combination of formulas with bounded semi-ladder index also has bounded semi-ladder index. The proof uses a Ramsey argument and is presented in Section 8.

Lemma 4. *Let $\varphi_1(\overline{x}; \overline{y}), \ldots, \varphi_k(\overline{x}; \overline{y})$ be formulas and let $\psi(\overline{x}; \overline{y})$ be a positive boolean combination of $\varphi_1, \ldots, \varphi_k$. Suppose G is a graph such that $\varphi_1(G), \ldots, \varphi_k(G)$ have semi-ladder index smaller than ℓ. Then $\psi(G)$ has semi-ladder index smaller than $R^k(\ell)$.*

We remark that the property of having bounded ladder index is preserved by taking arbitrary boolean combinations, not just positive ones.

We will later need the following variant of Lemma 4, which provides a sharper bound for formulas of a special form. Again, the proof relies on a Ramsey-like argument and is presented in Section 8.

Lemma 5. *Let $\psi(\overline{x};\overline{y}) = \bigvee_{j=1}^{k} \varphi(\overline{x}^j;\overline{y})$ for some $k \geqslant 2$, where $\varphi(\overline{x};\overline{y})$ is a formula and $\overline{x}^1,\ldots,\overline{x}^k$ are permutations of $\overline{x}$. Suppose G is a graph such that $\varphi(G)$ has semi-ladder index smaller than G. Then $\psi(G)$ has semi-ladder index smaller than $k^{\ell-1}$.*

Stability. The classes of bipartite graphs with bounded ladder index that are relevant to this paper are provided by the following result, due to Podewski and Ziegler [18].

Theorem 6 ([18], cf. [1, 17]). *Let $\mathscr{C}$ be a nowhere-dense class of graphs and let $\varphi(\overline{x};\overline{y})$ be any first-order formula. Then φ has bounded ladder index on $\mathscr{C}$.*

The above result was originally stated for *superflat graphs*. The connection with nowhere-denseness was observed in [1], and a proof providing explicit bounds was given in [17]. The observation of [1] is the starting point of this work, as it brings to light the connection between model theory and computer science.

3 Domination and independence problems

We consider subset problems, where in a given graph we look for a solution S of size k that satisfies some property, whose dissatisfaction can be witnessed by a small subset of vertices W. Moreover, checking a candidate solution S against a witness W can be expressed in first-order logic. Thus, a problem of interest can be expressed by a sentence of the form $\exists_{\overline{x}}\forall_{\overline{y}}\varphi(\overline{x};\overline{y})$, for a suitable formula $\varphi(\overline{x};\overline{y})$, where $\overline{x}$ is a tuple of k variables that represent a candidate S, while $\overline{y}$ is a tuple of ℓ variables that represent a witness W.

Example 7. The Distance-r Dominating Set problem for parameter k can be expressed as above using the formula $\delta_r^k(\overline{x};y) = \bigvee_{i=1}^{k} \delta_r(x_i,y)$, where $\delta_r(x,y)$ is a formula that checks whether $\mathrm{dist}(x,y) \leqslant r$, and $\overline{x} = (x_1,\ldots,x_k)$. Similarly, the Distance-r Independent Set problem for parameter k can be expressed using the formula $\eta_r^k(\overline{x};y) = \bigwedge_{1\leqslant i<j\leqslant k} \eta_r(x_i,x_j,y)$, where $\eta_r(x,x',y)$ is a formula that checks whether $\mathrm{dist}(x,y) + \mathrm{dist}(x',y) > r$.

Observe that a graph G satisfies the sentence $\exists_{\overline{x}}\forall_{\overline{y}}\varphi(\overline{x};\overline{y})$ if and only if the right part of the bipartite graph $H = \varphi(G)$ is covered by the left side, i.e., all vertices in the right part (witnesses) have a common neighbor (solution). We call this abstract problem — checking whether the right part of a given bipartite graph H is covered by the left side — the Coverage problem.

Note that the size of the bipartite graph $\varphi(G)$ is polynomial in the size of G, where the exponent depends on the number of free variables in φ, which is usually the parameter we are interested in. As we are aiming at fixed-parameter algorithms, we cannot afford to even construct the whole bipartite graph $\varphi(G)$. Therefore, we will design algorithms that solve the Coverage problem using an oracle access to the bipartite graph graph $H = \varphi(G)$, where the oracle calls will be implemented using subroutines on the original graph G. The running time of these algorithms, expressed in terms of the number of oracle calls, will be bounded only in terms of quantities (ladder indices, numbers governing Helly property, etc.) related to the class of bipartite graphs $\varphi(\mathscr{C})$, where $\mathscr{C}$ is the considered class of input graphs.

Therefore, to obtain an algorithm for solving the initial problem on a given graph class $\mathscr{C}$ we proceed in two steps:

- Prove that the class $\varphi(\mathscr{C})$ has a suitable Helly-type property and bounded ladder index.
- Design an algorithm for Coverage, for input bipartite graphs with suitable Helly-type properties and bounded ladder index, that uses only a bounded number of oracle calls.

In Section 4 we give two such algorithms solving Coverage: the *Semi-ladder Algorithm*, and the *Ladder Algorithm*. The Semi-ladder Algorithm requires that H has bounded semi-ladder index, whereas the Ladder Algorithm requires that H has bounded ladder index and the weak p-Helly property, for some fixed p. Note that by Lemma 2 and 3, boundedness of the semi-ladder index is equivalent to boundedness of the ladder index and having the strong p-Helly property, for some fixed p, so the prerequisites for the Semi-ladder Algorithm are stronger than for the Ladder Algorithm. See Figure 1 for an overview.

We postpone the discussion of the algorithms to Section 4, and for now we focus on exhibiting the suitable properties for various classes of bipartite graphs. Slightly more precisely, we prove that on certain graph classes, formulas corresponding to domination-type problems have bounded semi-ladder index, while those corresponding to independence-type problems have the weak Helly property and bounded ladder index. Hence, in the first case we will apply the Semi-ladder Algorithm, and in the second — the Ladder Algorithm.

Distance formulas and domination-type problems. We shall prove fixed-parameter tractability results not only for distance-r domination, but for a more general class of domination-type problems. Those can be expressed by suitable formulas, as explained next.

For $r \in \mathbb{N}$, let $\delta_r(x, y)$ be the formula checking whether $\mathrm{dist}(x, y) \leqslant r$. A *distance formula* is a formula $\varphi(\overline{x}; \overline{y})$ which is a boolean combination of *atoms* of the form $\delta_r(x, y)$, where the variable x occurs in $\overline{x}$, the variable y occurs in $\overline{y}$, and $r \in \mathbb{N}$ is any number. The *radius* of a distance formula is the maximal number r occurring in its atoms, whereas its *size* is the number of atoms occurring in it. A distance formula is *positive* if it is a positive boolean combination of atoms.

A *domination-type property* is a sentence ψ of the form $\exists_{\overline{x}} \forall_{\overline{y}} \varphi(\overline{x}; \overline{y})$, where φ is a positive distance formula. A *domination-type problem* is the computational problem of determining whether a given graph G satisfies a given domination-type property.

Example 8. Fix $r \in \mathbb{N}$ and let $\overline{x} = (x_1, \ldots, x_k)$ be a k-tuple of variable. Then the formula $\delta_r^k(\overline{x}; y)$ considered in Example 7 is a positive distance formula, hence the problem defined by the domination-type property $\exists_{\overline{x}} \forall_{\overline{y}} \delta_r^k(\overline{x}; y)$ (aka Distance-r Dominating Set) is a domination-type problem. Similarly, formulas $\varphi(\overline{x}; y)$ expressing the following properties give raise to natural domination-type problems:

- y is at distance at most r from at least two of the vertices $x_1, \ldots, x_k$; and
- the sum $\mathrm{dist}(x_1, y) + \mathrm{dist}(x_2, y) + \ldots + \mathrm{dist}(x_k, y)$ is at most r.

On the other hand, the formula $\eta_r^k(\overline{x}; y)$ considered in Example 7 is a distance formula, but it is not positive, and hence it does not yield a domination-type property.

From Lemma 4 and 5 and the remark about spurious variables not affecting the semi-ladder index, we immediately obtain the following.

Corollary 9. *Let $\varphi(\overline{x};\overline{y})$ be a positive distance formula of radius r and size s. If G is a graph such that the semi-ladder index of $\delta_q(G)$ is smaller than ℓ for all $q \leqslant r$, then the semi-ladder index of $\varphi(G)$ is smaller than $R^s(\ell)$. Moreover, if $\varphi = \delta_r^k$ as defined in Example 7 and $k \geqslant 2$, then the semi-ladder index of $\varphi(G)$ is smaller than $k^{\ell-1}$.*

Domination problems. We first consider domination-type problems and prove that they have bounded semi-ladder indices on any nowhere dense class. This result can actually be extended beyond nowhere denseness: to powers of nowhere dense classes, to map graphs, and to $K_{t,t}$-free graphs for radius $r = 1$. We define the former two concepts next.

For a graph G and $s \in \mathbb{N}$, let G^s denote the graph with the same vertex set as G, where two vertices are adjacent if and only if their distance in G is at most s. If $\mathscr{D}$ is a graph class, then $\mathscr{D}^s$ denotes the class $\{G^s \colon G \in \mathscr{D}\}$. Note that a power of a nowhere dense class is not necessarily nowhere dense, e.g., the square of the class of stars is the class of complete graphs.

A graph G is a map graph if one can assign to each vertex of G a closed, arc-connected region in the plane so that the interiors of regions are pairwise disjoint and two vertices of G are adjacent if and only if their regions share at least one point on their boundaries. Note that map graphs are not necessarily planar and may contain arbitrarily large cliques, as four or more regions may share a single point on their boundaries.

The following result will be used in the next section to obtain fixed-parameter tractability of domination-type problems over graph classes described above.

Theorem 10. *For any $r \in \mathbb{N}$ and nowhere dense graph class $\mathscr{C}$, the formula $\delta_r(x, y)$ has bounded semi-ladder index on $\mathscr{C}$. The same holds also when $\mathscr{C} = \mathscr{D}^s$ for some nowhere dense class $\mathscr{D}$ and $s \in \mathbb{N}$, when $\mathscr{C}$ is the class of map graphs, and when $r = 1$ and $\mathscr{C}$ is the class of $K_{t,t}$-free graphs, for any fixed $t \in \mathbb{N}$.*

We prove Theorem 10 in Section 6. For the case when $\mathscr{C}$ is nowhere dense we utilize the well-known characterization of nowhere denseness via *uniform quasi-wideness* [13]. In a nutshell, if for some $G \in \mathscr{C}$ the graph $\delta_r(G)$ has semi-ladder index ℓ, then in G we have vertices $a_1, \ldots, a_\ell$ and $b_1, \ldots, b_\ell$ such that $\mathrm{dist}(a_i, b_j) \leqslant r$ for all $i > j$ and $\mathrm{dist}(a_i, b_i) > r$ for all i. Then provided ℓ is huge, using uniform quasi-wideness we can find a large subset $A \subseteq \{a_1, \ldots, a_\ell\}$ of vertices that "communicate" with each other only through a set S of constant size — all paths of length at most $2r$ between vertices of A pass through S. Now the vertices from A have pairwise different distance-r neighborhoods within $\{b_1, \ldots, b_\ell\}$, but only a limited number of possible interactions with S (measured up to distance r). This quickly leads to a contradiction if A is large enough. The cases when $\mathscr{C}$ is a power of a nowhere dense class and when $\mathscr{C}$ is the class of map graphs follow as simple corollaries from the result for nowhere dense classes. The case when $\mathscr{C}$ is the class of $K_{t,t}$-free graphs is a simple observation: a large semi-ladder in $\delta_1(G)$ enforces a large biclique in G.

We remark that the argument used in the proof of Lemma 29 is similar to the reasoning that shows that graphs from a fixed nowhere dense class admit small *distance-r domination cores*: subsets of vertices whose distance-r domination forces distance-r domination of the whole graph. This property was first proved implicitly by Dawar and Kreutzer in their FPT algorithm for Distance-r Dominating Set on any nowhere dense class [5], also using uniform quasi-wideness. We refer the reader to [16, Chapter 3, Section 5] for an explicit exposition. We will discuss the existence of small cores in the more general setting given in this paper in Section 5.

Having established boundedness of the semi-ladder index of $\delta_r(x, y)$ on a class $\mathscr{C}$, we can use Corollary 9 to extend this to any positive distance formula. Therefore, by Theorem 10, Corollary 9, and Lemma 3 we immediately obtain the following.

Corollary 11. *Let $\mathscr{C}$ and r be as in Theorem 10 and let $\varphi(\overline{x};\overline{y})$ be a positive distance formula of radius at most r. Then the class $\varphi(\mathscr{C})$ has bounded semi-ladder index, so in particular it has the strong Helly property.*

In fact, Corollary 9 provides a better control of the semi-ladder index of $\varphi(\mathscr{C})$ in terms of the semi-ladder index of $\delta_r(\mathscr{C})$ and the size of φ. In the next section we will use these more refined bounds for a precise analysis of the running times.

Note that Corollary 11 does not generalize to arbitrary first-order formulas. Indeed, if $\mathscr{C}$ is the class of all edgeless graphs and $\varphi(x;y)$ is the formula $x \neq y$, then $\varphi(\mathscr{C})$ is the class of all complements of matchings, which does not even have the weak Helly property.

Independence problems. We now move to the Distance-r Independent Set problem: deciding whether a given graph contains a distance-r independent set of size k. This property is most naturally expressed using an existential sentence, and not as a sentence of the form $\exists_{\overline{x}}\forall_{\overline{y}}\varphi(\overline{x};\overline{y})$. However, in Example 7 we gave a suitable formula $\eta_r^k(\overline{x};y)$ that expresses the problem: the trick is to phrase the property that $x_1,\ldots,x_k$ are pairwise at distance more than r by saying that for every vertex y, for all $1 \leqslant i < j \leqslant k$ the sum of distances from y to x_i and x_j is larger than r. Thus, a vertex y that does not satisfy this condition may serve as a witness that a given tuple $\overline{x}$ does not form a distance-r independent set.

In Section 7 we prove the following.

Theorem 12. *Let $\mathscr{C}$ be a nowhere-dense class of graphs and let $k, r \in \mathbb{N}$. Then the class $\eta_r^k(\mathscr{C})$ has the weak p-Helly property, for some $p \in \mathbb{N}$ depending on k, r, and $\mathscr{C}$.*

It is easy to see that for any $k \geqslant 2$ and $r \geqslant 1$, the formula $\eta_r^k(x;y)$ does not have the strong Helly property on the class $\mathscr{C}$ of edgeless graphs. Thus, in general we cannot hope for boundedness of the semi-ladder index of $\eta_r^k(\mathscr{C})$ and use the Semi-Ladder Algorithm.

The proof of Theorem 12 is actually very different from the proof of Theorem 10, and presents a novel contribution of this work. Instead of uniform quasi-wideness, we use the characterization of nowhere denseness via the *Splitter game* [11]. The idea is that in case a graph $G \in \mathscr{C}$ does not have a distance-r independent set of size k, there is a small witness of this: a set W of size bounded in terms of k, r, and $\mathscr{C}$ such that for every vertex subset S of size k, some path of length at most r connecting two vertices of S crosses W. This exactly corresponds to the notion of witnessing expressed by η_r^k. Such a witness W is constructed recursively along Splitter's strategy tree in the Splitter game in G. We use the condition that G does not have a distance-r independent set of size k to prove that we can find a small (in terms of $k, r, \mathscr{C}$) set of "representative" moves of the Connector. Trimming the strategy tree to those moves bounds its size in terms of $k, r, \mathscr{C}$, yielding the desired upper bound on the witness size.

We remark that our proof of Theorem 12 can actually be turned into an algorithm for the Distance-r Independent Set problem on any nowhere dense class $\mathscr{C}$ with running time of $2^{\mathcal{O}(k\log k)} \cdot \|G\|$. However, this algorithm is much more complicated than the Ladder algorithm that we explain in the next section, and in particular it uses some black-box results from the theory of nowhere dense graph classes. Details can be found at the end of Section 7.

4 Algorithms

Model. We consider the following model of an algorithmic search for a solution in a bipartite graph representing the search space. Consider a bipartite graph $H = (L, R, E)$, where the left side L is the set of *candidates* and the right side R is the set of *witnesses*. An edge between a candidate $a \in L$ and a witness $b \in R$ is interpreted as that a and b *agree*: b agrees that a is a solution. Expressed in those terms,

Coverage is the problem of finding a *solution*: a candidate which agrees with all witnesses. We will use the terminology of candidates, witnesses, solutions, and agreeing as explained above, as this facilitates the understanding of the algorithms for Coverage in terms of the original problems.

As we explained, the considered bipartite graph H will typically be of the form $\varphi(G)$ for some formula $\varphi(\overline{x};\overline{y})$ expressing the considered problem. Thus, H shall represent the whole search space, so we allow our algorithms a restricted access to H via the following *oracles*.

Candidate Oracle: Given a set of witnesses $B \subseteq R$, the oracle either returns a candidate $a \in L$ that agrees with all witnesses of B, or concludes that no such candidate exists.

Weak Witness Oracle: Given a candidate $a \in L$, the oracle either concludes that a is a solution, or returns a witness $b \in R$ that does not agree with a.

Strong Witness Oracle: Given a set of candidates $A \subseteq L$ and a number $p \in \mathbb{N}$, the oracle either finds a set of witnesses $P \subseteq R$ such that $|P| \leqslant p$ and every candidate of A does not agree with some witness from P, or concludes that no such set P exists.

Note that the Weak Witness Oracle can be simulated by the Strong Witness Oracle applied to $A = \{a\}$. We now provide the two algorithms for Coverage announced in Section 3.

Semi-ladder Algorithm. The Semi-ladder Algorithm proceeds in a number of rounds, where each round consists of two steps: first the *Candidate Step*, and then the *Witness Step*. Also, the algorithm maintains a set B of witnesses gathered so far, initially set to be empty. The steps are defined as follows:

Candidate Step: Apply the Candidate Oracle to find a candidate $a \in L$ that agrees with all the witnesses in B. If no such candidate exists, terminate the algorithm returning that no solution exists. Otherwise, proceed to the Witness Step.

Witness Step: Apply the Weak Witness Oracle to find a witness $b \in R$ that does not agree with a. If there is no such witness, terminate the algorithm and return a as the solution. Otherwise, add b to B and proceed to the next round.

The correctness of the algorithm is obvious, while the running time can be bounded by the immediate observation that if the Semi-ladder Algorithm performs ℓ full rounds, then the candidates $a_1, \ldots, a_\ell \in L$ discovered in consecutive rounds, together with the witnesses $b_1, \ldots, b_\ell \in R$ added to B in consecutive rounds, form a semi-ladder in H.

Corollary 13. *The Semi-ladder Algorithm applied to a graph H with semi-ladder index ℓ terminates after performing at most ℓ full rounds. Consequently, it uses at most $\ell + 1$ Candidate Oracle Calls, each involving a set of witnesses B with $|B| \leqslant \ell$, and at most ℓ Weak Witness Oracle Calls.*

Ladder algorithm. As before, the Ladder Algorithm maintains the set B of witnesses gathered so far, but also the set A of candidates found so far. The algorithm is also given a parameter $p \in \mathbb{N}$. Again, the algorithm proceeds in rounds, each consisting of the Candidate step and the Witness step, with the following description:

Candidate Step: Apply the Candidate Oracle to find a candidate $a \in L$ that agrees with all the witnesses in B. If no such candidate exists, terminate the algorithm returning that no solution exists. Otherwise, add a to A and proceed to the Witness step.

Witness Step: Apply the Strong Witness Oracle to set A and parameter p, yielding either a set of witnesses $P \subseteq R$ such that $|P| \leqslant p$ and every candidate from A does not agree with some witness from P, or a conclusion that no such set P exists. In the former case, add P to B and proceed to the next round. In the latter case, terminate the algorithm returning that a solution exists.

Note that the algorithm actually never finds a solution, but only may claim its existence in the Witness Step, and this claim is not substantiated by having a concrete solution in hand. However, the observation is that assuming the weak p-Helly property, the structure discovered by the algorithm is sufficient to deduce the existence of a solution.

Lemma 14. *The Ladder Algorithm applied with parameter p in a bipartite graph with the weak p-Helly property is always correct.*

Proof. If the algorithm returns that there is no solution, this claim is justified by finding a set of witnesses that cannot simultaneously agree with any candidate, which implies that no solution exists. If the algorithm returns that there is a solution, then this is because it constructed a set of candidates A such that for every subset of witnesses $P \subseteq R$ with $|P| \leqslant p$, some candidate within A agrees with all the witnesses from P. If there was no solution, then by the p-Helly property there would exist a set P_0 of at most p witnesses, for which there would be no candidate simultaneously agreeing with all the witnesses of P_0; this would contradict the previous conclusion. Hence, the algorithm's claim that there exists a solution is correct. □

Finally, we show that if H has ladder index bounded by ℓ, then the algorithm terminates in a number of rounds bounded in terms of ℓ and p. For this we observe that during its execution, the algorithm in fact constructs a ladder in an auxiliary bipartite graph H' with candidates a on the left side and sets of witnesses P on the right side, and the ladder index of H' can be bounded in terms of p and the ladder index of H using a Ramsey argument.

Lemma 15. *The Ladder Algorithm applied with parameter p to a bipartite graph H with ladder index smaller than ℓ terminates after performing less than $R^p(2\ell)$ full rounds.*

Proof. Let H' be the bipartite graph constructed as follows: the left side of H' consists of the candidate set L, the right side of H' consists of the family of all sets of witnesses $P \subseteq R$ satisfying $|P| \leqslant p$, and a candidate $a \in L$ is adjacent in H' to a set of witnesses $P \subseteq R$ if and only if a agrees with all witnesses of P.

We claim that H' has ladder index smaller than $R^p(2\ell)$. Note that this will conclude the proof for the following reason: if $a_1, \ldots, a_t$ and $P_1, \ldots, P_t$ are candidates and witness sets discovered by the algorithm during consecutive rounds, where t is the number of full rounds performed, then $a_1, \ldots, a_t$ and $P_1, \ldots, P_t$ form a ladder in H'.

For contradiction, suppose that $a_1, \ldots, a_q$ and $P_1, \ldots, P_q$ form a ladder of order $q = R^p(2\ell)$ in H'. Arbitrarily enumerate the elements of each set P_i as $b_i^1, \ldots, b_i^p$, repeating elements in case $|P_i| < p$. Now, color every pair (i, j) satisfying $q \geqslant i > j \geqslant 1$ with any number $s \in \{1, \ldots, p\}$ such that a_j does not agree with b_i^s; such s exists because a_j and P_i are not adjacent in H'. By Ramsey's theorem, we may find an index subset $X \subseteq \{1, \ldots, q\}$ of size 2ℓ and a color $s \in \{1, \ldots, p\}$ such that all pairs (i, j) with $i > j$ and $i, j \in X$ have color s. Let $c_1, \ldots, c_{2\ell}$ be the elements of $\{a_i \colon i \in X\}$ ordered as in the sequence $a_1, \ldots, a_q$, and let $d_1, \ldots, d_{2\ell}$ be the elements of $\{b_i^s \colon i \in X\}$ ordered as in the sequence $b_1^s, \ldots, b_q^s$. Then it follows that $c_1, c_3, \ldots, c_{2\ell-1}$ and $d_2, d_4, \ldots, d_{2\ell}$ form a ladder of order ℓ in H, a contradiction. □

Corollary 16. *The Ladder Algorithm applied with parameter p to a graph H with ladder index smaller than ℓ and the weak p-Helly property, always returns the correct answer and terminates after performing at most $q = R^p(\ell) - 1$ full rounds. Consequently, it uses at most $q + 1$ Candidate Oracle Calls, each involving a set of witnesses B with $|B| \leqslant pq$, and at most q Strong Witness Oracle Calls, each involving a set of candidates A with $|A| \leqslant q$.*

Implementing the oracles. The last missing ingredient for obtaining our algorithmic results is an efficient implementation of the oracles for bipartite graphs of the form $\varphi(G)$, where G is the input graph and $\varphi(\overline{x}, \overline{y})$ is a formula expressing the considered problem. We describe such an implementation whenever φ is a distance formula.

We use the concept of *distance profiles* and *distance profile complexity*. Let G be a graph and let S be a set of its vertices. For a vertex v of G, the *distance-r profile* of v on S, denoted $\mathrm{profile}_r^{G,S}(v)$, is the function mapping S to $\{0, 1, \ldots, r, \infty\}$ such that for $s \in S$,

$$\mathrm{profile}_r^{G,S}(v)(s) = \begin{cases} \mathrm{dist}_G(v, s) & \text{if } \mathrm{dist}_G(v, s) \leqslant r, \\ \infty & \text{otherwise.} \end{cases}$$

The *distance-r profile complexity* of G is the function from $\mathbb{N}$ to $\mathbb{N}$ defined as

$$\nu_r^G(m) = \max_{S \subseteq V,\, |S| \leqslant m} |\{\mathrm{profile}_r^{G,S}(u) \colon v \in V(G)\}|.$$

That is, this is the maximum possible number of different functions from S to $\{0, 1, \ldots, r, \infty\}$ realized as distance-r profiles on S of vertices of G, over all vertex subsets S of size at most m. For a graph class $\mathscr{C}$, we denote $\nu_r^{\mathscr{C}}(m) = \sup_{G \in \mathscr{C}} \nu_r^G(m)$.

Note that for any graph G and $r, m \in \mathbb{N}$ we have $\nu_r^G(m) \leqslant (r + 2)^m$, as this is the total number of functions from a set of size m to $\{0, 1, \ldots, r, \infty\}$. This bound is exponential in m, however it is known that on nowhere dense classes an almost linear bound holds.

Lemma 17 ([10]). *Let $\mathscr{C}$ be a nowhere dense class of graphs. Then for every $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists a constant $c_{r,\varepsilon}$ such that $\nu_r^{\mathscr{C}}(m) \leqslant c_{r,\varepsilon} \cdot m^{1+\varepsilon}$ for all $m \in \mathbb{N}$.*

We remark that the conclusion of Lemma 17 still holds when $\mathscr{C}$ is any fixed power of a nowhere dense class, and when $\mathscr{C}$ is the class of map graphs. Moreover, when $\mathscr{C}$ is the class of $K_{t,t}$-free graphs for some $t \in \mathbb{N}$, then $\nu_1^{\mathscr{C}}(m) \leqslant \mathcal{O}(m^t)$. We prove these statements in Lemma 42 and 43 in Section 8.

We are ready to give implementations for the oracles. The main idea is that because we are working with a distance formula, when looking for, say, a candidate that agrees with all witnesses in a set B, the only information that is relevant about any vertex is its distance-r profile on the set S consisting of all vertices appearing in the tuples of B. Hence, there are only $\nu_r^G(|S|)$ different "types" of vertices, and instead of checking all k-tuples of vertices in the graph, we can check all k-tuples of types.

Lemma 18. *Fix a distance formula $\varphi(\overline{x}; \overline{y})$ of radius r and with $|\overline{x}| = c$ and $|\overline{y}| = d$. Then for an input graph $G = (V, E)$, there are implementations of oracle calls in $\varphi(G)$ that achieve the following running times:*

- *Candidate Oracle: time $\mathcal{O}(|B| \cdot \|G\| + |B| \cdot \nu_r^G(d|B|)^c)$ for a call to $B \subseteq V^{\overline{y}}$;*
- *Weak Witness Oracle: time $\mathcal{O}(\|G\| + \nu_r^G(c)^d)$ for a call to $\overline{a} \in V^{\overline{x}}$;*
- *Strong Witness Oracle: time $\mathcal{O}(|A| \cdot \|G\| + |A| \cdot \nu_r^G(c|A|)^{pd})$ for a call to $A \subseteq V^{\overline{x}}$ and $p \in \mathbb{N}$.*

Proof. Consider first the Candidate Oracle. Let $S \subseteq V$ be the set of all vertices contained in tuples from B; then $|S| \leqslant d|B| = \mathcal{O}(|B|)$. Apply a BFS from every vertex of S to compute, for every $u \in V$, a vector of distances from u to all the vertices of S. This takes time $\mathcal{O}(|S| \cdot \|G\|) = \mathcal{O}(|B| \cdot \|G\|)$. Given this information, for every vertex $u \in V$ we may compute its distance-r profile on S, and in particular we may compute the set $\mathcal{P}$ of all distance-r profiles on S realized in G. This can be done in time $\mathcal{O}(|S| \cdot |V|)$ by adding profiles of all the vertices to a trie. Note that $|\mathcal{P}| \leqslant \nu_r^G(d|B|)$.

Next, we look for a candidate $\overline{a} \in V^{\overline{x}}$ that agrees with all elements of B by considering all c-tuples of profiles in $\mathcal{P}$. Here, the crucial observation is that since φ is a distance formula, whether a candidate $\overline{a}$ agrees with a fixed witness $\overline{b} \in B$ depends only on the distance-r profiles of the entries of $\overline{a}$ on S. Hence, if suffices to consider only c-tuples of profiles in $\mathcal{P}$, and not all c-tuples of vertices of G. Checking a given c-tuple of profiles in $\mathcal{P}$ against a fixed witness $\overline{b} \in B$ can be done in constant time, so the whole search for a candidate can be executed in time $\mathcal{O}(|B| \cdot |\mathcal{P}|^c) \leqslant \mathcal{O}(|B| \cdot \nu_r^G(d|B|)^c)$. Note that a suitable candidate can be retrieved by storing together every profile of $\mathcal{P}$ any candidate $\overline{a} \in V^{\overline{x}}$ realizing this profile.

The implementation of the Strong Witness Oracle follows from applying exactly the same idea: S is the set of all vertices involved in tuples in A, there are at most $\nu_r^G(c|A|)$ relevant distance-r profiles on S, and we need to search through pd-tuples of such profile. The Weak Witness Oracle can be implemented by simply running the Strong Witness Oracle for a singleton set A and $p = 1$. □

Algorithmic consequences. We are ready to present our algorithmic corollaries, promised in Section 1. Throughout this section, when stating parameterized running times we use k to denote the target size of a solution (i.e., distance-r dominating or independent set). We start with the domination problems.

Theorem 19. *Fix $r \in \mathbb{N}$ and let $\mathscr{C}$ be a class of graphs such that for each $q \leqslant r$, the class $\delta_q(\mathscr{C})$ has finite semi-ladder index. Then, for any positive distance formula $\varphi(\overline{x}; \overline{y})$ of radius at most r and size k, the domination-type problem corresponding to φ can be solved on $\mathscr{C}$ in time $f(k) \cdot \|G\|$, for some function f.*

Proof. W.l.o.g. we can assume that $|\overline{x}|, |\overline{y}| \leqslant k$. Let $\ell \in \mathbb{N}$ be such that $\delta_q(\mathscr{C})$ has semi-ladder index smaller than ℓ, for all $q \leqslant r$. Given a graph G, we apply the Semi-Ladder Algorithm for the Coverage problem in the graph $\varphi(G)$ with implementations of oracles provided by Lemma 18. By Lemma 4 we conclude the semi-ladder index of $\varphi(\mathscr{C})$ is bounded by $R^k(\ell)$. Now the claimed running time follows immediately from Corollary 13 and Lemma 18. □

Remark 20. By Corollary 13 and Lemma 18, the running time is actually $\mathcal{O}(p \cdot \nu_r^{\mathscr{C}}(p)^k \cdot \|G\|)$, where p is the semi-ladder index of $\varphi(G)$. By Lemma 4, we have that $p \leqslant R^k(\ell)$, which is upper-bounded by $k^{k\ell-1}$ for $k \geqslant 2$ (see Lemma 41). Combining this with the trivial upper bound $\nu_r^{\mathscr{C}}(p) \leqslant (r+2)^p$ yields $f(k) \leqslant 2^{2^{\mathcal{O}(k \log k)}}$, where r and ℓ are considered fixed constants. However, if a priori we know for the graph class $\mathscr{C}$ that $\nu_r^{\mathscr{C}}(m)$ is polynomial in m, instead of exponential, then by the analysis above we obtain $f(k) \leqslant 2^{\mathcal{O}(k^2 \log k)}$. Finally, by Lemma 5, for $\varphi = \delta_r^k$ — the formula corresponding to the Distance-r Dominating Set problem — we can use a sharper bound of $p \leqslant k^{\ell-1}$. Thus, for this case we obtain an upper bound of $f(k) \leqslant 2^{\mathrm{poly}(k)}$ in the general setting, and $f(k) \leqslant 2^{\mathcal{O}(k \log k)}$ when $\nu_r^{\mathscr{C}}(m)$ is polynomial in m.

Now, using Theorem 19 together with combinatorial results stated in Section 2 we immediately obtain the algorithmic results promised in Section 1. Note that the results hold not only for Distance-r Dominating Set, but even for every domination-type problem of fixed radius r and size k that is considered the parameter.

Theorem 21. *Fix $r \in \mathbb{N}$. Then any domination-type problem defined by a positive distance formula of size k and radius at most r can be solved in time $2^{\mathcal{O}(k^2 \log k)} \cdot \|G\|$ on any graph class $\mathscr{C}$ such that either $\mathscr{C}$ is nowhere dense, or $\mathscr{C} = \mathscr{D}^s$ for a nowhere dense class $\mathscr{D}$ and some $s \in \mathbb{N}$, or $\mathscr{C}$ is the class of map graphs, or $r = 1$ and $\mathscr{C}$ is the class of $K_{t,t}$-free graphs for some fixed $t \in \mathbb{N}$. Moreover, if this domination-type problem is* Distance-r Dominating Set *for parameter k, then the running time can be improved to $2^{\mathcal{O}(k \log k)} \cdot \|G\|$.*

Proof. By Theorem 10, the class $\delta_r(\mathscr{C})$ has finite semi-ladder index. By Lemma 17 and its strengthenings, Lemma 42 and Lemma 43, $\nu_r^{\mathscr{C}}(m)$ is bounded by a polynomial in m. Hence, we may apply Theorem 19; the claimed running times follow from the remark following it. □

We now move to the independence problems, for which we apply the Ladder algorithm.

Theorem 22. *Let $r \in \mathbb{N}$ and let $\mathscr{C}$ be a class of graphs such that for any $k \in \mathbb{N}$, the class $\eta_r^k(\mathscr{C})$ has ladder index smaller than $\ell(k)$ and has the weak $p(k)$-Helly property, for some functions $\ell, p \colon \mathbb{N} \to \mathbb{N}$. Then the* Distance-$r$ Independent Set *problem on $\mathscr{C}$ can be solved in time $f(k) \cdot \|G\|$, for some function f.*

Proof. Given a graph G, we apply the Ladder Algorithm in the graph $\eta_r^k(\mathscr{C})$ with implementations of oracles provided by Lemma 18. The correctness of the algorithm and the running time bound follow directly from Corollary 16 and Lemma 18, where we may set $f(k) = \mathcal{O}(R^{p(k)}(2\ell(k)) \cdot \nu_r^{\mathscr{C}}(p(k) \cdot R^{p(k)}(2\ell(k))))^{p(k)\cdot k})$. □

Theorem 23. *For any $r \in \mathbb{N}$ and nowhere dense class $\mathscr{C}$, the* Distance-r Independent Set *problem on $\mathscr{C}$ can be solved in time $f(k) \cdot \|G\|$, for some function f.*

Proof. By Theorem 6 and 12, for every $k \in \mathbb{N}$ there are constants $\ell, p \in \mathbb{N}$, depending on k, such that the class $\eta_r^k(\mathscr{C})$ has ladder index bounded by ℓ and has the weak p-Helly property. This allows us to apply Theorem 22. □

Discussion of related results. Fixed-parameter tractability of both Distance-r Dominating Set and Distance-r Independent Set on any nowhere dense class follows from the general model-checking result for first-order logic of Kreutzer et al. [11]. The algorithms derived in this manner have running time $f(k) \cdot n^{1+\varepsilon}$ for any fixed $\varepsilon > 0$ and some function f, where n is the number of vertices of the input graph. In fact, an algorithm with running time $f(k) \cdot n^{1+\varepsilon}$ for the Distance-r Independent Set problem is one of the intermediate results used in [11]. A close inspection of this algorithm reveals that the polynomial factor is in fact $\|G\|$, improving the claimed $n^{1+\varepsilon}$, however this is not explicit in [11]. For the Distance-r Dominating Set problem, its fixed-parameter tractability on any nowhere dense class was established earlier by Dawar and Kreutzer [5], but their algorithm had at least a quadratic polynomial factor in the running time bound.

As far as Distance-r Dominating Set on powers of nowhere dense classes is concerned, we remark that the result provided in Theorem 21 would *not* follow immediately from applying the algorithm on the graph before taking the power, for radius rs instead of r. The reason is that the input consists only of the graph G^s, and it is completely unclear how to algorithmically find the preimage G if we are dealing with an arbitrary nowhere dense class $\mathscr{D}$. To the best of our knowledge, this result is a completely new contribution.

Regarding map graphs, the fixed-parameter tractability of the Distance-r Dominating Set problem on this class of graphs was established by Demaine et al. [7]. However, they use the recognition algorithm for map graphs of Thorup [21] to draw a map model of the graph; this algorithm has an estimated running time of at least $\mathcal{O}(n^{120})$ [2] and not all technical details have been published. Another way of obtaining a fixed-parameter algorithm would be to use the fact that map graphs have *locally bounded rank width*;

however, again achieving linear running time would be difficult due to the need of computing branch decompositions with approximately optimum rankwidth, for which the best known algorithms have cubic running time. In contrast, as we have shown, the Semi-ladder Algorithm solves the problem in linear fixed-parameter time without the need of having a map model provided.

Finally, the fixed-parameter tractability of Dominating Set on $K_{t,t}$-free graphs, where both k and t are considered parameters, was established by Telle and Villanger [20]. Thus, Theorem 21 reproves this result and also improves upon the running time: from $2^{\mathcal{O}(k^{t+2})} \cdot \|G\|$ of [20] to $2^{\mathcal{O}(k \log k)} \cdot \|G\|$.

5 Domination cores

As we remarked earlier, the argument used in the proof of Lemma 29 is similar to the reasoning that shows that graphs from a fixed nowhere dense class admit small *distance-r domination cores*: subsets of vertices whose distance-r domination forces distance-r domination of the whole graph; this concept was introduced by Dawar and Kreutzer [5]. In fact, we can lift this result to the more general setting, for domination-type properties with bounded semi-ladder indices, as follows.

Let $H = (L, R, E)$ be a bipartite graph. As before, we consider the left side L to be the set of candidates and the right side R to be the set of witnesses. An edge between a candidate $a \in L$ and a witness $b \in R$ is interpreted as that a and b agree, that is, b agrees that a is a solution. Here, the considered bipartite graph H will be of the form $\varphi(G)$ a some formula $\varphi(\overline{x}; \overline{y})$ expressing a domination-type problem.

A *coverage core* for a set of witnesses $S \subseteq R$ is a subset $C \subseteq S$ such that every candidate $a \in L$ which agrees with all the witnesses from C, actually agrees with all the witnesses from S. A coverage core for H is a coverage core for R.

Our goal is to give an algorithm that for domination-type problems computes small coverage cores. As before, we allow our algorithms a restricted access to H via an oracle.

Semi-ladder Extension Oracle: Given a set of witnesses $B \subseteq R$, the oracle either returns a candidate $a \in L$ and a witness $b \in R - B$ such that

- a and b do not agree, that is, $(a, b) \notin E$, and
- a agrees with all witnesses in B,

or concludes that no such pair a, b exists.

Before we show that Semi-ladder Extension Oracles for domination-type properties can be efficiently implemented, we present an algorithm that computes a coverage core for H, given that H has bounded semi-ladder index.

Coverage Core Algorithm. The Coverage Core Algorithm maintains two sequences $a_1, \ldots, a_n \in L$, $b_1, \ldots, b_n \in R$ which form a semi-ladder in H. Initially, both sequences are empty and $n = 0$. The algorithm repeats the following step.

Extension step: Let $B = \{b_1, \ldots, b_n\}$. Apply the Semi-ladder Extension Oracle to the set B. If the oracle does not return a pair, terminate and return B as a coverage core for H. Otherwise, let (a, b) be the pair returned by the oracle. Extend the semi-ladder by inserting a as the element a_{n+1} to the sequence $a_1, \ldots, a_n$ and b as the element b_{n+1} to the sequence $b_1, \ldots, b_n$. Note that the sequences again form a semi-ladder in H. Proceed to the next round.

The following lemma follows immediately from the fact that the Coverage Core Algorithm constructs in n steps a semi-ladder of length n.

Lemma 24. *The Coverage Core Algorithm applied to a bipartite graph H with semi-ladder index ℓ terminates after performing at most $\ell + 1$ rounds. Consequently, it uses at most $\ell + 1$ Semi-ladder Extension Oracle calls, each involving a set of candidates A with $|A| \leqslant \ell$ and a set of witnesses B with $|B| \leqslant \ell$.*

Lemma 25. *The Coverage Core Algorithm applied to a bipartite graph H returns a coverage core for H.*

Proof. The algorithm terminates if the Semi-ladder Extension Oracle does not return a candidate $a \in L$ and a witness $b \in R - B$ for the set $B = \{b_1, \ldots, b_n\}$. Unravelling the definition, this means that every candidate $a \in L$ which agrees with all witnesses from B also agrees with all witnesses from $R - B$. This means that B is a coverage core for H. □

Finally, we give an implementation of the Semi-ladder Extension Oracle.

Lemma 26. *Fix a distance formula $\varphi(\bar{x}; \bar{y})$ of radius r with $|\bar{x}| = c$ and $|\bar{y}| = d$. Then for an input graph $G = (V, E)$, there is an implementation of Semi-ladder Extension Oracle calls in $\varphi(G)$ with parameters $B \subseteq V^{\bar{y}}$ running in time*

$$\mathcal{O}\big(|G|^d \cdot (|B| \cdot \|G\| + |B| \cdot \nu_r^G(d|B| + d)^c)\big).$$

Proof. We iterate over all valuations $\bar{b} \in V^{\bar{y}} - B$, giving a factor $|G|^d$ in the running time. Fix such $\bar{b} \in V^{\bar{y}} - B$. Our task is to find $\bar{a} \in V^{\bar{x}}$ which agrees with all witnesses from B, but which does not agree with $\bar{b}$. This can be done by a slight modification of the Candidate Oracle (applied to $B \cup \{\bar{b}\}$), as presented in Lemma 18, running in time $\mathcal{O}(|B| \cdot \|G\| + |B| \cdot \nu_r^G(d|B| + d)^c)$. The modification boils down to ignoring c-tuples of profiles on the set of vertices involved in $B \cup \{\bar{b}\}$ that imply agreeing with $\bar{b}$. In total, we get the claimed running time. □

Corollary 27. *Fix $r \in \mathbb{N}$ and let $\mathscr{C}$ be a class of graphs such that for each $q \leqslant r$, the class $\delta_q(\mathscr{C})$ has finite semi-ladder index. Then, for any positive distance formula $\varphi(\bar{x}, \bar{y})$ of radius at most r, size at most k and $|\bar{y}| = d$, the domination-type problem corresponding to φ on $\mathscr{C}$ admits a coverage core of size $f(k)$ and such a core can be computed in time $f(k) \cdot |G|^d \cdot \|G\|$, for some function f.*

Proof. W.l.o.g. we can assume that $|\overline{x}| \leqslant k$. Let $\ell \in \mathbb{N}$ be such that $\delta_q(\mathscr{C})$ has semi-ladder index smaller than ℓ, for all $q \leqslant r$. Given a graph G, we apply the Coverage Core Algorithm in the graph $H = \varphi(G)$ with implementation of the Semi-ladder Extension Oracle provided by Lemma 26. By Lemma 4 we conclude the semi-ladder index of $\varphi(\mathscr{C})$ is bounded by $R^k(\ell)$. Now the claimed running time follows immediately from Lemma 24 and Lemma 26. □

6 Helly property for domination

In this section are going to present the proof of Theorem 10. We start with the case when the considered class $\mathscr{C}$ is nowhere dense, and for this we use the well-known characterization of nowhere denseness in terms of *uniform quasi-wideness*.

Definition 1. Let $s\colon \mathbb{N} \to \mathbb{N}$ and $N\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be functions. We say that a graph class $\mathscr{C}$ is *uniformly quasi-wide* with *margins* s and N if for all $r, k \in \mathbb{N}$, every graph $G \in \mathscr{C}$, and every vertex subset $W \subseteq V(G)$ of size larger than $N(r, k)$, there exist disjoint vertex subsets $S \subseteq V(G)$ and $A \subseteq W$ such that $|S| \leqslant s(r)$, $|A| > k$, and A is distance-r independent in $G - S$. A class $\mathscr{C}$ is *uniformly quasi-wide* if it is uniformly quasi-wide with some margins.

Theorem 28 ([12, 14]). *A graph class $\mathscr{C}$ is nowhere dense if and only if it is uniformly quasi-wide.*

We proceed to the nowhere dense case, which is encapsulated in the following lemma.

Lemma 29. *For every $r \in \mathbb{N}$ and nowhere dense class $\mathscr{C}$, the class $\delta_r(\mathscr{C})$ has a finite semi-ladder index.*

Proof. Fix $r \in \mathbb{N}$ and a graph $G \in \mathscr{C}$. Suppose vertices $a_1, \ldots, a_\ell$ and $b_1, \ldots, b_\ell$ form a semi-ladder of order n in $\delta_r(G)$, that is, we have that

- $\mathrm{dist}_G(a_i, b_i) > r$ for each $i \in \{1, \ldots, \ell\}$; and
- $\mathrm{dist}_G(a_i, b_j) \leqslant r$ for all $i, j \in \{1, \ldots, \ell\}$ with $i > j$.

We need to give a universal upper bound on ℓ, expressed only in terms of r and $\mathscr{C}$.

By Theorem 28, $\mathscr{C}$ is uniformly quasi-wide, say with margins $s\colon \mathbb{N} \to \mathbb{N}$ and $N\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Let $s = s(2r)$ and $t = 2 \cdot (r+2)^s$. Suppose from now on that $\ell > N(2r, t)$.

Let $W = \{a_1, \ldots, a_\ell\}$; then $|W| = \ell > N(2r, t)$, because vertices a_i are pairwise different. By uniform quasi-wideness, we can find disjoint vertex subsets $S \subseteq V(G)$ and $A \subseteq W$ such that $|S| \leqslant s$, $|A| > t$, and A is distance-$2r$ independent in $G - S$.

As $|S| \leqslant s$, there are at most $(r+2)^s$ possible distance-r profiles on S (called further just *profiles* for brevity). Since $|A| > t = 2 \cdot (r+2)^s$, we can find three indices $1 \leqslant \alpha < \beta < \gamma \leqslant \ell$ such that the vertices $x := a_\alpha, y := a_\beta, z := a_\gamma$ belong to A and have equal profiles. Denote $w := b_\alpha$. In particular, we have the following:

- the distance between y and z in $G - S$ is larger than $2r$;
- the vertices x, y, z have the same profiles; and
- $\mathrm{dist}_G(w, x) > r$, $\mathrm{dist}_G(w, y) \leqslant r$, and $\mathrm{dist}_G(w, z) \leqslant r$.

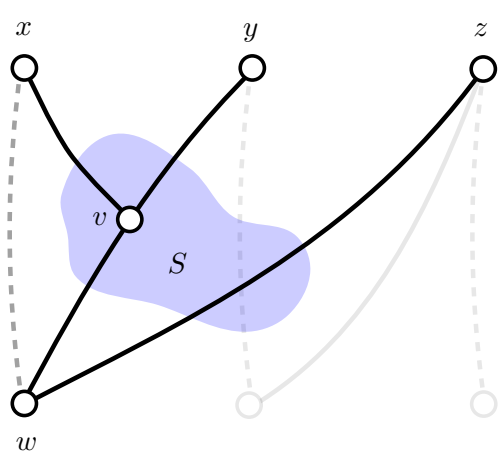

Figure 3: Proof of Lemma 29: contradiction looms.

We show that this leads to a contradiction (see Figure 3). Let P_{yw} be a path of length at most r connecting y and w, and let P_{wz} be a path of length at most r connecting w and z. In particular, the concatenation of P_{yw} and P_{wz} has length at most $2r$ and connects y and z. Since the distance between y and z in $G - S$ is larger than $2r$, at least one of the paths P_{yw}, P_{wz} must contain a vertex $v \in S$. Suppose

that it is P_{yw}, the other case being analogous. Then P_{yw} is split by v into two subpaths: P_{yv} and P_{vw}. Since x and y have the same profiles, we may find a path P_{xv} connecting x and v whose length is not larger than the length of P_{yv}. Now, the concatenation of paths P_{xv} and P_{vw} has length at most r and connects x with w, a contradiction with $\mathrm{dist}_G(w, x) > r$.

Therefore, we must have that $\ell \leqslant N(2r, 2 \cdot (r+2)^{s(2r)})$, which finishes the proof. □

The case when $\mathscr{C}$ is a power of a nowhere dense class now follows almost immediately.

Lemma 30. *For all $r, s \in \mathbb{N}$ and nowhere dense class $\mathscr{D}$, the class $\delta_r(\mathscr{D}^s)$ has a finite semi-ladder index.*

Proof. It suffices to observe that $\delta_r(\mathscr{D}^s) = \delta_{rs}(\mathscr{D})$ and use Lemma 29. □

For the case of map graphs, we use the following characterization of map graphs as *half-squares of planar graphs*, due to Chen et al. [3].

Theorem 31 ([3]). *If $G = (V, E)$ is a map graph, then there exists a bipartite planar graph H with V being one side of its bipartition such that G is the subgraph induced by V in H^2.*

Lemma 32. *Let $\mathscr{C}$ be the class of map graphs. Then for every $r \in \mathbb{N}$, the class $\delta_r(\mathscr{C})$ has a finite semi-ladder index.*

Proof. Let $\mathscr{P}$ be the class of planar graphs. Since $\mathscr{P}$ is nowhere dense, by Lemma 30 we have that the class $\delta_r(\mathscr{P}^2)$ has finite semi-ladder index. It now suffices to observe that by Theorem 31, every graph in $\delta_r(\mathscr{C})$ is an induced subgraph of a graph in $\delta_r(\mathscr{P}^2)$, and semi-ladder index is monotone under taking induced subgraphs. □

We are left with the case of $r = 1$ and classes excluding a fixed complete bipartite graph.

Lemma 33. *Let $t \in \mathbb{N}$ and let $\mathscr{C}$ be the class of $K_{t,t}$-free graphs. Then the semi-ladder index of the class $\delta_1(\mathscr{C})$ is smaller than $3t$.*

Proof. Suppose for the sake of contradiction that there exists a graph $G \in \mathscr{C}$ and vertices $a_1, \ldots, a_{3t}$ and $b_1, \ldots, b_{3t}$ that form a semi-ladder of order $3t$ in $\delta_1(G)$. Note here that $\delta_1(x, y)$ checks whether x and y are equal or adjacent, which means that a_i and b_j are equal or adjacent whenever $i > j$, whereas a_i and b_i are different and non-adjacent for all i. Observe that $b_i \neq b_j$ for all $1 \leqslant j < i \leqslant 3t$, because $\delta_1(a_i, b_j)$ holds, while $\delta_1(a_i, b_i)$ does not. Similarly, $a_i \neq a_j$ for all $1 \leqslant j < i \leqslant 3t$. Hence, among vertices $a_{t+1}, \ldots, a_{3t}$ there are at least t vertices different from the vertices $b_1, \ldots, b_s$. These vertices on one side, and vertices $b_1, \ldots, b_t$ on the other side, form a $K_{t,t}$ subgraph in G, contradicting the assumption that $G \in \mathscr{C}$. □

Lemma 29, 30, 32, and 33 together prove Theorem 10.

7 Helly property for independence

In this section we show that distance-r independence has the weak Helly property on every nowhere dense class of graphs. More precisely, we prove Theorem 12 and derive some auxiliary algorithmic results from it.

Let us start with clarifying the notation. Let G be a graph and let $r \in \mathbb{N}$. For a graph G, by $V(G)$ and $E(G)$ we denote the vertex and the edge set of G, respectively. For vertex $u \in V(G)$, the *distance-r neighborhood* of u is the set $N_r^G(u) = \{v \colon \mathrm{dist}_G(u, v) \leqslant r\}$. This notation is extended to subsets by setting $N_r^G(U) = \bigcup_{u \in U} N_r^G(u)$. A vertex subset $A \subseteq V(G)$ is *distance-r independent* if $\mathrm{dist}_G(u, v) > r$ for all distinct $u, v \in A$. A vertex subset $D \subseteq V(G)$ *distance-r dominates* A if $A \subseteq N_r^G(D)$. Then D is a *distance-r dominating set* in G if D distance-r dominates $V(G)$.

Throughout this section we will use distance-r profiles, which we discussed in Section 4. We extend the notation for profiles to subsets of vertices as follows: For $r \in \mathbb{N}$, a graph G, and vertex subsets $S, U \subseteq V(G)$ we define

$$\mathrm{profile}_r^{G,S}(U)(v) = \min_{u \in U} \mathrm{profile}_r^{G,S}(u)(v) \qquad \text{for all } v \in S.$$

Note that, again, $\mathrm{profile}_r^{G,S}(U)$ is a function from S to $\{0, 1, 2, \ldots, r, \infty\}$, and there are $(r+2)^{|S|}$ possibilities for such a function.

Dependence cores and pre-cores. We will work with the following combinatorial object that we call a *dependence core*.

Definition 2. Let G be a graph and let $r, k \in \mathbb{N}$. A *distance-r dependence core* for G and k is a subset of vertices $Q \subseteq V(G)$ with the following property: for every set $X \subseteq V(G)$ of size at most k there exists a path of length at most r containing a vertex of Q and connecting two elements of X.

Observe that a distance-r dependence core for G and k can only exist if G does not contain a distance-r independent set of size k. By definition of η_r^k, Theorem 12 is an immediate consequence of the following theorem.

Theorem 34. *Let $r, k \in \mathbb{N}$ and $\mathscr{C}$ be a nowhere dense graph class. Then there exists $p \in \mathbb{N}$, depending on r, k, and $\mathscr{C}$, such if $G \in \mathscr{C}$ does not contain a distance-r independent set of size k, then G has a distance-r dependence core of size at most p.*

Hence, it suffices to prove Theorem 34. Towards this goal, we introduce the following definitions.

Definition 3. Let G be a graph, let $Q, D \subseteq V(G)$, and let $a \in V(G)$. We shall say that Q *distance-r captures* the pair (D, a) if there exists a path of length at most r containing a vertex of Q and connecting a with a vertex d of D.

In the following we will always talk about distance-r capturing for the number r clear in the context, and hence for brevity we will just use the term *capture*.

Definition 4. Let G be a graph, $A \subseteq V(G)$ and let $r, k \in \mathbb{N}$. A *distance-r dependence pre-core* for G, A and k is a subset of vertices $Q \subseteq V(G)$ with the following property: for every set $D \subseteq V(G)$ that has size at most k and distance-r dominates A, the set Q captures (D, a) for each $a \in A$.

While dependence cores can only exist in the absence of independent sets of size k, pre-cores always exist (the whole vertex set is always a pre-core). In order to prove Theorem 34, we prove that in case that a graph G does not contain a distance-r independent set of size k, a distance-r dependence pre-core for $G, V(G)$ and $k-1$ is in fact a distance-r dependence core for k. We then proceed to prove that nowhere dense graph classes admit small pre-cores. In fact, the first author proves in his thesis that monotone nowhere dense graph classes are characterized by the existence of small pre-cores for all vertex subsets A and values of r.

A pre-core is a core, whenever possible. We first prove that every distance-r dependence pre-core for $G, V(G)$ and $k-1$ in absence of a distance-r independent set of size k is in fact a distance-r dependence core for k.

Lemma 35. *Let G be a graph and let $r, k \in \mathbb{N}$. Let Q be a distance-r dependence pre-core for $G, V(G)$ and $k-1$. If G does not contain a distance-r independent set of size k, then Q is a distance-r dependence core for G and k.*

Proof. Assume towards a contradiction that Q is not a distance-r dependence core for G and k. Hence, there exists $X \subseteq V(G)$ of size k such that Q does not intersect any path of length at most r between two distinct vertices of X. From all such choices for X, fix one that minimizes the function

$$f(X) = |\{w \in X\colon \text{ there exists } v \in X, v \neq w, \text{ with } \operatorname{dist}(v,w) \leqslant r\}|\,.$$

By assumption, there does not exist a distance-r independent set of size k in G, so there are $v, w \in X$, $v \neq w$, with $\operatorname{dist}(v,w) \leqslant r$.

If $X - \{w\}$ is a distance-r dominating set of G, then, by definition of a pre-core, for every vertex $u \in V(G)$ there exists a path of length at most r containing a vertex of Q and connecting u with a vertex of $X - \{w\}$. In particular, this holds for the vertex w. But then Q crosses some path of length at most r between two distinct vertices of X, a contradiction.

Otherwise, $X - \{w\}$ is not a distance-r dominating set of G. Let u be a vertex with $\operatorname{dist}(u,v) > r$ for all $v \in X - \{w\}$. We define $X' := (X \cup \{u\}) - \{w\}$ and claim that X' satisfies the condition on X: no path of length at most r connecting two distinct elements of X' is intersected by Q. To see this, assume that there exists such a path. As u is not within distance r to any other vertex of X', we have $x_1, x_2 \neq u$. Hence, the considered path connects two vertices from X, contradiction our assumption on X. Now observe that $f(X') < f(X)$, as by construction of X' we have $f(X') = f(X - \{w\}) = f(X) - 1$, contradicting the minimality of X. This finishes the proof. □

Now, Theorem 34 follows immediately from Lemma 35 and the following lemma, which we are going to prove next.

Lemma 36. *Let $r, k \in \mathbb{N}$ and $\mathscr{C}$ be a fixed nowhere dense graph class. Then there exists $p \in \mathbb{N}$, depending on r, k and $\mathscr{C}$ such that for every $G \in \mathscr{C}$ and every $A \subseteq V(G)$ there exists a distance-r dependence pre-core for G, A and k of size at most p. Moreover, given G, A, and k, such a distance-r dependence pre-core can be computed in time $f(r,k) \cdot \|G\|$, for some function $f(r,k)$ that is polynomial in k for fixed r.*

Splitter game. In order to prove Lemma 36 we appeal to a characterization of nowhere dense graph classes in terms of a game, called the splitter game, which was introduced in [11].

Definition 5 (Splitter game). Let G be a graph and let $\ell, r \in \mathbb{N}$. The ℓ-round radius-r *splitter game* on G is played by two players, *Connector* and *Splitter*, as follows. We let $G_0 := G$. In round $i+1$ of the game, Connector chooses a vertex $v_{i+1} \in G_i$. Then Splitter picks a vertex $w_{i+1} \in N_r^{G_i}(v_{i+1})$. We define $G_{i+1} := G_i[N_r^{G_i}(v_{i+1}) - \{w_{i+1}\}]$. Splitter wins if $G_{i+1} = \emptyset$. Otherwise the game continues on the graph G_{i+1}. If Splitter has not won after ℓ rounds, then Connector wins.

A strategy for Splitter is a function f that associates to every partial play $(v_1, w_1, \ldots, v_i, w_i)$ with associated sequence $G_0, \ldots, G_i$ of graphs and move $v_{i+1} \in V(G_i)$ by Connector a vertex $w_{i+1} \in N_r^{G_i}(v_{i+1})$. A strategy f is a winning strategy for Splitter in the ℓ-round radius-r splitter game on G if Splitter wins every play in which he follows the strategy f. We say that a winning strategy is computable in time T if for every partial play as above, the vertex w_{i+1} can be computed in time T.

Theorem 37 ([11]). *Let $\mathscr{C}$ be a class of graphs. Then $\mathscr{C}$ is nowhere dense if and only if for every $r \in \mathbb{N}$ there exists $\ell \in \mathbb{N}$, such that for every $G \in \mathscr{C}$, Splitter wins the ℓ-round radius-r splitter game on G. Furthermore, for fixed r, a winning strategy for splitter in the radius-r splitter game on any graph $G \in \mathscr{C}$ is computable in time $\mathcal{O}\big(\ell \cdot \|G\|\big)$.*

Now Lemma 36 is implied by Theorem 37 and the following lemma (applied with $s = 0$).

Lemma 38. *For all $\ell, k, r, s \in \mathbb{N}$ there exists $p \in \mathbb{N}$, depending polynomially on k for fixed ℓ, r, s, with the following property. For every graph G and $S \subseteq V(G)$ with $|S| \leqslant s$ and such that Splitter wins the ℓ-round radius-$3r$ splitter game on $G - S$, and for every $A \subseteq G$, there exists a distance-r dependence pre-core Q for G, A and k of size at most p. Furthermore, such a set Q is computable in time $f(\ell, k, r, s) \cdot \|G\|$, for some computable function f which is polynomial in k for fixed ℓ, r, s.*

Induction on the splitter's strategy. Before we prove Lemma 38, we need one more lemma, which expresses the outcome of applying a naive greedy strategy for finding a small distance-r dominating set.

Lemma 39. *Let G be a graph, $X \subseteq V(G)$ and $r, k \in \mathbb{N}$. Then either*

1. *there is $Y \subseteq X$ such that $|Y| > k$ and Y is distance-$2r$ independent in G, or*
2. *there is $Z \subseteq X$ such that $|Z| \leqslant k$ and $X \subseteq N_{2r}^G(Z)$.*

Moreover, there is an algorithm which for given G, X, r, k computes Y or Z as above in time $\mathcal{O}(k \cdot \|G\|)$.

Proof. We use a simple greedy algorithm, which aims to construct the set Y. Let us assume we have already constructed $Y \subseteq X$ such that $|Y| \leqslant k$ and Y is distance-$2r$ independent. There are two possibilities:

- If $X \subseteq N_{2r}(Y)$, then we set $Z := Y$ satisfying the second condition and terminate.
- Otherwise, there is $v \in X$ such that $v \notin N_{2r}(Y)$, so $Y \cup \{v\}$ is distance-$2r$ independent and larger, hence we continue with $Y \cup \{v\}$.

After $k+1$ repetition of this procedure, we constructed a set Y satisfying the first condition. Testing whether there exists $v \notin N_{2r}(Y)$ can be easily done in linear time, so the final running time is $\mathcal{O}(k \cdot (n + m))$. □

Proof (of Lemma 38). We prove the lemma by induction on the length ℓ of the splitter game. Furthermore, throughout the induction we maintain the following additional invariant: $Q \supseteq S$. If $\ell = 0$ we have $V(G) = S$ and we can set $p = s$ and $Q := V(G)$.

Denote by $\mathcal{P}$ the set of all distance-r profiles on S; then $|\mathcal{P}| \leqslant (r + 2)^s$. In the induction step, our goal will be to find a family

$$\mathcal{G} = \{(G_t, S_t, A_t)\}_{t \in T} \qquad \text{for some index set } T,$$

where G_t is a subgraph of G and $S_t, A_t \subseteq V(G_t)$ satisfying the following properties:

1. The size $|\mathcal{G}|$ can be bounded by a function of ℓ, k, r, s, which is polynomial in k for fixed ℓ, r, s.
2. We have that $S_t \supseteq S$ and $|S_t| \leqslant |S| + 1$ for all $t \in T$.
3. For each $t \in T$, Splitter wins the $\ell - 1$-round radius-$3r$ splitter game on $G_t - S_t$. So by the induction hypothesis, there exists a small distance-r dependence pre-core $Q_t \supseteq S_t$ for G_t, A_t, and k.
4. For every $a \in A$ and for every set $D \subseteq V(G)$ that has size at most k and distance-r dominates A, if the pair (D, a) is not captured by S, then (D, a) is captured by Q_t in some graph G_t.

As we will see, once we have constructed the family $\mathcal{G}$ we can output $Q := \bigcup_{t \in T} Q_t$. Note that then indeed $Q \supseteq S$, as $Q_t \supseteq S_t \supseteq S$ for all $t \in T$.

Each of the graphs G_t is constructed by simulating one step of the splitter game. That is, we consider some vertex $v_t \in G - S$ as the choice of Connector in the splitter game. By assumption, we can find

an answer $w_t \in N_{3r}^{G-S}(v_t)$ of Splitter, so that Splitter wins the $\ell - 1$-round radius-$3r$ splitter game on $G[N_{3r}^{G-S}(v_t) - \{w_t\}]$. We let $G_t := G[N_{3r}^{G-S}(v_t) \cup S]$ and $S_t := S \cup \{w_t\}$. Hence, the second and third of the above properties are satisfied for all constructed graphs G_t. Our task will be to identify a small set of moves v_t of Connector and sets A_t, so that also the first and fourth property are satisfied.

To construct such a set of moves of Connector we use a localization property for any pair (D, a) to be captured — we want to find a small set Z such that every pair (D, a) that needs to be captured will be close to some vertex of Z. This is made precise by the following statement, whose proof relies on Lemma 39.

Claim 1. *There exists a set $Z \subseteq V(G)$ with $|Z| \leqslant k \cdot (r+2)^s$, such that for every set D such that $|D| \leqslant k$ and D distance-r dominates of A, and every $a \in A$, if (D, a) is not captured by S, then $a \in N_{2r}^{G-S}(Z)$.*

Proof. Fix D and a as in the statement of the claim. Let $p := \mathrm{profile}_r^{G,S}(a)$ be the distance-r profile of a on S. Let

$$T_p := \{x \in A : \mathrm{profile}_r^{G,S}(x) = p\}.$$

We apply Lemma 39 with parameters $G - S, T_p, r$ and k. The output is either a set $Y_p \subseteq T_a$ of size greater than k which is distance-$2r$ independent in $G - S$, or a set Z_p of size at most k that distance-$2r$ dominates T_p. In case the output is a set Y_p, we let $Z_p := \emptyset$. We define

$$Z := \bigcup_{p \in \mathcal{P}} Z_p.$$

Then $|Z| \leqslant k \cdot |\mathcal{P}| \leqslant k \cdot (r+2)^s$, as required It remains to prove that $a \in N_{2r}^{G-S}(Z)$.

Assume first that $Z_p \neq \emptyset$. Then Z_p is distance-$2r$ dominates T_p in $G - S$, in particular, the vertex a (with profile p) lies in the distance-$2r$ neighborhood of Z_p.

To finish the proof we show that the case $Z_p = \emptyset$ cannot occur. Assume otherwise that there is $Y_p \subseteq T_p$ with $|Y_p| > k$ and which is distance-$2r$ independent, i.e. the distance-r neighborhoods $N_r^{G-S}(y)$, for all $y \in Y_p$, are pairwise disjoint. By assumption, $|D| \leqslant k$, so for at least one $y_0 \in Y_p$, the set $N_r^{G-S}(y_0)$ is disjoint with X. We claim that $\mathrm{dist}_G(y_0, D) > r$. We already know that this fact holds in $G - S$. The only remaining thing to show is the impossibility of connecting y_0 with D by a path of length at most r through S. But this is impossible by the assumption that S does not capture (D, a), because y and a have the same distance-r profiles on S. □

We now use the set Z provided by Claim 1 to generate the family $\mathcal{G}$. Indeed, if for each $z \in Z$ we let w_z be the Splitter's response for Connector's move z in the radius-$3r$ game on G, then we may define

$$G_z := G[N_{3r}^{G-S}(z) \cup S] \qquad \text{and} \qquad S_z := S \cup \{w_z\}.$$

With such definition, G_z and S_z satisfy the assertion of the induction hypothesis. However, for a single $z \in Z$ we will consider multiple sets A_t, as explained next.

Define the index set for the family $\mathcal{G}$ as $T = Z \times \mathcal{P}$. For $z \in Z$ and $p \in \mathcal{P}$, we set

$$A_{z,p} := \{a \in A \colon \mathrm{profile}_r^{G,S}(a)(v) + p(v) > r \text{ for all } v \in S\} \cap N_{2r}^{G-S}(z).$$

In other words, we define $A_{z,p}$ by taking $A \cap N_{2r}^{G-S}(z)$, and removing all vertices that are connected by a path of length at most r passing through S to any vertex with profile p on S. Note that in this definition we use distance $2r$ instead of $3r$. Now, for each $t = (z, p) \in T$, we put

$$(G_t, S_t, A_t) = (G_z, S_z, A_{z,p}).$$

This defines the family $\mathcal{G} = \{(G_t, S_t, A_t)\}_{t \in T}$.

Now, for each $t \in T$ apply the induction hypothesis to the triple G_t, S_t, A_t, yielding a suitably small pre-core Q_t for G_t, A_t and k. We define $Q = \bigcup_{t \in T} Q_t$ and verify that it has all the required properties.

Claim 2. *The set Q is a distance-r dependence pre-core for G, A and k.*

Proof. Let D be a set of size at most k that distance-r dominates A, and let $a \in A$. Since S is contained in Q, it suffices to prove the following: if (D, a) is not captured by S, then there is $t \in T$ such that (D, a) is captured by Q_t. By the construction of Z there exists $z \in Z$ with $\mathrm{dist}_{G-S}(z, a) \leqslant 2r$. Let $p = \mathrm{profile}_r^{G,S}(D)$. Observe that since (D, a) is not captured by S, we have $a \in A_{z,p}$. We claim that (D, a) is captured by Q_t for $t = (z, p)$.

Let $D_t := D \cap V(G_t)$. We verify that D_t distance-r dominates A_t in G_t. For this, take any $b \in A_t$. Since D distance-r dominates A in G, there is some $d \in D$ such that $\mathrm{dist}_G(b, d) \leqslant r$. As $b \in A_t$, in fact we must have $\mathrm{dist}_{G-S}(b, d) \leqslant r$. This, together with the assertion $\mathrm{dist}_{G-S}(z, b) \leqslant 2r$ following from $b \in A_t \subseteq N_{2r}^{G-S}(z)$, entails that $d \in N_{3r}^{G-S}(z)$, which in turns implies that $d \in V(G_t)$ and d distance-r dominates b in G_t. Therefore $d \in D_t$ and, consequently, D_t distance-r dominates every $b \in A_t$ in G_t.

Applying the induction assumption we conclude that Q_t captures (D_t, a) in G_t. This implies that Q captures (D, a) and we are done. □

It remains to bound the size of Q, which we do as follows as follows. Let $c(r, k, d, s)$ be the smallest number for which the statement of the lemma holds. From the proof we obtain the following recursive bound on the function c:

$$c(r, k, \ell, s) \leqslant |Z| \cdot (r+2)^s \cdot c(r, k, \ell - 1, s+1) \leqslant k \cdot (r+2)^{2s} \cdot c(r, k, \ell - 1, s+1);$$
$$c(r, k, 0, s) \leqslant s.$$

This gives an upper bound $c(r, k, \ell, s) \leqslant k^\ell \cdot (r+2)^{2\ell(s+\ell)} \cdot (s+\ell)$, which is indeed polynomial in k for fixed r, ℓ, s.

Furthermore, the provided proof is not only constructive, but effectively computable by a recursive algorithm. In the following, by when speaking about linear time we mean time of the form $f(k, \ell, r, s) \cdot \|G\|$ for a function f that is polynomial in k for fixed ℓ, r, s.

- Computing Z using Lemma 39 can be done in linear time.
- Computing Splitter's response w_z to each possible Connector's move $z \in Z$ can be done in linear time. Computing the resulting graphs G_z and sets S_z can be done in linear time.
- For each resulting pair G_z, S_z, the sets $A_{z,p}$ for $p \in \mathcal{P}$ can be computed in linear time, by applying breadth-first search from each vertex of S.
- Finally, for each resulting triple (G_t, S_t, A_t) we make a recursive subcall.

Hence, we have a recursion of depth at most ℓ with a branching of order $|T|$, which is bounded by a function that is polynomial in k. Hence, in total, we have a running time $f(k, \ell, r, s) \cdot \|G\|$ for a function f which is polynomial in k. □

As argued, Lemma 38 implies Lemma 36, which implies Theorem 34, which implies Theorem 12, and we are done.

A linear time algorithm for Distance-r Independent Set **in nowhere dense classes.** We now we apply the obtained results to construct an algorithm for Distance-r Independent Set on nowhere dense graph classes via a direct dependence pre-core search. Precisely, we prove the following theorem.

Theorem 40. *Let $\mathscr{C}$ be a nowhere dense class of graphs an let $r \in \mathbb{N}$. Then there exists an algorithm which on input $G \in \mathscr{C}$ and $k \in \mathbb{N}$ decides in time $2^{\mathcal{O}(k \log k)} \cdot \|G\|$ whether G contains a distance-r independent and outputs such a set, or correctly decides that no such set exists.*

Proof. By Lemma 36, in time $\mathrm{poly}(k) \cdot \|G\|$ we can compute a distance-r pre-core Q for G, $V(G)$, and $k-1$ of size $\mathrm{poly}(k)$. By Lemma 35, whether G contains a distance-r independent set is equivalent to whether Q is a distance-r core for G and k.

We can check the latter assertion as follows. Observe that Q is a core as above if the following condition holds for every set $X \subseteq V(G)$ with $|X| = k$: there exist $x_1, x_2 \in X$, $x_1 \neq x_2$, and $y \in Q$ such that $\mathrm{dist}(x_1, y) + \mathrm{dist}(x_2, y) \leqslant r$. This condition depends only on the multiset $\{\{\mathrm{profile}_r^{G,Q}(x) \colon x \in X\}\}$, and given such a multiset of size k it can be decided in time $\mathcal{O}(|Q| \cdot k^2)$ whether the corresponding set X indeed satisfies the condition. Observe that we can construct the multiset $\mathcal{P} = \{\{\mathrm{profile}_r^{G,Q}(u) \colon u \in V(G)\}\}$ in time $\mathcal{O}(|Q| \cdot \|G\|)$ by running BFS from every vertex of Q, reading for every vertex $u \in V(G)$ its distance-r profile on Q, and then counting how many times each profile is realized. Now, it remains to check all submultisets of $\mathcal{P}$ of size k. By Lemma 17, the number of different distance-r profiles on Q that are actually realized in G is bounded polynomially in $|Q|$, so the number of such multisets is at most $|Q|^{\mathcal{O}(k)}$. Since $|Q| = \mathrm{poly}(k)$, we have $2^{\mathcal{O}(k \log k)}$ multisets to check, and checking each of them takes time $\mathcal{O}(|Q| \cdot k^2) = \mathrm{poly}(k)$. Hence, the total running time is $2^{\mathcal{O}(k \log k)} \cdot \|G\|$.

Observe that the above reasoning only gives a decision procedure and does not construct the actual distance-r independent set. Such a set can be constructed as follows. Using the algorithm described above we can find a set X with $|X| = k$ such that $\mathrm{dist}(x_1, y) + \mathrm{dist}(x_2, y) > r$ for all distinct $x_1, x_2 \in X$ and $y \in Q$, or conclude that no such set X exists. In the latter case, by the reasoning above we conclude that there is no distance-r independent of size k in G. Otherwise, we emulate the reasoning presented in the proof of Lemma 35: supposing X is not a distance-r independent set yet, we can find another set X' satisfying the same condition and with $f(X') < f(X)$, where f is defined as in the proof of Lemma 35, and apply the reasoning again replacing X with X'. The number of iterations until the procedure terminates is bounded by k and each iteration can be easily implemented in time $\mathcal{O}(k^2\|G\|)$. Hence, a distance-r independent set can be constructed using $\mathcal{O}(k^3\|G\|)$ additional time. □

8 Omitted proofs

Finally, in this section we present all proofs that we omitted from the main body of the text for the sake of a concise presentation.

Lemma 41. *For all $c, \ell \in \mathbb{N}$ with $c \geqslant 2$, we have $R^c(\ell) \leqslant c^{c\ell-1}$.*

Proof. We first prove the following claim: if G is a complete graph on a set V of c^{p-1} vertices, then there is a sequence $u_1, \ldots, u_p$ of different vertices from V with the following property: for every $i \in \{1, \ldots, p\}$ all edges connecting u_i with vertices appearing earlier in the sequence, i.e., u_j for $j < i$, are of the same color. Note that this color may vary for different vertices u_i.

We proceed by induction on p, where the base case $p = 1$ holds by taking u_i to be any vertex of p. For the induction step, let v be any vertex of V. Since edges between v and $V - \{v\}$ are colored with c colors, there is a color $s \in \{1, \ldots, c\}$ such that v has at least $\left\lceil \frac{c^{p-1}-1}{c} \right\rceil = c^{p-2}$ neighbors adjacent via an edge of

color s. Let H be the complete graph induced by those neighbors in G. Then by the induction hypothesis, in H we can find a suitable sequence of vertices $u_1, \ldots, u_{p-1}$. It now suffices to extend this sequence with $u_p = v$.

We proceed to the main proof. Using the claim, within any complete graph with $c^{c\ell-1}$ vertices and with edges colored with c colors we may find a sequence $u_1, \ldots, u_{c\ell}$ such that for every $i \in \{1, \ldots, c\ell\}$, all edges connecting u_i with vertices u_j for $j < i$ are of the same color, say s_i. Since there are c colors in total, for some color s there are at least ℓ indices $i \in \{1, \ldots, c\ell\}$ with $s_i = s$. Then vertices u_i corresponding to these indices i form a monochromatic clique of color s. □

Proof (of Lemma 2). The left-to-right implication is immediate: every ladder of order n and every co-matching of order n are also semi-ladders of order n, so the semi-ladder index is always an upper bound on both the co-matching and the ladder index.

For the right-to-left implication, we prove that if a bipartite graph $G = (L, R, E)$ has both the ladder and the co-matching index smaller than some $\ell \in \mathbb{N}$, then its semi-ladder index is smaller than $q = R^2(\ell)$. Suppose for contradiction that $a_1, \ldots, a_q \in L$ and $b_1, \ldots, b_q \in R$ form a semi-ladder of order q in G. Color all pairs (i, j) with $1 \leqslant i < j \geqslant q$ red or blue, depending on whether $(a_i, b_j) \in E$ or not. By Ramsey's theorem, there is a subset of $\{1, \ldots, q\}$ of size ℓ which is monochromatic, i.e., which only spans red edges, or only spans blue edges. In the first case, the corresponding vertices a_i and b_i form a co-matching of order ℓ in G, and in the latter case we analogously exhibit a ladder of order ℓ in G. In any case, this is a contradiction. □

Proof (of Lemma 3). Let $G = (L, R, E)$. For the left-to-right implication, suppose $a_1, \ldots, a_q \in L$ and $b_1, \ldots, b_q \in R$ form a co-matching of order q in G, for some $q \in \mathbb{N}$. Then $A = \{a_1, \ldots, a_q\}$ and $B = \{b_1, \ldots, b_q\}$ do not have the q-Helly property, implying that $q \leqslant p$. This means that the co-matching index of G is at most p.

For the right-to-left implication, it is enough to show that if G has co-matching index at most p, then it has the weak p-Helly property. Indeed, from this it follows that in fact G must have the strong p-Helly property, since we can apply the same argument to every induced subgraph of G, which also has co-matching index at most p.

Thus, assume that G has co-matching index at most p and R is not covered (otherwise we are done). Let $B \subseteq R$ any inclusion-minimal set which is not covered by L; we show that $|B| \leqslant p$. By minimality, for every $b \in B$ there is some $a_b \in L$ which is adjacent to all vertices in $B - \{b\}$ and not to b. This means that B together with $A = \{a_b \colon b \in B\}$ form a co-matching of order $|B|$ in G, which implies that $|B| \leqslant p$ by the assumption on the co-matching index of G. □

Proof (of Lemma 4). Let G be a graph with vertex set V and suppose that $\psi(G)$ has semi-ladder index at least $R^k(\ell)$. Then there are tuples $\overline{a}_1, \ldots, \overline{a}_q \in V^{\overline{x}}$ and $\overline{b}_1, \ldots, \overline{b}_q \in V^{\overline{y}}$, where $q = R^k(\ell)$, such that for all $i, j \in \{1, \ldots, q\}$, if $i > j$ then $\psi(\overline{a}_i; \overline{b}_j)$ holds in G, and if $i = j$, then $\psi(\overline{a}_i; \overline{b}_i)$ does not hold in G.

Since ψ is a positive boolean combination of $\varphi_1, \ldots, \varphi_k$, whenever $(\overline{a}, \overline{b}), (\overline{a}', \overline{b}') \in V^{\overline{x}} \times V^{\overline{y}}$ are two pairs of tuples, one satisfying ψ in G and the other satisfying $\neg\psi$ in G, then there must be some $p \in \{1, \ldots, k\}$ such that $(\overline{a}, \overline{b})$ satisfies φ_p and $(\overline{a}', \overline{b}')$ satisfies $\neg\varphi_p$.

Hence, for all $q \geqslant i > j \geqslant 1$, we may color the pair (i, j) with some number $p \in \{1, \ldots, k\}$ such that $\varphi_p(\overline{a}_i; \overline{b}_j)$ holds in G and $\varphi_p(\overline{a}_i; \overline{b}_i)$ does not hold in G.

By Ramsey's theorem, there is a set $X \subseteq \{1, \ldots, q\}$ of size ℓ, and a color $p \in \{1, \ldots, k\}$, such that each pair $(i, j) \in X^2$ with $i > j$ is colored with color p. It follows that $\varphi_p(a_i; b_j)$ holds in G for all $i, j \in X$ with $i > j$, and $\varphi_p(\overline{a}_i; \overline{b}_i)$ does not hold in G for all $i \in X$. This shows that $\varphi_p(G)$ contains a semi-ladder of order ℓ, contrary to the assumption. □

Proof (of Lemma 5). We prove the following statement: if for some graph $G = (V, E)$, in $\psi(G)$ we find a semi-ladder of order $q = k^{\ell-1}$, say formed by $\overline{a}_1, \ldots, \overline{a}_q \in V^{\overline{x}}$ and $\overline{b}_1, \ldots, \overline{b}_q \in V^{\overline{y}}$, then in $\varphi(G)$ we can find a semi-ladder $\overline{c}_1, \ldots, \overline{c}_\ell$ and $\overline{d}_1, \ldots, \overline{d}_\ell$, where $\overline{c}_i$-s are permutations of different tuples from $\{\overline{a}_1, \ldots, \overline{a}_q\}$, and $\overline{d}_i$-s are different tuples from $\{\overline{b}_1, \ldots, \overline{b}_q\}$. For $j \in \{1, \ldots, k\}$ and a tuple $\overline{a} \in V^{\overline{x}}$, by $\overline{a}^j$ we denote the tuple from $V^{\overline{x}^j}$ that is obtained by permuting $\overline{a}$ as in the permutation that maps $\overline{x}$ to $\overline{x}^j$.

We proceed by induction on ℓ. For the base case $\ell = 1$, we can take $\overline{c}_1 = \overline{a}_1^1$ and $\overline{d}_1 = \overline{b}_1$.

We move to the inductive step. Since $\psi(\overline{a}_q; \overline{b}_t)$ holds for all $t < q$, there exists an index $j \in \{1, \ldots, k\}$ such that $\varphi(\overline{a}_q^j; \overline{b}_t)$ holds for $\lceil \frac{q-1}{k} \rceil = k^{\ell-2}$ indices $t \in \{1, \ldots, q-1\}$. Restricting sequences $\overline{a}_1, \ldots, \overline{a}_q$ and $\overline{b}_1, \ldots, \overline{b}_q$ to those indices t, we obtain a semi-ladder in $\psi(G)$ of order $k^{\ell-2}$. By applying the inductive assumption to this semi-ladder, we can find a semi-ladder $\overline{c}_1, \ldots, \overline{c}_{\ell-1}$ and $\overline{d}_1, \ldots, \overline{d}_{\ell-1}$ in $\varphi(G)$, where $\overline{c}_i$-s are permutations of different tuples from $\{\overline{a}_1, \ldots, \overline{a}_q\}$ and $\overline{d}_i$-s are different tuples from $\{\overline{b}_1, \ldots, \overline{b}_q\}$, all of which satisfy $\varphi(\overline{a}_q^j; \overline{d}_i)$. It now remains to extend this semi-ladder by appending $\overline{c}_\ell = \overline{a}_q^j$ and $\overline{d}_\ell = \overline{b}_q$. □

Lemma 42. *The conclusion of Lemma 17 holds also when $\mathscr{C} = \mathscr{D}^s$ for any nowhere dense class $\mathscr{D}$ and fixed $s \in \mathbb{N}$, and when $\mathscr{C}$ is the class of map graphs.*

Proof. Consider first the case when $\mathscr{C} = \mathscr{D}^s$, where $\mathscr{D}$ is nowhere dense. Take any graph from $\mathscr{C}$, say G^s for $G = (V, E) \in \mathscr{D}$. Observe that for any two vertices $u, v \in V$ we have

$$\mathrm{dist}_{G^s}(u, v) = \left\lceil \frac{\mathrm{dist}_G(u, v)}{s} \right\rceil.$$

It follows that for every $r \in \mathbb{N}$, vertex $u \in V$, and vertex subset $S \subseteq V$, $\mathrm{profile}_r^{G^s,S}(u)$ is uniquely determined by $\mathrm{profile}_{rs}^{G,S}(u)$. Consequently,

$$\nu_r^{\mathscr{C}}(m) \leqslant \nu_{rs}^{\mathscr{D}}(m) \qquad \text{for all } m \in \mathbb{N}.$$

The claim now follows from Lemma 17 applied to the class $\mathscr{D}$ and radius parameter rs.

Consider now the case when $\mathscr{C}$ is the class of map graphs. Take any map graph $G = (V, E)$. By Theorem 31, there exists a bipartite planar graph H such that one part of its bipartition is V and G is the subgraph of H^2 induced by V. It follows that for any $u, v \in V$, we have $\mathrm{dist}_G(u, v) = \mathrm{dist}_H(u, v)/2$. Therefore, for any $r \in \mathbb{N}$, $u \in V$, and $S \subseteq V$, $\mathrm{profile}_r^{G,S}(u)$ is uniquely defined by $\mathrm{profile}_{2r}^{H,S}(u)$, implying

$$\nu_r^{\mathscr{C}}(m) \leqslant \nu_{2r}^{\mathscr{P}}(m) \qquad \text{for all } m \in \mathbb{N},$$

where $\mathscr{P}$ is the class of planar graphs. We may conclude as before using Lemma 17, because $\mathscr{P}$ is nowhere dense. □

Lemma 43. *Fix $t \in \mathbb{N}$ and let $\mathscr{C}$ be the class of $K_{t,t}$-free graphs. Then $\nu_1^{\mathscr{C}}(m) \leqslant \mathcal{O}(m^t)$.*

Proof. Let $G = (V, E)$ be a $K_{t,t}$-free graph and let $S \subseteq V$ be any subset of vertices with $|S| \leqslant m$. Observe that for two vertices $u, v \in V - S$, we have $\mathrm{profile}_1^{G,S}(u) = \mathrm{profile}_1^{G,S}(v)$ if and only if $N(u) \cap S = N(v) \cap S$. Since there are at most m vertices in S itself, it suffices to bound the size of the set system $\mathcal{F} = \{N(u) \cap S \colon u \in V - S\}$ by $\mathcal{O}(m^t)$.

To this end, we first note that $\mathcal{F}$ obviously contains at most $\binom{m}{0} + \binom{m}{1} + \ldots + \binom{m}{t-1} = \mathcal{O}(m^{t-1})$ sets of size smaller than t. To bound the number of sets in $\mathcal{F}$ of size at least t, do the following. For every vertex $u \in V - S$ with at least t neighbors in S, pick an arbitrary set A_u consisting of t neighbors of u in S. If any subset $A \subseteq S$ with $|A| = t$ was picked as A_u for at least t vertices $u \in V - S$, then A together with those vertices would form a $K_{t,t}$ subgraph in G, a contradiction. Therefore, by the pigeonhole

principle, the total number of vertices in $V - S$ that have at least t neighbors in S is bounded by $(t-1)m^t$, yielding the same upper bound on the number of sets in $\mathcal{F}$ of size at least t. Consequently, we have that $|\mathcal{F}| \leqslant \mathcal{O}(m^{t-1}) + (t-1)m^t = \mathcal{O}(m^t)$. □

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