From Dynamic Matrix Inverse to Dynamic Shortest Distances

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"Hardness of Approximation the Shortest Vector Problem in Lattices"
Subhash Khot

Find the shortest vector in
\[ \sum_{i=1}^{n} a_i \sim b_i \] where \( \sim \) is in \( \mathbb{R}^m \).

Let \( p > 1 \), it is hard to approximate shortest vector in \( l_p \) norm within an arbitrarily constant factor.
"Hardness of Approximation the Shortest Vector Problem in Lattices"

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FOCS’04 Highlights

Every OWF (resp., PRG) computable in NC$^0$, can be compiled into a corresponding OWF (resp., PRG) in NC$^0$.4
“Cryptography in NC$^0$”

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Every OWF (resp., PRG) computable in NC$^1$, can be compiled into a corresponding OWF (resp., PRG) in NC$^0$. 
"An Approximate Max-Steiner-Tree-Packing Min-Steiner-Cut Theorem"

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First polynomial time constant factor approximation algorithm.
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First polynomial time constant factor approximation algorithm.
Problems

How to dynamically compute the inverse of a matrix?

How to dynamically compute the transitive closure of a graph?

How to dynamically compute the shortest distances in a graph?
Outline

- Dynamic Algebraic Functions
- Dynamic Matrix Inverse for Column Updates
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- Dynamic Matrix Inverse for Column Updates
- Dynamic Matrix Inverse for Element Updates
- Dynamic Transitive Closure
- Dynamic Matrix Inverse over Rings
- Dynamic Shortest Distances
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- **query**$(k)$: return the value of the output $k$. 
Dynamic Matrix Functions

We consider the following problems:

- **determinant**: input $A$, output $\text{det}(A)$,
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Dynamic Matrix Functions

We consider the following problems:
- **determinant**: input $A$, output $\det(A)$,
- **adjoint**: input $A$, output $\text{adj}(A)$, where $\text{adj}(A)_{ij} = (-1)^{i+j} \det(A^{ji})$ and $A^{ji}$ is the $(n-1) \times (n-1)$ matrix obtained from $A$ by deleting $j$'th row and $i$'th column,
Dynamic Matrix Functions

We consider the following problems:
- **determinant**: input $A$, output $\text{det}(A)$,
- **adjoint**: input $A$, output $\text{adj}(A)$,
- **inverse**: input $A$, output $A^{-1}$,
Dynamic Matrix Functions

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- **determinant**: input $A$, output $\det(A)$,
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- **linear system of equations**: input $A$ and $b$, output $A^{-1}b$. 

Lower bounds — $W(n)$ (Frandsen, Hansen and Miltersen STACS'99).
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*Lower bounds — $\Omega(n)$ (Frandsen, Hansen and Miltersen STACS’99).*
Let $\omega(\epsilon)$ be the exponent of multiplication of an $n \times n^\epsilon$ matrix by an $n^\epsilon \times n$ matrix.

For $\epsilon = 1$ we get the square matrix multiplication exponent, known to be $\omega(1) < 2.376$.

For $\epsilon < 0.294$ we have $\omega(\epsilon) = 2$.

Matrix inverse can be computed in the matrix multiplication time.
Dynamic Matrix Inverse

Let us assume that during the updates the matrix remains nonsingular.

We want to change the column $i$’th to $v$. The new matrix is given by:

$$A' = A + (v - (A)_i)e_i^T,$$

where $(A)_i$ is the $i$’th column of $A$, and $e_i$ is the basis vector.
Dynamic Matrix Inverse

We will compute a matrix $B$ such, that:

$$A' = A \cdot B.$$ 

Then matrix $A'^{-1}$ can be computed with:

$$A'^{-1} = B^{-1} A^{-1}.$$
We substitute $B := I + be_i^T$ into $A' = A \cdot B$: 
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Dynamic Matrix Inverse

We substitute $B := I + be_i^T$ into $A' = A \cdot B$:

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\]

\[
(v - (A)_i)e_i^T = A \cdot be_i^T.
\]

\[
v - (A)_i = Ab,
\]

\[
b = A^{-1}(v - (A)_i).
\]
Dynamic Matrix Inverse

The inverse of $B$:

$$B^{-1} = (I + be_i^T)^{-1} = \begin{bmatrix}
1 & \ldots & 0 & -\frac{b_1}{1+b_i} & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 1 & -\frac{b_{i-1}}{1+b_i} & 0 & \ldots & 0 \\
0 & \ldots & 0 & (1+b_i)^{-1} & 0 & \ldots & 0 \\
0 & \ldots & 0 & -\frac{b_{i+1}}{1+b_i} & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & -\frac{b_n}{1+b_i} & 0 & \ldots & 1
\end{bmatrix}.$$ 

$A'^{-1} = B^{-1} A^{-1}$ can be computed in $O(n^2)$ arithmetic operations.
Dynamic Matrix Inverse

Theorem 1 The problem of dynamic matrix inverse, with non-singular column updates, can be solved with the following costs:

- **Initialization** $O(n^\omega)$ arithmetic operations,
- **Update** $O(n^2)$ arithmetic operations (worst-case),
- **Query** $O(1)$ arithmetic operations (worst-case).
Dynamic Matrix Inverse

**Theorem 1** The problem of dynamic matrix inverse with non-singular column updates, can be solved with the following costs:

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- **Query** $O(1)$ arithmetic operations (worst-case).

After an update $O(n^2)$ entries of the inverse can change.
Theorem 2 The problem of dynamic matrix inverse with non-singular element updates can be solved with the following costs:
- initialization $O(n^\omega)$ arithmetic operations,
- update $O(n^{1.575})$ arithmetic operations (worst-case),
- query $O(n^{0.575})$ arithmetic operations (worst-case).
Theorem 3: The problems of dynamic matrix inverse, with non-singular element updates can be solved with the following costs:

- **Initialization**: $O(n^\omega)$ arithmetic operations,
- **Update**: $O(n^{1.495})$ arithmetic operations (worst-case),
- **Query**: $O(n^{1.495})$ arithmetic operations (worst-case).
## Dynamic Transitive Closure

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td><strong>Henzinger and King ’95</strong></td>
<td>$\tilde{O}(nm^{0.58})$</td>
<td>$\Theta(n / \log n)$</td>
</tr>
<tr>
<td><strong>King and Sagert ’99</strong></td>
<td>$O(n^{2.26})$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td><strong>King ’99</strong></td>
<td>$O(n^2 \log n)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td><strong>Demetrescu and Italiano ’00</strong></td>
<td>$O(n^2)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td><strong>Roditty and Zwick ’02</strong></td>
<td>$O(m \sqrt{n})$</td>
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</tr>
<tr>
<td><strong>Roditty and Zwick ’04</strong></td>
<td>$O(m + n \log n)$</td>
<td>$O(n)$</td>
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Let $A$ be a DAG adjacency matrix, then the number of paths from one vertex to another is given by a matrix:

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because $A^n = 0$. 
Let $A$ be a DAG adjacency matrix, then the number of paths from one vertex to another is given by a matrix:

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In order to count paths in dynamic DAG we can maintain the inverse of the matrix $I - A$. 
Dynamic Transitive Closure

Can this approach be used in general case?

$$A^1 = \text{adj}(A) \times \frac{1}{\det(A)}$$
Can this approach be used in general case?

Yes, but instead of matrix inverse we have to use matrix adjoint.

\[ A^{-1} = \frac{\text{adj}(A)}{\det(A)}. \]
Theorem 4 Let $A$ be the adjacency matrix of a graph $G$. Substitute distinct variables for non-zero elements of $A$, than:

$$\text{adj}(I_A)_{ij} \neq 0$$

is non-zero over finite field $\mathbb{Z}_p$. For random substitution a non-zero polynomial has a non-zero value with high probability.
**Theorem 4** Let $A$ be the adjacency matrix of a graph $G$. Substitute distinct variables for non-zero elements of $A$, than:

- There exists a path in $G$ from $i$ to $j$, iff $\text{adj}(I - A)_{ij}$ is non-zero over finite field $\mathbb{Z}_p$.

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Reduction

\[ \det(X) = \]

\[ = \sum_{p \in \Pi_n} \text{sgn}(p) \prod_{i=0}^{n} x_{i,p_i} \]

The permutation \( p \) is a cycle cover of a graph.
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Dynamic Transitive Closure

Theorem 5
Dynamic Matrix Inverse
Update $O(n^\alpha)$ operations
Query $O(n^\beta)$ operations
one may assume non-singularity

Dynamic Transitive Closure
Update $O(n^\alpha)$ time
Query $O(n^\beta)$ time
randomized with one-sided error
Matrix Determinant over Rings

Gaussian elimination cannot be performed without divisions.
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The determinant of a matrix can be computed without divisions in $\tilde{O}(n^{\omega+1})$ ring operations (Strassen ’73).
Strassen’s Idea

The idea is to work in $\mathcal{R}[[u]]$ — the ring of formal power series over $\mathcal{R}$. 

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The idea is to work in $R[[u]]$ — the ring of formal power series over $R$.

Let $A$ be the matrix over the ring $R$, we define

$$A(u) = I + u(A - I)$$
Strassen’s Idea

The idea is to work in \( \mathcal{R}[[u]] \) — the ring of formal power series over \( \mathcal{R} \).

Let \( A \) be the matrix over the ring \( R \), we define

\[
A(u) = I + u(A - I)
\]

We have

\[
A = A(u) \big|_{u=1},
\]

\[
\det(A) = \det(A(u)) \big|_{u=1},
\]

\[
\adj(A) = \adj(A(u)) \big|_{u=1}.
\]
Strassen’s Idea

The elements of the form $1 - z$, where $z \in uR[[u]]$ are invertible:

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots = (1 + z)(1 + z^2)(1 + z^4) \ldots .$$

It is possible to compute this quantity without inverting elements of $R$. 
Strassen’s Idea

Determinant of
\[ A(u) = I + u(A - I) \]
can be computed with Gaussian Elimination.

The elements on the diagonal are of the form 1 – z.
Strassen’s Idea

The result of the evaluation det(\( A(u) \)) is a polynomial of degree \( n \) in \( u \), so the computations can be carried modulo \( u^{n+1} \).
Strassen’s Idea

The result of the evaluation $\det(A(u))$ is a polynomial of degree $n$ in $u$, so the computations can be carried modulo $u^{n+1}$.

The elements of $R[[u]]$ can be multiplied with use of $O(n \log(n) \log(\log(n)))$ operations (Strassen ’71).
Strassen’s Idea

The result of the evaluation $\det(A(u))$ is a polynomial of degree $n$ in $u$, so the computations can be carried modulo $u^{n+1}$.

The elements of $R[[u]]$ can be multiplied with use of $O(n \log(n) \log(\log(n)))$ operations (Strassen ’71).

We obtain a complexity of $\tilde{O}(n^{\omega+1})$ for computing the determinant without divisions.
The matrix $A(u)$ is non-singular for all $A$

$$A(u)^{-1} = \frac{1}{I + u(A - i)} = \sum_{i=0}^{\infty} (-u(A - I))^i.$$
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We can dynamically maintain the inverse of $A(u)$. 
Dynamic Matrix Inverse

We want to change the column $i$'th to $v$. The new matrix is given by:

$$A' = A + (v - (A)_i)e_i^T,$$

and

$$A'(u) = I + u \left( A + (v - (A)_i)e_i^T - I \right).$$
We compute matrix $B$ such that

$$A'(u) = A(u) \cdot B.$$
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Let us substitute $B = I + be_i^T$, then:

$$I + u \left( A + (v - (A)_i)e_i^T - I \right) = A(u) \cdot (I + be_i^T).$$
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$$I + u \left( A + (v - (A)_i)e_i^T - I \right) = A(u) \cdot (I + be_i^T).$$

$$A(u) + u(v - (A)_i)e_i^T = A(u) \cdot (I + be_i^T),$$
Dynamic Matrix Inverse over Rings

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Dynamic Matrix Inverse over Rings

\[ A(u) + u(v - (A)_{i})e_{i}^{T} = A(u) \cdot (I + be_{i}^{T}), \]

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A(u) + u(ν - (A)_i)e_i^T = A(u) \cdot (I + be_i^T),

u(ν - (A)_i)e_i^T = A(u) \cdot be_i^T.

u(ν - (A)_i) = A(u) \cdot b.
Dynamic Matrix Inverse over Rings

\[ A(u) + u(v - (A)_i)e_i^T = A(u) \cdot (I + be_i^T), \]

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\[ u(v - (A)_i) = A(u) \cdot b. \]

\[ b = A(u)^{-1}u(v - (A)_i), \]
Dynamic Matrix Inverse over Rings

\[ b = A(u)^{-1}u(\nu - (A)_i), \]

and:

\[ B = I + uA(u)^{-1}(\nu - (A)_i). \]
Dynamic Matrix Inverse over Rings

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We can show that the matrix \( B \) is invertible.

\[ B^{-1} = \sum_{k=0}^{\infty} (-ube_i^T)^k = \]

\[ = \sum_{k=0}^{\infty} (-uA(u)^{-1}(\nu - (A)_i)e_i^T)^k. \]
The vector $b$ can be computed in $O(n^2)$ operations in $\mathcal{R}[[u]]$.

The matrix $B^{-1}$ in $O(n^2)$ operations.

The matrix $B^{-1}$ has only $O(n)$ non-zero entries, so the multiplication $A'^{-1} = B^{-1}A^{-1}$ can be done with $O(n^2)$. 
Theorem 6  The problems of dynamic determinant and matrix adjoint over commutative ring $R$ with row and column updates can be solved with the following costs:

- **initialization** $\tilde{O}(n^{\omega+1})$ ring operations,
- **update** $\tilde{O}(n^3)$ ring operations (worst-case),
- **query** $O(1)$ ring operations (worst-case).
Dynamic Matrix Inverse over Rings

Notice, that \( \det(A'(u)) = (1 + b_i) \det(A(u)) \) and the determinant of \( A(u) \) can be also updated.
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We have \( \text{adj}(A(u))_{ij} = \det(A(u))(A(u)^{-1})_{ij} \).

To answer queries in constant number of operations we recompute the matrix \( \text{adj}(A) = \text{adj}(A(u))|_{u=1} \) after every update.
Dynamic Matrix Inverse over Rings

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This requires \( \tilde{O}(n^3) \) ring operations.
## Shortest Distances

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<thead>
<tr>
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<tbody>
<tr>
<td><strong>Henzinger et al. ’97</strong></td>
<td>$O(n^{\frac{4}{3}} \log(nC))$</td>
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<td>$O(n^{2.5} \sqrt{S \log^3 n})$</td>
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<td><strong>Real Weights</strong></td>
<td></td>
<td></td>
</tr>
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<td>$\tilde{O}(n^2)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td><strong>This work</strong></td>
<td>$O(n^{1.932})$</td>
<td>$O(n^{1.288})$</td>
</tr>
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</table>
Shortest Distances

**Theorem 7** Let $A$ be the adjacency matrix of a graph $G$. Substitute distinct variables for non-zero elements of $A$, then:

The length of the shortest path in $G$ from $i$ to $j$ is equal to the degree of the smallest degree term in $u$ in $\text{adj}(I - A)_{ij}$.

Moreover all non-zero terms in $\text{adj}(I - A)_{ij}$ are also non-zero over finite field $\mathbb{Z}_p$. 
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Reduction

\[ \text{det}(X) = \]

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\[ \text{adj}(I - uA)_{ij} = \]
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Generate random adjacency matrix $\tilde{A}$ from the adjacency matrix $A$ of $G$ by substituting each nonzero entry in $A$ with a random number in the range $1, \ldots, p - 1$. Now maintain dynamically the adjoint of the matrix $I_uB$ over polynomials of degree $n$, answer queries by finding the smallest degree term in $\text{adj}(I_u\tilde{B})_{ij}$. In this way we get an algorithm that supports updates in $\tilde{O}(n^{2.575})$ time and queries in $O(n^{0.575})$ time.
Dynamic Shortest Distances

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In this way we get algorithm that supports updates in $\tilde{O}(n^{2.575})$ time and queries in $O(n^{0.575})$ time.
Theorem 8 There exist an algorithm for dynamically computing shortest distances up to $k$ in unweighted graph supporting updates in $\tilde{O}(kn^{1.575})$ time and queries in $O(kn^{0.575})$ time.

The algorithm is randomized and with small probability may return wrong higher values.
Theorem 9 (Ullman and Yannakakis ’90)

Let $H \subseteq V$ be a set of vertices chosen uniformly at random.
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Theorem 9 (Ullman and Yannakakis ’90)

- Let $H \subseteq V$ be a set of vertices chosen uniformly at random.
- Let $P_n$ be the probability that a given simple path $P$ has a sequence of more than $\frac{cn}{|H|} \log n$ vertices, none of which are from $H$,
- For any $c > 0$, and for sufficiently large $n$, $P_n$ is bounded by $2^{-\alpha c}$ for some positive $\alpha$. 
Short Long Path Decomposition

The distances up to $k$ are computed exactly — matrix $D$. 
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Distance between $i$ and $j$ is given by a distance from $i$ to $H$ and then from $H$ to $j$. 
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A set $H \subseteq V$ of vertices uniformly at random, with $|H| = \frac{cn}{n^\mu} \log n = O(n^{1-\mu})$, for any constant $c > 0$, where $c \log n \leq n^\mu \leq n$. 

- An $n \times n$ matrix $D$ such that $D_{ij}$ is the shortest distance from $i$ to $j$ in $G$ using at most $n^\mu$ edges.

- An $j \times j$ matrix $B$ obtained from $D$ by choosing only columns and rows corresponding to $H$.

- The Kleene closure $B^*$ of matrix $B$, i.e., the distance matrix obtained from $B$. 

Short Long Path Decomposition - Updates
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Computing the matrix $B^*$ takes $- O(n^{3-3\mu})$ time.

For $\mu = 0.5$ we get $O(n^{2.075})$ update time.
Dynamic Shortest Distances

The distance from \( i \) to \( j \) can be computed with:

\[
\min \left\{ D_{ij}, \min_{p,q \in H} \left\{ D_{ip} + B^*_{pq} + D_{qj} \right\} \right\}.
\]
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- we have to query out row and column from \( H \),
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Dynamic Shortest Distances

The distance from $i$ to $j$ can be computed with:

$$\min \left\{ D_{ij}, \min_{p,q \in H} \left\{ D_{ip} + B_{pq}^* + D_{qj} \right\} \right\}.$$

- we have to query out row and column from $H$ — $O(n^\mu n^{0.575} n^{1-\mu})$ time,
- to compute the minimum over $p, q$ — $O(n^{2-2\mu})$ time.
Dynamic Shortest Distances

The distance from i to j can be computed with:

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- to compute the minimum over \( p, q \) — \( O(n^{2-2\mu}) \) time.

For \( \mu = 0.5 \) we obtain \( O(n^{1.575}) \) query time.
Subquadratic Dynamic Distances

We have to query out a large submatrix $B$ of the matrix $D$. 

\[ O\left(n^{w(1,m,e,1,m)}\right) \text{arithmetic operations.} \]

\[ O\left(n^{m/n^{1.575}} + n^{3/3^{m}}\right) \text{update time.} \]
Subquadratic Dynamic Distances

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We can use fast matrix multiplication to do this — $O(n^{\omega(1-\mu,\epsilon,1-\mu)})$ arithmetic operations.
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$$O(n^\mu n^{1.575} + n^\mu n^{\omega(1-\mu, 0.575, 1-\mu)} + n^{3-3\mu})$$
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$$O(n^\mu n^{1.575} + n^\mu n^{\omega(1-\mu, 0.575, 1-\mu)} + n^{3-3\mu})$$

For $\mu = \frac{3-1.575}{4} = 0.357$ we obtain $O(n^{1.932})$ update time.
Because $1 - \mu > 0.575$, we have

\[
O(n^\mu n^{\omega(1 - \mu, 0.575, 1 - \mu)}) = O(n^\mu n^{(1 - \mu)\omega}) = \\
= O(n^{1.885}) = O(n^{1.932})
\]
Because $1 - \mu > 0.575$, we have

$$O(n^\mu n^{\omega(1-\mu,0.575,1-\mu)}) = O(n^{\mu n^{(1-\mu)\omega}}) =$$

$$= O(n^{1.885}) = O(n^{1.932})$$

The query cost is unchanged — $O(n^{1.575})$ time.
Theorem 10 There exists an randomized algorithm for the dynamic shortest distances problem in unweighted graph $G = (V, E)$ supporting edge updates in $O(n^{1.932})$ and queries in $O(n^{1.288})$ time.
Other Applications

The dynamic matrix inverse algorithm can be used also for:
- counting spanning trees in graphs,
- testing for allowed edges,
- computing shortest paths lengths.
Summary

Dynamic algorithms for computing:
- determinant,
- matrix inverse,
- adjoint,
- solution to a linear system of equation,
- transitive closure.