On Weak Tail Domination of Random Vectors

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Abstract

Motivated by a question of Krzysztof Oleszkiewicz we study a notion of weak tail domination of random vectors. We show that if the dominating random variable is sufficiently regular then weak tail domination implies strong tail domination. In particular positive answer to Oleszkiewicz question would follow from the so-called Bernoulli conjecture. We also prove that any unconditional logarithmically concave distribution is strongly dominated by a product symmetric exponential measure.

1. Introduction. This note is inspired by the following problem about Rademacher series, posed by Krzysztof Oleszkiewicz (private comunication):

Problem. *Suppose that (ε_i) is a Rademacher sequence (i.e. a sequence of independent symmetric ± 1 r.v.'s) and x_i, y_i are vectors in some Banach space F such that the series $\sum_i x_i\varepsilon_i$ and $\sum_i y_i\varepsilon_i$ are a.s. convergent and*

$$\forall_{x^*\in F^*}\forall_{t>0}\ \mathbf{P}\Big(\Big|x^*\Big(\sum_i x_i\varepsilon_i\Big)\Big| \geq t\Big) \leq \mathbf{P}\Big(\Big|x^*\Big(\sum_i y_i\varepsilon_i\Big)\Big| \geq t\Big).$$

Does it imply that

$$\mathbf{E}\Big\|\sum_i x_i\varepsilon_i\Big\| \leq L\mathbf{E}\Big\|\sum_i y_i\varepsilon_i\Big\|,$$

for some universal constant $L < \infty$?

Motivated by the above question we introduce a notion of weak tail domination of random vectors. We prove that if the dominating vector has a regular distribution (including the Gaussian case) then weak tail domination yields strong tail domination (Theorem 1). In particular, Oleszkiewicz's question has positive answer provided that the so-called Bernoulli conjecture holds. We also show that in general weak tail domination does not yield

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comparison of means or medians of norms even if the distribution of the dominated vector is Gaussian.

In the last part we use Theorem 1 to extend the result of Bobkov and Nazarov [3] and show that unconditional logaritmically concave distributions are strongly dominated by a product symmetric exponential measure.

2. Weak and strong tail domination. Let us begin with the following definition.

Definition 1. Let X and Y be random vectors with values in some Banach space F. We say that *tails of Y are weakly dominated by tails of X* and write $Y \prec_\omega X$ if

$$\mathbf{P}(|x^*(Y)| \geq t) \leq \mathbf{P}(|x^*(X)| \geq t) \quad \text{for all } x^* \in F^*, t > 0.$$

The following regularity property of random vectors will give us a tool to go from weak to strong comparison. Its roots are in the paper [8], where a modification of the majorizing measure idea was introduced for processes controlled by a family of distances.

Definition 2. We say that a random vector X with values in F is K*-regular* for some $K < \infty$ if there exists a sequence $(x_n^*) \subset F^*$ such that

$$\|x_n^*(X)\|_{\log(n+2)} = (\mathbf{E}|x_n^*(X)|^{\log(n+2)})^{1/\log(n+2)} \leq K\mathbf{E}\|X\| \text{ for } n = 1, 2, \ldots.$$

and

$$B_{F^*} = \{x^* \in F^* \colon \|x^*\| \leq 1\} \subset \mathrm{cl}_X(\mathrm{conv}\{\pm x_n^* \colon n \geq 1\}),$$

where for $A \subset F^*$, $\mathrm{cl}_X(A)$ denotes the closure of A with respect to the L^2 distance $d_X(x^*, y^*) := (\mathbf{E}|x^*(X) - y^*(X)|^2)^{1/2}$.

Proposition 1. *If X is K-regular and $Y \prec_\omega X$, then*

$$\mathbf{E}\|Y\| \leq 20K\mathbf{E}\|X\|.$$

Proof. Let x_n^* be as in Definition 2. We have for any $t > 0$,

$$\begin{aligned}\mathbf{P}\Big(\sup_{n\geq 1} |x_n^*(Y)| \geq t\Big) &\leq \sum_{n\geq 1} \mathbf{P}(|x_n^*(Y)| \geq t) \leq \sum_{n\geq 1} t^{-\log(n+2)}\mathbf{E}|x_n^*(Y)|^{\log(n+2)} \\ &\leq \sum_{n\geq 1} t^{-\log(n+2)}\mathbf{E}|x_n^*(X)|^{\log(n+2)} \\ &\leq \sum_{n\geq 1} \Big(\frac{K\mathbf{E}\|X\|}{t}\Big)^{\log(n+2)}.\end{aligned}$$

Notice that $d_Y(x^*, y^*) \leq d_X(x^*, y^*)$, hence B_F^* is contained also in the closure of absolute convex hull of $\pm x_n^*$ with respect to d_Y metric and thus

$$\mathbf{E}\|Y\| \leq \mathbf{E}\sup_{n\geq 1}|x_n^*(Y)| \leq K\mathbf{E}\|X\|\Big(e^2 + \int_{e^2}^{\infty}\mathbf{P}(\sup_{n\geq 1}|x_n^*(Y)| \geq tK\mathbf{E}\|X\|)dt\Big)$$
$$\leq K\mathbf{E}\|X\|\Big(e^2 + \sum_{n=1}^{\infty}\int_{e^2}^{\infty} t^{-\log(n+2)}dt\Big) \leq 20K\mathbf{E}\|X\|.$$

□

Theorem 1. *Let $X_1, X_2, \ldots$ be independent copies of a symmetric random vector X. Suppose that there exist constants $K < \infty$ and $\alpha, \beta > 0$ such that for all $n = 1, 2, \ldots$*
(i) the random vector $(X_1, \ldots, X_n)$ with values in $l_\infty^n(F)$ is K-regular,
(ii) $\mathbf{P}(\max_{i\leq n}\|X_i\| \geq \alpha\mathbf{E}\max_{i\leq n}\|X_i\|) \geq \beta$.
Then for any random vector Y such that $Y \prec_\omega X$ we have for all $t > 0$

$$\mathbf{P}(\|Y\| \geq t) \leq \frac{2}{\beta}\mathbf{P}\Big(\|X\| \geq \frac{\alpha t}{80K}\Big).$$

The main idea of how to derive comparison of tails from comparison of means is not new - it goes back at least to the paper of Asmar and Montgomery-Smith [1].

Proof. We may obviously assume that Y is symmetric; we will denote by $Y_1, Y_2, \ldots$ independent copies of Y. Let $n \geq 2$ be such that

$$\frac{2}{n} \geq \mathbf{P}(\|Y\| \geq t) \geq \frac{1}{n}.$$

Then $\mathbf{P}(\max_{i\leq n}\|Y_i\| \geq t) \geq 1 - (1 - 1/n)^n \geq 1/2$, hence $\mathbf{E}\max_{i\leq n}\|Y_i\| \geq t/2$. Let η be r.v. independent of (Y_i) such that $\mathbf{P}(\eta = 1) = \mathbf{P}(\eta = 0) = 1/2$. Then by Theorem 3.2.1 of [5], $\eta(Y_1, \ldots, Y_n) \prec_\omega (X_1, \ldots, X_n)$, where both sides are considered as random vectors in $l_\infty^n(F)$. By Proposition 1,

$$\frac{t}{4} \leq \mathbf{E}\max_{i\leq n}\|\eta Y_i\| = \mathbf{E}\|\eta(Y_1, \ldots, Y_n)\|_{l_\infty^n(F)} \leq 20K\mathbf{E}\|(X_1, \ldots, X_n)\|_{l_\infty^n(F)}$$
$$= 20K\mathbf{E}\max_{i\leq n}\|X_i\|.$$

Property (ii) yields

$$\beta \leq \mathbf{P}\Big(\max_{i\leq n}\|X_i\| \geq \frac{\alpha t}{80K}\Big) \leq n\mathbf{P}\Big(\|X\| \geq \frac{\alpha t}{80K}\Big),$$

so $\mathbf{P}(\|X\| \geq \alpha t/(80K)) \geq \beta/n \geq \beta\mathbf{P}(\|Y\| \geq t)/2$. □

Remark 1. The comparison of first and second moments of maxima,

$$\mathbf{E}\max_{i\leq n}\|X_i\|^2 \leq C(\mathbf{E}\max_{i\leq n}\|X_i\|)^2 \tag{1}$$

implies by the Paley-Zygmund inequality (cf. [5, Lemma 0.2.1]) property (ii) of the previous theorem with $\alpha = 1/2$ and $\beta = 1/(4C)$.

Remark 2. Proofs of both Proposition 1 and Theorem 1 show that both statements hold if we replace the condition $Y \prec_\omega X$ by the condition

$$\|x^*(Y)\|_p \leq \|x^*(X)\|_p \quad \text{for all } x^* \in F^*, p \geq 1. \tag{2}$$

Moreover, if $\|x^*(X)\|_{2p} \leq C\|x^*(X)\|_p$ for all $p \geq 1$ and $x^* \in F^*$ it is enough to check (2) for p being even integer (constants $20K$ and $80K$ would change into $20CK$ and $80CK$).

Let us give a few examples of random vectors satisfying the assumptions of Theorem 1.

Example 1. Any centered Gaussian vector on a separable Banach space is L-regular with universal L. This is a consequence of majorizing measure theorem (cf. [7] and [9, Theorem 2.1.8]). Since a product of Gaussian measures is again Gaussian, property (i) holds with $K = L$. Moments of Gaussian vectors are comparible so by Remark 1 also property (ii) holds with $\alpha = 1/2$ and universal β.

Example 2. Let (η_i) be a sequence of independent symmetric real r.v.'s with logarithmically concave tails satisfying the Δ_2 condition and let $v_i \in F$ be such that $X = \sum_i v_i\eta_i$ is a.s. convergent. Then X is K-regular with a constant K depending only on the Δ_2 constant ([6, Theorem 3]). The random variable $(X_1, \ldots, X_n)$ has an analogous series representation in $l_\infty^n(F)$, so property (i) holds. It can also be checked that (1) is satisfied with a universal C.

Remark 3. Positive answer to the Bernoulli conjecture ([9, Chapter 4]) would imply the L-regularity of Rademacher series. Since (1) holds for X being a Rademacher sum with vector coefficients, Theorem 1 would give positive answer to Oleszkiewicz's question.

We conclude this section with an example showing that weak tail domination does not yield any comparison of strong parameters even if the dominated vector has Gaussian distribution.

Example 3. Let $F = l_\infty^n$, $Y = \sum_{i=1}^n g_i e_i$ and $X = 9(|g_1|+1)\sum_{i=1}^n \eta_i e_i$, where g_i are i.i.d. $\mathcal{N}(0,1)$ and η_i are i.i.d. r.v.'s with uniform distribution on $[-1,1]$, independent of g_1.

To show that tails of Y are weakly dominated by tails of X it is enough to check that

$$\mathbf{P}(|\langle u, Y\rangle| \geq t) \leq \mathbf{P}(|\langle u, X\rangle| \geq t) \quad \text{for } u \in S^{n-1},\ t \geq 0. \tag{3}$$

Let us fix $u \in S^{n-1}$. For $t > 0$ we have

$$\mathbf{P}(|\langle u, Y\rangle| \geq t) = \mathbf{P}(|g_1| \geq t).$$

By the Paley-Zygmund inequality,

$$\begin{aligned}\mathbf{P}\Big(\Big|\sum_{i=1}^n u_i\eta_i\Big| \geq \frac{1}{3}\Big) &= \mathbf{P}\Big(\Big|\sum_{i=1}^n u_i\eta_i\Big|^2 \geq \frac{1}{3}\mathbf{E}\Big|\sum_{i=1}^n u_i\eta_i\Big|^2\Big) \\ &\geq \big(1-\frac{1}{3}\big)^2\frac{(\mathbf{E}|\sum_{i=1}^n u_i\eta_i|^2)^2}{\mathbf{E}|\sum_{i=1}^n u_i\eta_i|^4} \geq \frac{4}{27},\end{aligned}$$

thus

$$\mathbf{P}(|\langle u, X\rangle| \geq t) \geq \frac{4}{27}\mathbf{P}(3(|g_1|+1) \geq t) \geq \frac{4}{27}\mathbf{P}\Big(|g_1| \geq \frac{t}{3}\Big).$$

Using the easy estimate $2t\exp(-(2t)^2/2)/\sqrt{2\pi} \leq \mathbf{P}(|g| \geq t) \leq \exp(-t^2/2)$, we immediately get (3) for $t \geq 3$. For $0 \leq t \leq 3$ we have

$$\begin{aligned}\mathbf{P}(|\langle u, X\rangle| \leq t) &\leq \mathbf{P}\Big(9\Big|\sum_{i=1}^n u_i\eta_i\Big| \leq t\Big) \leq \frac{\sqrt{2}t}{9} \leq t\frac{\mathbf{P}(|g_1| \leq 3)}{3} \\ &\leq \mathbf{P}(|g_1| \leq t) = \mathbf{P}(|\langle u, Y\rangle| \leq t),\end{aligned}$$

where to get the second inequality we used Ball's upper bound on cube sections [2]. Hence (3) holds also for $t \in [0,3]$.

Thus $Y \prec_\omega X$. However $\mathbf{E}\|Y\| = \mathbf{E}\max_{i\leq n}|g_i| \geq \sqrt{\log(n+1)}/L$ and $\mathbf{E}\|X\| \leq 9\mathbf{E}(|g_1|+1) \leq 18$.

3. Unconditional logarithmically concave distributions. A Borel probability measure μ on $\mathbb{R}^n$ is called *logaritmically concave* if for any nonempty compact sets A, B and $\lambda \in [0,1]$, $\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}$. If the support of μ has full dimension, μ is logconcave if and only if μ has the density of the form e^{-g}, where $g\colon \mathbb{R}^n \to (-\infty, \infty]$ is convex [4]. A random vector $X = (X_1, \ldots, X_n)$ is *unconditional* if for any sequence of signs $s_i = \pm 1$, $(s_1X_1, \ldots, s_nX_n)$ has the same distribution as X.

Theorem 2. *Suppose that $Y = (Y_1, \ldots, Y_n)$ is an unconditional vector with logarithmically concave distribution such that $\mathbf{E}Y_i^2 = 1$, and $X = (X_1, \ldots, X_n)$, where the X_i are independent, symmetric, exponential random variables with variance 1 (i.e. the density $2^{-1/2}\exp(-\sqrt{2}|x|)$). Then for any norm on $\mathbb{R}^n$,*

$$\mathbf{P}(\|Y\| \geq t) \leq L\mathbf{P}(\|X\| \geq t/L) \quad \text{for } t > 0,$$

where L is a universal constant.

Proof. By Example 2 the vector X satisfies the assumption of Theorem 1. Moreover $\|\sum_i a_i X_i\|_{2p} \leq C\|\sum_i a_i X_i\|_p$ for $p \geq 1$ and a universal C. Thus by Remark 2 it is enough to show that for any $a_1, \ldots, a_n$ and any positive even integer k,

$$\Big(\mathbf{E}\Big|\sum_{i=1}^n a_i Y_i\Big|^k\Big)^{1/k} \leq L_1\Big(\mathbf{E}\Big|\sum_{i=1}^n a_i X_i\Big|^k\Big)^{1/k}. \tag{4}$$

The result of Bobkov and Nazarov [3, Proposition 3.3] gives that for $t_1, \ldots, t_n \geq 0$,

$$\mathbf{P}(|Y_1| \geq L_1 t_1, \ldots, |Y_n| \geq L_1 t_n) \leq \mathbf{P}(|X_1| \geq t_1, \ldots, |X_n| \geq t_n),$$

thus after integration by parts we get

$$\mathbf{E}(|Y_1|^{2l_1} \cdots |Y_n|^{2l_n}) \leq L_1^{2(l_1+\ldots+l_n)}\mathbf{E}(|X_1|^{2l_1} \cdots |X_n|^{2l_n})$$

and (4) immediately follows. $\square$

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References

[1] N. Asmar and S. Montgomery-Smith, *On the distribution of Sidon series*, Ark. Mat. 31 (1993), 13–26.

[2] K. Ball, *Cube slicing in R^n*, Proc. Amer. Math. Soc. 97 (1986), 465–473.

[3] S.G. Bobkov and F.L. Nazarov, *On convex bodies and log-concave probability measures with unconditional basis*,in: Lecture Notes in Math. 1807, Springer, Berlin, 2003, 53–69.

[4] C. Borell, *Convex set functions in d-space*, Period. Math. Hungar. 6 (1975), 111–136.

[5] S. Kwapień and W. Woyczyński, *Random series and stochastic integrals: single and multiple*, Birkhäuser, Boston, 1992.

[6] R. Latała, *Sudakov minoration principle and supremum of some processes*, Geom. Funct. Anal. 7 (1997), 936–953.

[7] M. Talagrand, *Regularity of Gaussian processes*, Acta Math. 159 (1987), 99–149.

[8] M.Talagrand, *The supremum of some canonical processes,* Amer. J. Math. 116 (1994), 284-325.

[9] M. Talagrand *The generic chaining. Upper and lower bounds of stochastic processes*, Springer, Berlin, 2005.

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