A rigorous setting for the reinitialization of first order level set equations*

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1 Introduction

Evolution of fronts with prescribed velocity. Let $\Gamma_0 \subset \mathbb{R}^n$, be a surface represented as the zero level set of a Lipschitz continuous function u_0, i.e

$$\Gamma_0 = \{x \in \mathbb{R}^n : u_0(x) = 0\}. \tag{1.1}$$

We are interested in the evolution of Γ_0 under a given normal velocity field. It is well known that through the *Level Set Method* the the evolution $\{\Gamma_t\}_{t\geq 0}$ is given as the zero level sets of the solution of the Hamilton-Jacobi equation

$$\begin{cases} w_t = H(x,t,\nabla w) & \text{in } \mathbf{R}^n \times (0,T), \quad (1.2a) \\ w(x,0) = u_0(x) & \text{in } \mathbf{R}^n. \quad (1.2b) \end{cases}$$

namely $\Gamma_t = \{x \in \mathbb{R}^n : w(x,t) = 0\}$. A very useful function for the *Level Set Method* is the distance function. For $\Omega \subset \mathbb{R}^n$ open, the distance function is defined as $\operatorname{dist}(x,\partial\Omega) = \inf_{y\in\partial\Omega}|x-y|$ and it is a viscocity solution of the eikonal equation

$$|\nabla u(x)| = 1, \quad \text{in } \Omega. \tag{1.3}$$

Usually as initial condition u_0 in (1.2b) we chose the *signed distance function* of Γ_0 associated with an orientation or else a function u_0 as in (1.1), which is given by

$$d(x) = d_{u_0}(x) := \begin{cases} \operatorname{dist}(x,\Gamma_0) & \text{if } x \in \{u_0 > 0\}, \\ -\operatorname{dist}(x,\Gamma_0) & \text{if } x \in \{u_0 \leq 0\}, \end{cases} \tag{1.4}$$

similarly we define the signed distance $d(x,t) = d_w(x,t)$ associated with the level sets Γ_t. Note that different functions can give the same signed distance function.

*This talk is based on the paper [2]

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Numerical errors/Reinitialization. As observed in [4, Chapter 7.2], numerical errors may occur when the gradient of the solution of (1.2) becomes too flat or too steep, but as mentioned in the previous paragraph different functions can give the same zero level set. In order to avoid numerical errors we stop the evolution Γ_t before the gradient becomes too steep or too flat and initialize the equation (1.2a) to a new function with better gradient. This process is referred in the literature as the reinitialization algorithm, see [5]. According to [4] the reinitialization algorithm is a "powerful numerical tool". Another application of reinitialization is presented in [3], where the authors introduce an algorithm for calculating the motion of multiple junctions using level set methods; according to the authors the reinitialization is needed in order to get the right results.
In general we cannot expect that solutions of (1.2a) will preserve the distance function. The theory of viscosity solutions provide the perfect framework for studying the well posedeness as well as properties of solutions of (1.2); for example the unique viscosity solution of the problem

$$\begin{cases} u_t = x \cdot |u_x| & \text{in } \mathbb{R} \times (0,T), \\ u(x,0) = 1 - |x| & \text{in } \mathbb{R}, \end{cases} \tag{1.5}$$

is

$$u(x,t) = \begin{cases} 1 - xe^{-t} & \text{for } x \geq 0, \\ 1 + xe^{t} & \text{for } x < 0, \end{cases}$$

where we can see that the gradient of the solution flattens for $x > 0$ and gets steeper for $x < 0$.
Stopping the equation (1.2a) and calculating the distance function from the zero level set is very costly; for this reason in [5] the authors solve a different equation at the stopping time, namely

$$u_t = \frac{u}{\sqrt{\varepsilon_0^2 + u^2}}(1 - |\nabla u|), \tag{1.6}$$

for some fixed $\varepsilon_0 > 0$. The function $\frac{u}{\sqrt{\varepsilon_0^2+u^2}}$ is a smoother version of the sign function. The solution of this equation asymptotically converges to a steady state $|\nabla u| = 1$, which is a characteristic property of the distance function. The purpose of the sign function in (1.6) is to control the gradient. In the region where u is positive, the equation is $u_t = 1 - |\nabla u|$. Thus, the monotonicity of u is prescribed by the order of 1 and $|\nabla u|$. This forces $|\nabla u|$ to be close to 1 as time passes. Also, by the equation (1.6) we get $u_t = 0$ on the zero level, which guarantees that the initial zero level set will not get distorted.
The idea, as in [5], is to solve (1.2a) and (1.6) periodically in time, the first for a period of $k_1 \Delta t$ and the second for $k_2 \Delta t$, where $k_1, k_2, \Delta t > 0$ and one period will be completed at a time step of length $\varepsilon = (k_1 + k_2)\Delta t$. We are thus led to define the following combined Hamiltonian

$$\overline{H}(x,t,\tau,r,p) := \begin{cases} H(x, \frac{t}{1+\frac{k_2}{k_1}}, p) & \text{if } (i-1) < \tau \leq (i-1) + \frac{k_1 \Delta t}{\varepsilon}, \\ \frac{u}{\sqrt{\varepsilon_0^2+u^2}}(1 - |\nabla u|) & \text{if } (i-1) + \frac{k_1 \Delta t}{\varepsilon} < \tau \leq i \end{cases}$$

for $i = 1, ..., \lceil \frac{T}{\varepsilon} \rceil$. Here by $\lceil x \rceil$ we denote the smallest integer which is not smaller than $x \in \mathbf{R}$. The rescaling of the Hamiltonian H in time is required since certain time intervals are reserved for the corrector equation. More precisely, H is solved in time length $k_1 \Delta t \lceil \frac{T}{\varepsilon} \rceil \sim T \frac{k_1}{k_1+k_2} = \frac{T}{1+\frac{k_2}{k_1}}$. One would expect that solving the two equations infinitely often would force the solution of the reinitialization algorithm to converge to the signed distance function to Γ_t; we denote it by d. Therefore we are led to study the limit as $\varepsilon \to 0$ of the solutions of

$$\begin{cases} u_t^\varepsilon = \overline{H}\left(x, t, \frac{t}{\varepsilon}, u^\varepsilon, \nabla u^\varepsilon\right) & \text{in } \mathbf{R}^n \times (0, T), \\ u^\varepsilon(x, 0) = u_0(x) & \text{in } \mathbf{R}^n. \end{cases} \tag{1.7}$$

This is a homogenization problem with the Hamiltonian $\overline{H}$ being 1-periodic and discontinuous in the fast variable $\tau = t/\varepsilon$. Since the limit above is taken for $\Delta t \to 0$ (and consequently $\varepsilon \to 0$), two free parameters still remain, namely k_1 and k_2. In fact, we show that the solutions of (1.7) converge, as $\varepsilon \to 0$ and after rescaling, to the solution u^θ of

$$\begin{cases} u_t^\theta = H(x, t, \nabla u^\theta) + \theta\beta(u^\theta)(1 - |\nabla u^\theta|) & \text{in } \mathbf{R}^n \times (0, T), \\ u^\theta(x, 0) = u_0(x) & \text{in } \mathbf{R}^n. \end{cases} \tag{1.8}$$

Here $\theta = k_2/k_1$ is the ratio of length of the time intervals in which the equations (1.2a) and (1.6) are solved. If we solve the corrector equation (1.6) in a larger interval than the one we solve the original (1.2a), we can expect the convergence to a steady state. For this reason we study the limit as $\theta \to \infty$ of the solutions of (1.8).

2 Hamilton-Jacobi equations

As usual, in order to guarantee well posedeness of (1.2) we assume that $H : \mathbb{R}^n \times [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is continuous, with $\sup_{\substack{(x,t,p) \in \mathbf{R}^n \times [0,T] \times \mathbf{R}^n \\ |p|=1}} |H_1(x, t, p)| < \infty$. Note that the last condition is only used to construct the appropriate barriers for the Perron's existence technique, if these barriers are a priori known, as in example (1.5), this assumption is not needed. Furthermore, instead of the classical uniform continuity conditions we will assume the stronger

$$|H(x, t, p) - H(y, t, p)| \leq L_1(1 + |p|)|x - y|, \;\; |H(x, t, p) - H(x, t, q)| \leq L_2|p - q|$$

for some $L_1, L_2 > 0$, and the geometricity

$$H(x, t, \lambda p) = \lambda H(x, t, p) \;\text{ for all } \lambda > 0.$$

The reason we are assuming Lipschitz continuity of H in the gradient is to guarantee the property of finite propagation of the interface, in this case the speed of the interface will be L_2. As we will see solutions of (1.8) are uniformly continuous with a uniform modulus in space. For this reason we have to precise the convergence of u^θ to the signed distance. The distance function is continuous in space, it is also well known that is continuous from below in time, see [1, Proposition 2.1 (ii)], it is not true however that it is continuous in

general; example (2.1) shows that when there is an *extinction point* the distance function is discontinuous. The assumption of finite propagation implies that there are no *emerging points* which is a key point into proving the continuity of the distance function from below in time, whereas if we assume that there are no extinction points we can get the continuity of the distance function from above.
We can write the signed distance function $d = d_+ - d_-$, where d_+ and d_- stand for the plus and the minus parts of d respectively and are positive functions. In order to simplify the presentation of our results we will assume in what follows that the function d is positive, for this reason we choose initial data u_0 in (1.2b) to be positive, then the solution w of (1.2) is also positive by the comparison principle. This implies that $d(x,t) = \mathrm{dist}(x, \Gamma_t) \geq 0$. In the general case where d can take negative values we split it into plus and minus parts and study the evolution of the two parts separately.

Example 2.1. We study (1.2a) with $H(x,t,p) = |p|$ and initial condition $u_0(x) = \max\{(1 - |x-2|)_+,\ (1-|x+2|)_+\}$. Then the solution of (1.2) is given by the formula

$$w(x,t) = \max_{|x-y|\leq t} u_0(y),$$

which gives after a few calculations

$$w(x,t) = \min\{\max\{(t+1-|x-2|)_+,\ (t+1-|x+2|)_+\}, 1\},$$

and

$$d(x,t) = \begin{cases} \max\{(t+1-|x-2|)_+,\ (t+1-|x+2|)_+\} & \text{if } t \leq 1, \\ (t+3-|x|)_+ & \text{if } t > 1. \end{cases}$$

Then the point $0 \in \Gamma_1$ is an extinction point of the interface, see Figure (1).

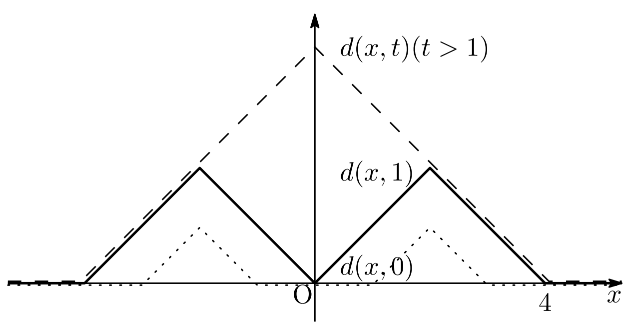

Figure 1: The graph of d.

The preservation of the zero levels by the solution u^θ of (1.8) follows from the following Proposition.

Proposition 2.1 (Barriers)**.** *There exist $\varepsilon > 0, L > 0$, independent of $\theta > 0$, such that*

$$\varepsilon w \leq u^\theta \leq Ld \quad \text{in } \mathbb{R}^n \times (0,T). \tag{2.1}$$

In order to prove this we show that εw and Ld are, respectively, a subsolution and a supersolution of (1.8) and apply the comparison principle.

3 Convergence results

Continuous distance function. To illustrate the idea of the proof, we first present a convergence result to a continuous distance function d, which is much easier to handle than a general (possibly discontinuous) distance function. When d is continuous, it is uniformly approximated by the solutions u^θ of (1.8).

Theorem 3.1. *If d is continuous, then u^θ converges to d locally uniformly in $\mathbb{R}^n \times (0,T)$ as $\theta \to \infty$.*

In the proof we use the comparison principle and the half-relaxed limits of u^θ defined as

$$\overline{u}(x,t) := \limsup_{(y,s,\theta)\to(x,t,\infty)} u^\theta(y,s), \quad \underline{u}(x,t) := \liminf_{(y,s,\theta)\to(x,t,\infty)} u^\theta(y,s).$$

We show that $\overline{u}(\cdot,t)$, $\underline{u}(\cdot,t)$ are, respectively, a subsolution and a supersolution of the eikonal equation (1.3). But in order to apply the comparison principle we need to know the boundary data of the half-relaxed limits. To this end, we use Proposition 2.1 together with the continuity of the distance function and get

$$\{w(\cdot,t)=0\} = \{\overline{u}(\cdot,t)\} = \{\underline{u}(\cdot,t)\}.$$

Then $\overline{u} \le d \le \underline{u}$ from which we get the desired result since we always have $\underline{u} \le \overline{u}$.

General distance function. If the distance function d is discontinuous, we cannot expect that the continuous solutions u^θ of (1.8) will converge to d locally uniformly. Since d is always continuous from below, one can show that $d \le \underline{u}$; it is the inequality $\overline{u} \le d$ that is not always true, due to the fact that d might not be continuous from above. In this case we can generalize the notion of convergence to d; namely we show a uniform convergence to d from below in time.

Theorem 3.2.

$$\lim_{\substack{(y,s,\theta)\to(x,t,\infty)\\ s\le t}} u^\theta(y,s) = d(x,t) \textit{ for all } (x,t)\in\mathbb{R}^n\times(0,T).$$

We call the limit in Theorem 3.2 *uniform limit from below in time* and denote it by $\overline{u}'(x,t)$. This way we always have that $\{\overline{u}'(\cdot,t)=0\} = \{w(\cdot,t)=0\}$ which allows us to compare $\overline{u}'$ and d. We show the following Lemma

Lemma 3.3. *$\overline{u}'$ is a subsolution of (1.3) in $\{w(\cdot,t)\ge 0\}$.*

The key point for the proof of this Lemma is the extension of the viscosity inequality for $\overline{u}'$ up to the terminal time in the intervals $(0,t)$ for $t\in(0,T)$. Theorem 3.2 now follows. By the proof of Lemma 3.3 we also get

- for every $t\in(0,T)$, $u^\theta(\cdot,t)$ converges to $d(\cdot,t)$ locally uniformly in $\mathbb{R}^n$ as $\theta\to\infty$;
- $\underline{u} = d$ in $\mathbb{R}^n\times(0,T)$.

The above can be understood as a locally uniform convergence for every fixed time and a convergence in the sense of lower half-relaxed limit.

Another equation. Finally, we study the convergence of solutions of the following equation

$$u_t^\theta(x,t) = H(x,t,\nabla u^\theta(x,t)) + \theta\beta(u^\theta(x,t))(1-|\nabla u^\theta(x,t)|)_+ \quad \text{in } \mathbb{R}^n\times(0,T). \tag{3.1}$$

As before we define the half-relaxed limits $\overline{u}, \underline{u}$, then we have $\underline{u} = \sup_{\theta>0} u^\theta$ and $\overline{u} = \inf_{\theta>0} u^\theta$. In the general case where we consider not necessarily positive initial data we define

$$\tilde{u} := \begin{cases} \sup_{\theta>0} u^\theta & \text{in } \{w>0\}, \\ \inf_{\theta>0} u^\theta & \text{for } \{w\le 0\}. \end{cases} \tag{3.2}$$

Although in this case we cannot show convergence to the signed distance function, the following Theorem holds.

Theorem 3.4. *Let $\tilde{u}$ be as in (3.2), then $\tilde{u}(\cdot,t)$ is Lipschitz continuous in $\mathbb{R}^n$ for $t\in(0,T)$ with $|\nabla u(\cdot,t)|\ge 1$ a.e, and $\{\tilde{u}(\cdot,t)=0\}=\{w(\cdot,t)=0\}$.*

References

[1] L. C. Evans, H. M. Soner, and P. E. Souganidis. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.*, 45(9):1097–1123, 1992.

[2] N. Hamamuki, E. Ntovoris. A rigorous setting for the reinitialization of first order level set equations. *preprint*, arXiv:1507.00217, *Hokkaido University Preprint Series in Mathematics* # 1074 (http://eprints3.math.sci.hokudai.ac.jp/2372/).

[3] B. Merriman, J. K. Bence, and S. J. Osher. Motion of multiple Junctions: a level set approach. *J. Comput. Phys.*, 112(2):334–363, 1994.

[4] S. Osher and R. Fedkiw. *Level set methods and dynamic implicit surfaces*, volume 153 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2003.

[5] M. Sussman, P. Smereka, and S. Osher. A level set approach for computing solutions to incompressible two-phase flow. *Journal of Computational Physics*, 114(1):146 – 159, 1994.