Moments of unconditional logarithmically concave vectors *

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Abstract

We derive two-sided bounds for moments of linear combinations of coordinates od unconditional log-concave vectors. We also investigate how well moments of such combinations may be approximated by moments of Gaussian random variables.

1 Introduction

The aim of this paper is to study moments of linear combinations of coordinates of unconditional, log-concave vectors $X = (X_1, \ldots, X_n)$. A nondegerate random vector X is *log-concave* if it has a density of the form $g = e^{-h}$, where $h\colon \mathbb{R} \to (-\infty, \infty]$ is a convex function. We say that a random vector X is *unconditional* if the distribution of $(\eta_1 X_1, \ldots, \eta_n X_n)$ is the same as X for any choice of signs $\eta_1, \ldots, \eta_n$.

Typical example of uncoditional log-concave vector is a vector distributed uniformly in an unconditional convex body K, i.e. such convex body that $(\pm x_1, \ldots, \pm x_n) \in K$ whenever $(x_1, \ldots, x_n) \in K$.

A random vector X is called *isotropic* if it has identity covariance matrix, i.e. $\mathrm{Cov}(X_i, X_j) = \delta_{i,j}$. Notice that uncoditional vector X is isotropic if and only if its coordinates have variance one, in particular if X is unconditional with nondenerate coordinates then the vector $(X_1/\mathrm{Var}^{1/2}(X_1), \ldots, X_n/\mathrm{Var}^{1/2}(X_n))$ is isotropic and unconditional.

In [3] Gluskin and Kwapień derived two-sided estimates for moments of $\sum_{i=1}^n a_i X_i$ if X_i are independent, symmetric random variables with log-concave tails (coordinates of log-concave vector have log-concave tails). In Section 2 we derive similar result for arbitrary unconditional log-concave vectors X.

In [6] Klartag obtained powerful Berry-Essen type estimates for isotropic, unconditional, log-concave vectors X, showing in particular that if $\sum_i a_i^2 = 1$ and all a_i's are small then the distribution of $S = \sum_{i=1}^n a_i X_i$ is close to the

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standard Gaussian distribution $\mathcal{N}(0,1)$. In Section 3 we investigate how well moments of S may be approximated by moments of $\mathcal{N}(0,1)$.

Notation. By $\varepsilon_1, \varepsilon_2, \ldots$ we denote a Bernoulli sequence, i.e. a sequence of independent symmetric variables taking values ± 1. We assume that the sequence (ε_i) is independent of other random variables.

For a random variable Y and $p > 0$ we write $\|Y\|_p = (\mathbb{E}|Y|^p)^{1/p}$. For a sequence (a_i) and $1 \le q < \infty$, $\|a\|_q = (\sum_i |a_i|^q)^{1/p}$ and $\|a\|_\infty = \max_i |a_i|$. We set $B_q^n = \{a \in \mathbb{R}^n : \|a\|_q \le 1\}$, $1 \le q \le \infty$. By $(a_i^*)_{1\le i\le n}$ we denote the nonincreasing rearrangement of $(|a_i|)_{1\le i\le n}$.

We use letter C (resp. $C(\alpha)$) for universal constants (resp. constants depending only on parameter α). Value of a constant C may differ at each occurence. Whenever we want to fix the value of an absolute constant we will use letters $C_1, C_2, \ldots$. For two functions f and g we write $f \sim g$ to signify that $\frac{1}{C} f \le g \le Cf$.

2 Estimation of moments

It is well known and easy to show that if X has a uniform distribution over a symmetric convex body K in $\mathbb{R}^n$ then for any $p \ge n$, $\|\sum_{i\le n} a_i X_i\|_p \sim \|a\|_{K^o} = \sup\{|\sum_{i\le n} a_i x_i| : x \in K\}$. Our first proposition generalizes this statement to arbitrary log-concave symmetric distributions.

Proposition 1. *Suppose that X has a symmetric n-dimensional log-concave distribution with the density g. Then for any $p \ge n$ we have*

$$\Big\| \sum_{i=1}^n a_i X_i \Big\|_p \sim \|a\|_{K_p^\circ},$$

where

$$K_p := \{x : g(x) \ge e^{-p} g(0)\} \quad \textit{and} \quad \|a\|_{K_p^\circ} = \sup\Big\{ \sum_{i=1}^n a_i x_i : x \in K_p \Big\}.$$

Proof. First notice that there exists an absolute constant C_1 such that

$$\mathbb{P}(X \in C_1 K_p) \ge 1 - e^{-p} \ge \frac{1}{2}.$$

For $n \le p \le 2n$ this follows by Corollary 2.4 and Lemma 2.2 in [7]. For $p \ge 2n$ we may either adjust arguments from [7] or take any log-concave symmetric $m = \lfloor p \rfloor - n$ dimensional vector Y independent of X with density g' and consider the set $K' = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : g(x)g'(y) \ge e^{-p} g(0) g'(0)\}$. Then K_p is a central n-dimensional section of K', hence $\mathbb{P}(X \in C_1 K_p) \ge \mathbb{P}((X,Y) \in C_1 \tilde{K}') \ge 1 - e^{-p}$.

Observe that for any $z \in K_p$,

$$\Big|\Big\{ x \in K_p : \Big| \sum_{i=1}^n a_i x_i \Big| \ge \frac{1}{2} \sum_{i=1}^n a_i z_i \Big\}\Big| \ge 2^{-n} |K_p| \ge (2C_1)^{-n} \mathbb{P}(X \in C_1 K_p)/g(0),$$

therefore choosing z such that $\sum_{i=1}^n a_i z_i = \|a\|_{K_p^o}$ we get

$$\Big\|\sum_{i=1}^n a_i X_i\Big\|_p \geq 2^{-1/p}\|a\|_{K_p^o} e^{-1} g(0)^{1/p}\Big|\Big\{x\in K_p\colon \Big|\sum_{i=1}^n a_i x_i\Big| \geq \frac{1}{2}\sum_{i=1}^n a_i z_i\Big\}\Big|^{1/p}$$
$$\geq 2^{-1/p}\|a\|_{K_p^o} e^{-1}(2C_1)^{-n/p}\mathbb{P}(X\in C_1K_p)^{1/p} \geq \frac{1}{4eC_1}\|a\|_{K_p^o}.$$

To get the upper estimate notice that

$$\mathbb{P}\Big(\Big|\sum_{i=1}^n a_i X_i\Big| > C_1\|a\|_{K_p^o}\Big) \leq \mathbb{P}(X\notin C_1K_p) \leq e^{-p}.$$

Together with the symmetry and log-concavity of $\sum_{i=1}^n a_i X_i$ this gives

$$\mathbb{P}\Big(\Big|\sum_{i=1}^n a_i X_i\Big| > C_1 t\|a\|_{K_p^o}\Big) \leq e^{-tp} \text{ for } t\geq 1.$$

Integration by parts yields $\Big\|\sum_{i=1}^n a_i X_i\Big\|_p \leq C\|a\|_{K_p^o}$. □

Remark. The same argument as above shows that if $\alpha \geq e$ then

$$\Big\|\sum_{i=1}^n a_i X_i\Big\|_p \geq \frac{1}{4\alpha C_1}\sup\Big\{\sum_{i=1}^n a_i x_i\colon g(x)\geq \alpha^{-p}g(0)\Big\}.$$

From now on till the end of this section we assume that vector X is unconditional, log-concave and isotropic. Jensen's inequality and Hitczenko estimates for moments of Rademacher sums [5] (see also [12]) imply that for $p\geq 2$,

$$\Big\|\sum_i a_i X_i\Big\|_p = \Big\|\sum_i a_i\varepsilon_i|X_i|\Big\|_p \geq \Big\|\sum_i a_i\varepsilon_i\mathbb{E}|X_i|\Big\|_p$$
$$\geq \frac{1}{C}\Big(\sum_{i\leq p} a_i^* + \sqrt{p}\Big(\sum_{i>p}|a_i^*|^2\Big)^{1/2}\Big). \tag{1}$$

The result of Bobkov and Nazarov [2] yields for $p\geq 2$,

$$\Big\|\sum_i a_i X_i\Big\|_p \leq C\Big\|\sum_i a_i E_i\Big\|_p \leq C\Big(p\max_i|a_i| + \sqrt{p}\Big(\sum_i a_i^2\Big)^{1/2}\Big), \tag{2}$$

where (E_i) is a sequence of independent symmetric exponential random variables with variance 1 and to get the second inequality we used the result of Gluskin and Kwapień [3].

Estimates (1) and (2) together with Proposition 1 give

$$\frac{1}{C}(\sqrt{p}B_2^n\cap B_\infty^n) \subset \Big\{x\colon g(x)\geq e^{-p}g(0)\Big\} \subset C(\sqrt{p}B_2^n + pB_1^n) \quad \text{ for } p\geq n. \tag{3}$$

Corollary 2. *Let $X = (X_1, \ldots, X_n)$ be an unconditional log-concave isotropic vector with the density g. Then for any $p \geq n$ we have*

$$\Big\| \sum_{i=1}^n a_i X_i \Big\|_p \sim \sup\Big\{ \sum_{i=1}^n a_i x_i \colon g(x) \geq e^{-p} g(0) \Big\}$$
$$\sim \sup\Big\{ \sum_{i=1}^n a_i x_i \colon g(x) \geq e^{-5p/2} \Big\}$$
$$\sim \sup\Big\{ \sum_{i=1}^n |a_i| t_i \colon \mathbb{P}(|X_1| \geq t_1, \ldots, |X_n| \geq t_n) \geq e^{-p} \Big\}.$$

Proof. We have $g(0) = L_X^n$, where L_X is the isotropic constant of vector X. Unconditionality of X implies boundedness of L_X, thus

$$e^{-3n/2} \leq (2\pi e)^{-n/2} \leq g(0) \leq C_2^n,$$

where C_2 is an absolute constant (see for example [2]). Hence

$$\{x \colon g(x) \geq e^{-p} g(0)\} \subset \{x \colon g(x) \geq e^{-5p/2}\} \subset \{x \colon g(x) \geq (e^{5/2} C_2)^{-p} g(0)\} \quad (4)$$

and first two estimates on moments follows by Proposition 1 (see also remark after it).

For any $t_1, \ldots, t_n \geq 0$,

$$\mathbb{E}\Big| \sum_{i=1}^n a_i X_i \Big|^p \geq \Big(\sum_{i=1}^n |a_i| t_i \Big)^p 2^{-n} \mathbb{P}(|X_i| \geq t_1, \ldots, |X_n| \geq t_n),$$

therefore

$$\Big\| \sum_{i=1}^n a_i X_i \Big\|_p \geq \frac{1}{2e} \sup\Big\{ \sum_{i=1}^n a_i t_i \colon \mathbb{P}(|X_1| \geq t_1, \ldots, |X_n| \geq t_n) \geq e^{-p} \Big\}.$$

To prove the opposite estimate we use already proven bound and take x such that $g(x) \geq e^{-5p/2}$ and $\sum_{i=1}^n a_i x_i \geq \frac{1}{C_2} \| \sum_{i=1}^n a_i X_i \|_p$. By the unconditionality without loss of generality we may assume that all a_i's and x_i's are nonnegative. Notice that by (3) and (4) we have $g(1/C_3, \ldots, 1/C_3) \geq e^{-5p/2}$. Hence by log-concavity of g we also have $g(y) \geq e^{-5p/2}$ for $y_i = (x_i + 1/C_3)/2$. Notice that g is coordinate increasing on $\mathbb{R}_+^n$, therefore

$$\mathbb{P}\Big(X_1 \geq \frac{y_1}{2}, \ldots, X_n \geq \frac{y_n}{2} \Big) \geq g(y) \prod_{i=1}^n \frac{y_i}{2} \geq e^{-5p/2} (4C_3)^{-n} \geq (4e^{5/2} C_3)^{-p}.$$

Function $F(s_1, \ldots, s_n) = -\ln \mathbb{P}(X_1 \geq s_1, \ldots, X_n \geq s_n)$ is convex on $\mathbb{R}_+^n$, $F(0) = n \ln 2$, therefore

$$\mathbb{P}\Big(|X_1| \geq \frac{y_1}{C_4}, \ldots, |X_n| \geq \frac{y_n}{C_4} \Big) = 2^n \mathbb{P}\Big(X_1 \geq \frac{y_1}{C_4}, \ldots, X_n \geq \frac{y_n}{C_4} \Big) \geq e^{-p}$$

for suffiently large C_4. To conclude it is enough to notice that

$$\sum_{i=1}^{n} a_i \frac{y_i}{C_4} \geq \frac{1}{2C_4} \sum_{i=1}^{n} a_i x_i \geq \frac{1}{2C_2 C_4} \Big\| \sum_{i=1}^{n} a_i X_i \Big\|_p .$$

□

Theorem 3. *Suppose that X is an unconditional log-concave isotropic random vector in $\mathbb{R}^n$. Then for any $p \geq 2$,*

$$\begin{aligned}
\Big\| \sum_{i=1}^{n} a_i X_i \Big\|_p &\sim \sup \Big\{ \sum_{i \in I_p} a_i x_i \colon\ g_{I_p}(x) \geq e^{-p} g_{I_p}(0) \Big\} + \sqrt{p} \Big(\sum_{i \notin I_p} a_i^2 \Big)^{1/2}, \\
&\sim \sup \Big\{ \sum_{i \in I_p} a_i x_i \colon\ g_{I_p}(x) \geq e^{-5p/2} \Big\} + \sqrt{p} \Big(\sum_{i \notin I_p} a_i^2 \Big)^{1/2} \\
&\sim \sup \Big\{ \sum_{i \in I_p} |a_i| t_i \colon\ \mathbb{P}\Big(\forall_{i \in I_p}\ |X_i| \geq t_i \Big) \geq e^{-p} \Big\} + \sqrt{p} \Big(\sum_{i \notin I_p} a_i^2 \Big)^{1/2},
\end{aligned}$$

where g_{I_p} is the density of $(X_i)_{i \in I_p}$ and I_p is the set of indices of $\min\{\lceil p \rceil, n\}$ *largest values of $|a_i|$'s.*

Proof. By Corollary 2 it is enough to show that

$$\begin{aligned}
\frac{1}{C} \Big(\Big\| \sum_{i \in I_p} a_i X_i \Big\|_p + \sqrt{p} \Big(\sum_{i \notin I_p} a_i^2 \Big)^{1/2} \Big) &\leq \Big\| \sum_{i=1}^{n} a_i X_i \Big\|_p \\
&\leq C \Big(\Big\| \sum_{i \in I_p} a_i X_i \Big\|_p + \sqrt{p} \Big(\sum_{i \notin I_p} a_i^2 \Big)^{1/2} \Big). \quad (5)
\end{aligned}$$

Observe also that $\sum_{i \notin I_p} a_i^2 = \sum_{i > p} |a_i^*|^2$.

Unconditionality of X_i implies that $\| \sum_{i=1}^{n} a_i X_i \|_p \geq \| \sum_{i \in I_p} a_i X_i \|_p$. Hence the lower estimate in (5) follows by (1).

Obviously we have

$$\Big\| \sum_{i=1}^{n} a_i X_i \Big\|_p \leq \Big\| \sum_{i \in I_p} a_i X_i \Big\|_p + \Big\| \sum_{i \notin I_p} a_i X_i \Big\|_p .$$

Estimate (1) and (2) imply

$$\begin{aligned}
\Big\| \sum_{i \notin I_p} a_i X_i \Big\|_p &\leq C \Big(p \max_{i \notin I_p} |a_i| + \sqrt{p} \Big(\sum_{i \notin I_p} a_i^2 \Big)^{1/2} \Big) \\
&\leq C \Big(\Big\| \sum_{i \in I_p} a_i X_i \Big\|_p + \sqrt{p} \Big(\sum_{i \notin I_p} a_i^2 \Big)^{1/2} \Big)
\end{aligned}$$

and upper bound in (5) follows. □

Example 1. Let X_i be independent symmetric log-concave r.v's. Define $N_i(t) := -\mathbb{P}(|X_i| \geq t)$, then $\mathbb{P}(X_i \geq t_i \text{ for } i \in I_p) = \exp(-\sum_{i \in I_p} N_i(t_i))$ and Theorem 3 yields the Gluskin-Kwapień estimate

$$\Big\| \sum_{i=1}^n a_i X_i \Big\|_p \sim \sup\Big\{ \sum_{i \in I_p} |a_i| t_i \colon \sum_{i \in I_p} N_i(t_i) \leq p \Big\} + \sqrt{p} \Big(\sum_{i \notin I_p} a_i^2 \Big)^{1/2}.$$

Example 2. Let X be uniformly distriputed on $r_{n,q} B_q^n$ with $1 \leq q < \infty$, where $r_{n,q}$ is chosen in such a way that X is isotropic. Then it is easy to check that $r_{n,q} \sim n^{1/q}$. Since all k-dimensional sections of B_q^n are homogenous we immediately obtain that for $I \subset \{1, \ldots, n\}$ and $x \in \mathbb{R}^I$, $g_I(x)/g_I(0) = (1 - (\|x\|_q / r_{n,q})^q)^{n-|I|}$. Hence for $1 \leq p \leq n/2$ we get that

$$\sup\Big\{ \sum_{i \in I_p} a_i x_i \colon g_{I_p}(x) \geq e^{-p} g_{I_p}(0) \Big\} \sim \sup\Big\{ \sum_{i \in I_p} a_i x_i \colon \|x\|_q \leq p \Big\}.$$

Since for $p \geq n/2$, $\| \sum_{i=1}^n a_i X_i \Big\|_p \sim \| \sum_{i=1}^n a_i X_i \Big\|_{n/2}$, we recover the result from [1] and show that for $p \geq 2$,

$$\Big\| \sum_{i=1}^n a_i X_i \Big\|_p \sim \min\{p, n\}^{1/q} \Big(\sum_{i \leq p} |a_i^*|^{q'} \Big)^{1/q'} + \sqrt{p} \Big(\sum_{i > p} |a_i^*|^2 \Big)^{1/2},$$

where $1/q' + 1/q = 1$.

Remark. In the case of vector coefficients the following conjecture seems reasonable. There exists a universal constant C such that for any isotropic unconditional log-concave vector $X = (X_1, \ldots, X_n)$ any vectors $v_1, \ldots, v_n$ in a normed space $(F, \| \cdot \|)$ and $p \geq 1$,

$$\Big(\mathbb{E} \Big\| \sum_{i=1}^n v_i X_i \Big\|^p \Big)^{1/p} \sim \Big(\mathbb{E} \Big\| \sum_{i=1}^n v_i X_i \Big\| + \sup_{\|\varphi\|_* \leq 1} \Big(\mathbb{E} \Big| \sum_{i=1}^n \varphi(v_i) X_i \Big|^p \Big)^{1/p} \Big).$$

The nontrivial part is the upper bound for $(\mathbb{E} \| \sum_{i=1}^n v_i X_i \|^p)^{1/p}$. It is known that the above conjecture holds if the space $(F, \| \cdot \|)$ has nontrivial cotype – see [9] for this and some related results.

Remark. Let $S = \sum_{i=1}^n a_i X_i$, where X is as in Theorem 3. Then $\mathbb{P}(|S| \geq e\|S\|_p) \leq e^{-p}$ by the Chebyshev's inequality. Moreover $\|S\|_{2p} \leq C \|S\|_p$ for $p \geq 2$, hence Paley-Zygmund inequality yields $\mathbb{P}(|S| \geq \|S\|_p / C) \geq \min\{1/C, e^{-p}\}$. This way Theorem 3 may be also used to get two-sided estimates for tails of S.

3 Gaussian approximation of moments

Let $\gamma_p = \|\mathcal{N}(0,1)\|_p = 2^{p/2} \Gamma(\frac{p+1}{2}) / \sqrt{\pi}$. In [8] it was shown that for independent symmetric random variables $X_1, \ldots, X_n$ with log-concave tails (notice that log-concave symmetric random variables have log-concave tails) and variance 1,

$$\Big| \Big\| \sum_{i=1}^n a_i X_i \Big\|_p - \gamma_p \|a\|_2 \Big| \le p\|a\|_\infty \quad \text{for } a \in \mathbb{R}^n,\ p \ge 3 \tag{6}$$

(see also [11] for $p \in [2,3)$). The purpose of this section is to discuss similar statements for general log-concave isotropic vectors X.

The lower estimate of moments is easy. In fact it holds for more general class of unconditional vectors with bounded fourth moment.

Proposition 4. *Suppose that X is an isotropic unconditional n-dimensional vector with finite fourth moment. Then for any nonzero $a \in \mathbb{R}^n$ and $p \ge 2$,*

$$\begin{aligned} \Big\| \sum_{i=1}^n a_i X_i \Big\|_p &\ge \gamma_p \|a\|_2 - \frac{p}{\sqrt{2}\|a\|_2} \Big(\sum_{i=1}^n a_i^4 \mathbb{E} X_i^4 \Big)^{1/2} \\ &\ge \gamma_p \|a\|_2 - \frac{p}{\sqrt{2}} \max_i (\mathbb{E} X_i^4)^{1/2} \|a\|_\infty. \end{aligned}$$

Proof. Let us fix $p \ge 2$. By the homogenity we may and will assume that $\|a\|_2 = 1$.

Corollary 1 in [8] gives

$$\Big\| \sum_{i=1}^n b_i \varepsilon_i \Big\|_p \ge \gamma_p \Big(\sum_{i \ge \lceil p/2 \rceil} |b_i^*|^2 \Big)^{1/2} \quad \text{for } b \in \mathbb{R}^n,$$

where (b_i^*) denotes the nonicreasing rearrangement of $(|b_i|)_{i \le n}$. Therefore

$$\begin{aligned} \Big\| \sum_{i=1}^n a_i X_i \Big\|_p^p &= \mathbb{E} \Big| \sum_{i=1}^n a_i \varepsilon_i X_i \Big|^p \ge \gamma_p^p \mathbb{E} \Big(\sum_{i=1}^n a_i^2 X_i^2 - \max_{\#I < p/2} \sum_{i \in I} a_i^2 X_i^2 \Big)^{p/2} \\ &\ge \gamma_p^p \Big(\mathbb{E} \Big(\sum_{i=1}^n a_i^2 X_i^2 - \max_{\#I < p/2} \sum_{i \in I} a_i^2 X_i^2 \Big) \Big)^{p/2} = \gamma_p^p \Big(1 - \mathbb{E} \max_{\#I < p/2} \sum_{i \in I} a_i^2 X_i^2 \Big)^{p/2}. \end{aligned}$$

We have

$$\begin{aligned} \mathbb{E} \max_{\#I < p/2} \sum_{i \in I} a_i^2 X_i^2 &\le \mathbb{E} \max_{\#I < p/2} \sqrt{\#I} \Big(\sum_{i \in I} a_i^4 X_i^4 \Big)^{1/2} \le \sqrt{\frac{p}{2}} \mathbb{E} \Big(\sum_{i=1}^n a_i^4 X_i^4 \Big)^{1/2} \\ &\le \sqrt{\frac{p}{2}} \Big(\sum_{i=1}^n a_i^4 \mathbb{E} X_i^4 \Big)^{1/2}. \end{aligned}$$

Since $\sqrt{1-x} \ge 1-x$ for $x \ge 0$ and $\gamma_p \le \sqrt{p}$ the assertion easily follows. □

Since $\mathbb{E} Y^4 \le 6$ for symmetric log-concave random variables Y we immediately get the following.

Corollary 5. *Let X be an isotropic unconditional n-dimensional log-concave vector. Then for any $a \in \mathbb{R}^n \setminus \{0\}$ and $p \geq 2$,*

$$\Big\| \sum_{i=1}^n a_i X_i \Big\|_p \geq \gamma_p \|a\|_2 - \frac{p}{\|a\|_2} \Big(3 \sum_{i=1}^n a_i^4 \Big)^{1/2} \geq \gamma_p \|a\|_2 - \sqrt{3} p \|a\|_\infty .$$

Now we turn our attention to the upper bound. Notice that for unconditional vectors X and $p \geq 2$,

$$\Big\| \sum_{i=1}^n a_i X_i \Big\|_p = \Big\| \sum_{i=1}^n a_i \varepsilon_i X_i \Big\|_p \leq \gamma_p \Big\| \Big(\sum_{i=1}^n a_i^2 X_i^2 \Big)^{1/2} \Big\|_p, \tag{7}$$

where the last inequality follows by the Khintchine inequality with optimal constant [4]. First we will bound moments of $(\sum_{i=1}^n a_i^2 X_i^2)^{1/2}$ using the result of Klartag [6].

Proposition 6. *For any isotropic unconditional n-dimensional log-concave vector X, $p \geq 2$ and nonzero $a \in \mathbb{R}^n$ we have*

$$\Big\| \sum_{i=1}^n a_i X_i \Big\|_p - \gamma_p \|a\|_2 \leq C p^{5/2} \frac{1}{\|a\|_2} \Big(\sum_{i=1}^n |a_i|^4 \Big)^{1/2} \leq C p^{5/2} \|a\|_\infty .$$

Proof. By homogenity we may assume that $\|a\|_2 = 1$. We have

$$\Big\| \Big(\sum_{i=1}^n a_i^2 X_i^2 \Big)^{1/2} \Big\|_p \leq 1 + \Big\| \Big(\Big(\sum_{i=1}^n a_i^2 X_i^2 \Big)^{1/2} - 1 \Big)_+ \|_p .$$

Notice that

$$\sum_{i=1}^n a_i^2 (X_i^2 - 1) = \Big(\Big(\sum_{i=1}^n a_i^2 X_i^2 \Big)^{1/2} - 1 \Big) \Big(\Big(\sum_{i=1}^n a_i^2 X_i^2 \Big)^{1/2} + 1 \Big),$$

thus

$$\Big\| \Big(\Big(\sum_{i=1}^n a_i^2 X_i^2 \Big)^{1/2} - 1 \Big)_+ \|_p \leq \Big\| \sum_{i=1}^n a_i^2 (X_i^2 - 1) \Big\|_p .$$

Lemma 4 in [6] gives

$$\Big\| \sum_{i=1}^n a_i^2 (X_i^2 - 1) \Big\|_2^2 = \mathrm{Var} \Big(\sum_{i=1}^n a_i^2 X_i^2 \Big) \leq \frac{8}{3} \sum_{i=1}^n a_i^4 \mathbb{E} X_i^4 \leq 16 \sum_{i=1}^n a_i^4 .$$

Comparison of moments of polynomials with respect to log-concave distributions [13] implies

$$\Big\| \sum_{i=1}^n a_i^2 (X_i^2 - 1) \Big\|_p \leq (Cp)^2 \Big\| \sum_{i=1}^n a_i^2 (X_i^2 - 1) \Big\|_2 \leq C p^2 \Big(\sum_{i=1}^n a_i^4 \Big)^{1/2} .$$

□

We may improve $p^{5/2}$ term if we assume some concentration properties of vector X. We say that vector X satisfies *exponential concentration* with constant κ if

$$\mathbb{P}(X \in A) \geq \frac{1}{2} \ \Rightarrow\ \mathbb{P}(X \in A + tB_2^n) \geq 1 - e^{-t/\kappa}.$$

Proposition 7. *Let X be an isotropic unconditional vector that satisfies exponential concentration with constant κ. Then for any $p \geq 2$ and $a \in \mathbb{R}^n$,*

$$\Big\| \sum_{i=1}^n a_i X_i \Big\|_p \leq \gamma_p \|a\|_2 + C\kappa p^{3/2} \|a\|_\infty.$$

Proof. Notice that

$$\sup\Big\{ \Big(\sum_{i=1}^n a_i^2 y_i^2 \Big)^{1/2} : y \in tB_2^n \Big\} = t\|a\|_\infty.$$

Using standard arguments we may therefore show that exponential concentration implies for $p \geq 2$,

$$\Big\| \Big(\sum_{i=1}^n a_i^2 X_i^2 \Big)^{1/2} \Big\|_p \leq \Big\| \Big(\sum_{i=1}^n a_i^2 X_i^2 \Big)^{1/2} \Big\|_2 + C\kappa p \|a\|_\infty = \|a\|_2 + C\kappa p \|a\|_\infty.$$

We conclude using (7). □

Since by the result of Klartag [6] unconditional log-concave vectors satisfy exponential concentration with constant $C \log n$ we get

Corollary 8. *Let X be isotropic unconditional logconcave vector. Then for any $p \geq 2$ and $a \in \mathbb{R}^n$,*

$$\Big\| \sum_{i=1}^n a_i X_i \Big\|_p \leq \gamma_p \|a\|_2 + C p^{3/2} \log n \|a\|_\infty.$$

To get the factor p instead of $p^{3/2}$ we need a stronger notion than exponential concentration. We say that a random vector X satisfies *two level concentration with constant* κ if

$$\mathbb{P}(X \in A) \geq \frac{1}{2} \ \Rightarrow\ \mathbb{P}(X \in A + \sqrt{t} B_2^n + tB_1^n) \geq 1 - e^{-t/\kappa}.$$

Since it is enough to consider $t \geq 1$ two level concentration is indeed stronger than exponential concentration.

Proposition 9. *Suppose that X is an isotropic unconditional vector that satisfies two level concentration with constant κ. Then for any $p \geq 2$ and $a \in \mathbb{R}^n$,*

$$\Big\| \sum_{i=1}^n a_i X_i \Big\|_p \leq \gamma_p \|a\|_2 + C\kappa p \|a\|_\infty.$$

Proof. For $p \geq 2$ define a norm $\|\!|\cdot|\!\|_p$ on $\mathbb{R}^n$ by $\|\!|x|\!\|_p = \|\sum_{i=1}^n x_i\varepsilon_i\|_p$. Notice that $\|\!|x|\!\|_p \leq \gamma_p\|x\|_2$, hence

$$\mathbb{E}\|\!|(a_iX_i)|\!\|_p^2 \leq \gamma_p^2\|a\|_2^2.$$

Observe also that

$$\begin{aligned}
&\sup\{\|\!|(a_ix_i)|\!\|_p \colon x \in \sqrt{t}B_2^n + tB_1^n\} \\
&\qquad \leq \sqrt{t}\sup\{\|\!|(a_ix_i)|\!\|_p \colon x \in B_2^n\} + t\sup_{j\leq n}\|\!|(a_i\delta_{i,j})|\!\|_p \\
&\qquad \leq \sqrt{t}\gamma_p\sup\{\|(a_ix_i)\|_2 \colon x \in B_2^n\} + t\|a\|_\infty = (\sqrt{t}\gamma_p + t)\|a\|_\infty.
\end{aligned}$$

Let $M_p = \mathrm{Med}(\|\!|(a_iX_i)|\!\|_p)$, two level concentration (applied twice to sets $A = \{\|\!|(a_iX_i)|\!\|_p \leq M_p\}$ and $A = \{\|\!|(a_iX_i)|\!\|_p \geq M_p\}$) implies that

$$\mathbb{P}\Big(\Big|\|\!|(a_iX_i)|\!\|_p - M_p\Big| \geq (\sqrt{t}\gamma_p + t)\|a\|_\infty\Big) \leq 2\exp(-t/\kappa).$$

Integrating by parts this gives for $p \geq q \geq 2$,

$$\Big\|\|\!|(a_iX_i)|\!\|_p - M_p\Big\|_q \leq C\kappa(\sqrt{q}\gamma_p + q)\|a\|_\infty \leq C\kappa p\|a\|_\infty.$$

Hence

$$\Big\|\sum_{i=1}^n a_iX_i\Big\|_p = \|\|\!|(a_iX_i)|\!\|_p\|_p \leq \|\|\!|(a_iX_i)|\!\|_p\|_2 + C\kappa p\|a\|_\infty \leq \gamma_p\|a\|_2 + C\kappa p\|a\|_\infty.$$

□

Unfortunately we do not know many examples of random vectors satisfying two level concentration with a good constant. Using estimate (2) it is not hard to see that infimum convolution inequality investigated in [10] implies two level concentration. In particular isotropic log-concave unconditional vectors with independent coordinates and isotropic vectors uniformly distributed on the (suitably rescaled) B_p^n balls satisfy two level concentration with an absolute constant.

The last approach to the problem of Gaussian approximation of moments we will discuss is based on the notion of negative association. We say that random variables $(Y_1, \ldots, Y_n)$ are *negatively associated* if for any disjoint sets I_1, I_2 in $\{1, \ldots, n\}$ and any bounded functions $f_i \colon \mathbb{R}^{I_i} \to \mathbb{R}$, $i = 1, 2$ that are coordinate nondecreasing we have

$$\mathrm{Cov}\Big(f_1((X_i)_{i\in I_1}), f_2((X_i)_{i\in I_2})\Big) \leq 0.$$

Our next result is an unconditional version of Theorem 1 in [16].

Theorem 10. *Suppose that $X = (X_1, \ldots, X_n)$ is an unconditional random vector with finite second moment and random variables $(|X_i|)_{i=1}^n$ are negatively* *associated. Let $X_1^*,\dots,X_n^*$ be independent random variables such that X_i^* has the same distribution as X_i. Then for any nonnegative function f on $\mathbb{R}$ such that f'' is convex and any $a_1,\dots,a_n$ we have*

$$\mathbb{E}f\Big(\sum_{i=1}^n a_iX_i\Big) \le \mathbb{E}f\Big(\sum_{i=1}^n a_iX_i^*\Big). \tag{8}$$

In particular

$$\mathbb{E}\Big|\sum_{i=1}^n a_iX_i\Big|^p \le \mathbb{E}\Big|\sum_{i=1}^n a_iX_i^*\Big|^p \qquad \text{for } p\ge 3.$$

Proof. Since random variables $|a_iX_i|$ are also negatively associated, it is enough to consider the case when $a_i=1$ for all i. We may also assume that variables X_i^* are independent of X. Assume first that random variables X_i are bounded.

Let $Y=(Y_1,\dots,Y_n)$ be independent copy of X and $2\le k\le n$. To shorten the notation put for $1\le l\le n$, $S_l=\sum_{i=1}^l \varepsilon_i|X_i|$ and $\tilde{S}_l=\sum_{i=1}^l\varepsilon_i|Y_i|$ (recall that ε_i denotes a Bernoulli sequence independent of other variables).

We have

$$\begin{aligned} f(S_k)&+f(\tilde{S}_k)-f(S_{k-1}+\varepsilon_k|Y_k|)-f(\tilde{S}_{k-1}+\varepsilon_k|X_k|)\\ &=\int_{|Y_k|}^{|X_k|}\varepsilon_k(f'(S_{k-1}+\varepsilon_kt)-f'(\tilde{S}_{k-1}+\varepsilon_kt))dt\\ &=\int_{-\infty}^{\infty}\varepsilon_k(f'(S_{k-1}+\varepsilon_kt)-f'(\tilde{S}_{k-1}+\varepsilon_kt))(\mathbf{I}_{\{|X_k|\ge t\}}-\mathbf{I}_{\{|Y_k|\ge t\}})dt. \end{aligned} \tag{9}$$

Define for $t>0$, $g_t(x)=\mathbb{E}\varepsilon_kf'(x+\varepsilon_kt)=(f'(x+t)-f'(x-t))/2$ and

$$h_t(|x_1|,\dots,|x_{k-1}|)=\mathbb{E}_\varepsilon\varepsilon_kf'\Big(\sum_{i=1}^{k-1}\varepsilon_i|x_i|+\varepsilon_kt\Big)=\mathbb{E}g_t\Big(\sum_{i=1}^{k-1}\varepsilon_i|x_i|\Big).$$

Taking the expectation in (9) and using the unconditionality we get

$$\begin{aligned} 2\Big(\mathbb{E}f\Big(\sum_{i=1}^k X_i\Big)&-\mathbb{E}f\Big(\sum_{i=1}^{k-1}X_i+X_k^*\Big)\Big)\\ &=\mathbb{E}\int_{-\infty}^{\infty}\varepsilon_k(f'(S_{k-1}+\varepsilon_kt)-f'(\tilde{S}_{k-1}+\varepsilon_kt))(\mathbf{I}_{\{|X_k|\ge t\}}-\mathbf{I}_{\{|Y_k|\ge t\}})dt\\ &=\int_{-\infty}^{\infty}\mathbb{E}\big[\big(h_t(|X_1|,\dots,|X_k|)-h_t(|Y_1|,\dots,|Y_k|)\big)\big(\mathbf{I}_{\{|X_n|\ge t\}}-\mathbf{I}_{\{|Y_n|\ge t\}}\big)\big]dt\\ &=\int_{-\infty}^{\infty}\mathrm{Cov}\big(h_t(|X_1|,\dots,|X_{k-1}|),\mathbf{I}_{\{|X_k|\ge t\}}\big)dt. \end{aligned}$$

Convexity of f'' implies that the function g_t is convex on $\mathbb{R}$, therefore the function h_t is coordinate increasing on $\mathbb{R}_+^{k-1}$. So by the negative association we get

$$\mathbb{E}f\Big(\sum_{i=1}^k X_i\Big)\le\mathbb{E}f\Big(\sum_{i=1}^{k-1}X_i+X_k^*\Big) \tag{10}$$

The same inequality holds if we change the function f into the function $f(\cdot+h)$ for any $h \in \mathbb{R}$. Therefore applying (10) conditionally we get

$$\mathbb{E}f\Big(\sum_{i=1}^{k} X_i + \sum_{i=k+1}^{n} X_i^*\Big) \leq \mathbb{E}f\Big(\sum_{i=1}^{k-1} X_i + \sum_{i=k}^{n} X_i^*\Big)$$

and inequality (8) easily follows in the bounded case.

To settle the unbounded case first notice that random variables $|X_i| \wedge m$ are bounded and negatively associated for any $m > 0$. Hence we know that

$$\mathbb{E}f\Big(\sum_{i=1}^{n} \varepsilon_i |X_i| \wedge m\Big) \leq \mathbb{E}f\Big(\sum_{i=1}^{n} \varepsilon_i |X_i^*| \wedge m\Big).$$

We have $\liminf_{m\to\infty} f(\sum_{i=1}^{n} \varepsilon_i |X_i| \wedge m) \geq \mathbb{E}f(\sum_{i=1}^{n} \varepsilon_i |X_i|)$, so it is enough to show that , $\liminf_{m\to\infty} \mathbb{E}f(\sum_{i=1}^{n} \varepsilon_i |X_i^*| \wedge m) \leq \mathbb{E}f(\sum_{i=1}^{n} \varepsilon_i |X_i^*|)$.

Let us define $u(x) = f(x) - \frac{1}{2}f''(0)x^2$, function u'' is convex and $u''(0) = 0$. Since $\mathbb{E}|X_i|^2 = \mathbb{E}|X_i^*|^2 < \infty$ it is enough to show that for any $m > 0$,

$$\mathbb{E}u\Big(\sum_{i=1}^{n} \varepsilon_i |X_i^*| \wedge m\Big) \leq \mathbb{E}u\Big(\sum_{i=1}^{n} \varepsilon_i |X_i^*|\Big). \tag{11}$$

Let for $s \in \mathbb{R}$, $v_s(t) := \mathbb{E}u(\varepsilon_1 s + \varepsilon_2 t)$, then $v_s''(t) = \mathbb{E}u''(\varepsilon_1 s + \varepsilon_2 t) \geq u''(\mathbb{E}(\varepsilon_1 s + \varepsilon_2 t)) = 0$ and $v_s'(0) = 0$, hence v_s is nondecreasing on $[0, \infty)$. Thus for any $x \in \mathbb{R}^n$,

$$\mathbb{E}_\varepsilon u\Big(\sum_{i=1}^{n} \varepsilon_i |x_i| \wedge m\Big) \leq \mathbb{E}_\varepsilon u\Big(\sum_{i=1}^{n} \varepsilon_i |x_i|\Big)$$

and (11) immediately follows. □

Corollary 11. *Suppose that X is an isotropic undonditional n-dimensional log-concave vector such that variables $|X_i|$ are negatively associated. Then for any $a_1, \ldots, a_n$ and $p \geq 3$,*

$$-\sqrt{3}p\|a\|_\infty \leq \Big\|\sum_{i=1}^{n} a_i X_i\Big\|_p - \gamma_p \|a\|_2 \leq p\|a\|_\infty.$$

In particular the above inequality holds if X has a uniform distribution on a (suitably rescaled) Orlicz ball.

Proof. First inequality follows by Corollary 5, second by Theorem 10 and (6). The last part of the statement is a consequence of the result of Pilipczuk and Wojtaszczyk [14] (see also [15] for a simpler proof and a slightly more general class of unconditional log-concave measures with negatively associated absolute values of coordinates). □

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