

Between Sobolev and Poincaré *

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Abstract

Let $a \in [0, 1]$ and $r \in [1, 2]$ satisfy relation $r = 2/(2 - a)$. Let $\mu(dx) = c_r^n \exp(-(|x_1|^r + |x_2|^r + \dots + |x_n|^r)) dx_1 dx_2 \dots dx_n$ be a probability measure on the Euclidean space $(R^n, \|\cdot\|)$. We prove that there exists a universal constant C such that for any smooth real function f on R^n and any $p \in [1, 2)$

$$E_\mu f^2 - (E_\mu |f|^p)^{2/p} \leq C(2 - p)^a E_\mu \|\nabla f\|^2.$$

We prove also that if for some probabilistic measure μ on R^n the above inequality is satisfied for any $p \in [1, 2)$ and any smooth f then for any $h : R^n \rightarrow R$ such that $|h(x) - h(y)| \leq \|x - y\|$ there is $E_\mu |h| < \infty$ and

$$\mu(h - E_\mu h > \sqrt{C} \cdot t) \leq e^{-Kt^r}$$

for $t > 1$, where $K > 0$ is some universal constant.

Let us begin with few definitions.

Definition 1 Let (Ω, μ) be a probability space and let f be a measurable, square integrable non-negative function on Ω . For $p \in [1, 2)$ we define the p -variance of f by

$$Var(p)_\mu(f) = \int_\Omega f(x)^2 \mu(dx) - \left(\int_\Omega f(x)^p \mu(dx) \right)^{2/p} = E_\mu f^2 - (E_\mu f^p)^{2/p}.$$

Note that $Var(1)_\mu(f) = D_\mu^2(f) = Var_\mu(f)$ coincides with classical notion of variance, while

$$\lim_{p \rightarrow 2^-} \frac{Var(p)_\mu(f)}{2 - p} = \frac{1}{2} (E_\mu f^2 \ln(f^2) - E_\mu f^2 \cdot \ln(E_\mu f^2)) = \frac{1}{2} Ent_\mu(f^2),$$

where Ent_μ denotes a classical entropy functional (see [L] for a nice introduction to the subject).

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Definition 2 Let \mathcal{E} be a non-negative functional on some class \mathcal{C} of non-negative functions from $L^2(\Omega, \mu)$. We will say that $f \in \mathcal{C}$ satisfies

- the Poincaré inequality with constant C
if $\text{Var}_\mu(f) \leq C \cdot \mathcal{E}(f)$,
- the logarithmic Sobolev inequality with constant C
if $\text{Ent}_\mu(f^2) \leq C \cdot \mathcal{E}(f)$,
- the inequality $I_\mu(a)$ (for $0 \leq a \leq 1$) with constant C
if $\text{Var}(p)_\mu(f) \leq C \cdot (2-p)^a \cdot \mathcal{E}(f)$ for all $p \in [1, 2)$.

Lemma 1 For a fixed $f \in \mathcal{C}$ and $p \in [1, 2)$ let

$$\varphi(p) = \frac{\text{Var}(p)_\mu(f)}{1/p - 1/2}.$$

Then φ is a non-decreasing function.

Proof. Hölder's inequality yields that $\alpha(t) = t \ln(E_\mu f^{1/t})$ is a convex function for $t \in (1/2, 1]$. Hence also $\beta(t) = e^{2\alpha(t)} = (E_\mu f^{1/t})^{2t}$ is convex and therefore $\frac{\beta(t) - \beta(1/2)}{t - 1/2}$ is non-decreasing on $(1/2, 1]$. Observation that

$$\varphi(p) = \frac{\beta(1/2) - \beta(1/p)}{1/p - 1/2}$$

completes the proof. \square

Corollary 1 For $f \in \mathcal{C}$ the following implications hold true:

- f satisfies the Poincaré inequality with constant C
if and only if f satisfies $I_\mu(0)$ with constant C ,
- if f satisfies the logarithmic Sobolev inequality with constant C
then f satisfies $I_\mu(1)$ with constant C ,
- if f satisfies $I_\mu(1)$ with constant C
then f satisfies the logarithmic Sobolev inequality with constant $2C$,
- if f satisfies $I_\mu(a)$ with constant C and $0 \leq \alpha \leq a \leq 1$
then f satisfies $I_\mu(\alpha)$ with constant C .

Proof.

- To prove the first part of Corollary 1 it suffices to note that $p \mapsto \text{Var}(p)_\mu(f)$ is a non-increasing function.

- The second part of Corollary 1 follows easily from the fact that

$$\lim_{p \rightarrow 2^-} \frac{\text{Var}(p)_\mu(f)}{2-p} = \frac{1}{2} \cdot \text{Ent}_\mu(f^2).$$

- To prove the third part of Corollary 1 use Lemma 1 and note that for $p \in [1, 2)$ we have

$$\frac{\text{Var}(p)_\mu(f)}{2-p} = \frac{\varphi(p)}{2p} \leq \frac{\lim_{p \rightarrow 2^-} \varphi(p)}{2} = \text{Ent}_\mu(f^2).$$

- The last part of statement is trivial. \square

Corollary 1 shows that inequalities $I_\mu(a)$ interpolate between Poincaré and logarithmic Sobolev inequalities. Note that $I_\mu(a)$ for $a < 0$ would be equivalent to the Poincaré inequality and the only functions satisfying $I_\mu(a)$ for $a > 1$ would be the constant functions (because in this case $I_\mu(a)$ would imply the logarithmic Sobolev inequality with constant 0). Therefore restriction to $a \in [0, 1]$ is natural.

Definition 3 *Given probability space (Ω, μ) , a class $\mathcal{C} \subseteq L^2_+(\Omega, \mu)$ and non-negative functional \mathcal{E} on \mathcal{C} we will say that a pair (μ, \mathcal{E}) satisfies $I(a)$ (respectively the Poincaré or the logarithmic Sobolev) inequality if every $f \in \mathcal{C}$ satisfies $I_\mu(a)$ (resp. the Poincaré or the logarithmic Sobolev) inequality with constant C (for these particular μ and \mathcal{E}). For the sake of brevity we will assume that μ identifies probability space and \mathcal{E} carries information about \mathcal{C} .*

An obvious modification of Corollary 1 for pairs (μ, \mathcal{E}) follows. In some cases we can establish the precise relation between best possible constants in $I(1)$ and logarithmic Sobolev inequalities.

Let $m : (-a, a) \rightarrow \mathbb{R}$ be an even, strictly positive continuous density of some probability measure μ on $(-a, a)$, where $0 < a \leq \infty$ and assume that $\int_{-a}^a x^2 m(x) dx < \infty$. For $f \in C_0^\infty(-a, a)$ put

$$(Lf)(x) = xf'(x) - u(x)f''(x),$$

where $u(x) = \frac{\int_x^a tm(t) dt}{m(x)} \geq 0$. General theory (see [KLO] for detailed references and some related results) yields that L can be extended to a positive definite self-adjoint operator (denoted by the same symbol), defined on a dense subspace $\text{Dom}(L)$ of $L^2((-a, a), \mu)$, whose spectrum $\sigma(L)$ is contained in $\{0\} \cup [1, \infty)$. Moreover $P_t = e^{-tL}$ ($t \geq 0$) is a Markov semigroup with invariant measure μ . Put $\mathcal{E}(f) = \|L^{1/2}f\|_2^2$ (we accept $\mathcal{E}(f) = +\infty$ for f which do not belong to $\text{Dom}(L^{1/2})$) and take $\mathcal{C} = L^2_+((-a, a), \mu)$.

Lemma 2 *Under the above assumptions the following equivalence holds true:*
 (μ, \mathcal{E}) satisfies the inequality $I(1)$ with constant C
if and only if
 (μ, \mathcal{E}) satisfies the logarithmic Sobolev inequality with constant $2C$.

Proof. If (μ, \mathcal{E}) satisfies the inequality $I(1)$ with constant C then by Corollary 1 it satisfies the logarithmic Sobolev inequality with constant $2C$. Now let us assume that (μ, \mathcal{E}) satisfies the logarithmic Sobolev inequality with constant $2C$. Then for any $f \in L^2((-a, a), \mu)$ we have

$$Ent_\mu(f^2) = Ent_\mu(|f|^2) \leq 2C\mathcal{E}(|f|) \leq 2C\mathcal{E}(f)$$

(the last inequality is a well known property of Dirichlet forms of Markov semi-groups - see for example Theorem 1. 3. 2 of [D]). Therefore classical hypercontractivity result [G] yields

$$\|P_{t(p)}f\|_2 \leq \|f\|_p,$$

where $t(p) = \frac{C}{2} \ln(\frac{1}{p-1})$ for $p \in [1, 2)$; if $p = 1$ then we put $t(p) = \infty$ and $P_\infty(f) = E_\mu f$. Hence

$$Efe^{-2t(p)L}f \leq (Ef^p)^{2/p}$$

or equivalently

$$Ef^2 - (Ef^p)^{2/p} \leq Ef(Id - e^{-2t(p)L})f$$

for any $f \in \mathcal{C}$. Now it suffices to prove that for any $\lambda \in \sigma(L)$ we have

$$1 - e^{-2t(p)\lambda} \leq (2-p)C\lambda,$$

i.e.

$$1 - (2-p)C\lambda \leq (p-1)^{C\lambda}.$$

For $\lambda = 0$ and $p \in (1, 2)$ the inequality is trivial. It is known that if (μ, \mathcal{E}) satisfies the logarithmic Sobolev inequality with constant $2C$ then (under the assumptions of Lemma 2) $C \geq 1$ - to see this consider the logarithmic Sobolev inequality for functions of the form $f(x) = |1 + \varepsilon x|$ with ε tending to zero (this is a special case of more general observation which says that, for functionals \mathcal{E} satisfying certain natural conditions, if (μ, \mathcal{E}) satisfies the logarithmic Sobolev inequality with constant $2C$ then it also satisfies the Poincaré inequality with constant C). We can restrict our considerations to the case $\lambda \geq 1$ since $\sigma(L) \setminus \{0\} \subseteq [1, \infty)$. Therefore $(p-1)^{C\lambda}$ is a convex function of p and to prove that

$$h(p) = (p-1)^{C\lambda} + (2-p)C\lambda - 1 \geq 0$$

for $p \in [1, 2)$ it suffices to check that $h(2) = h'(2) = 0$ which is obvious. The case $p = 1$ (omitted when $\lambda = 0$ because $(p-1)^{C\lambda}$ was not well defined) follows easily since the function $p \mapsto (Ef^p)^{2/p}$ is continuous for $p \in [1, 2]$. \square

Corollary 2 *If μ is a $\mathcal{N}(0, 1)$ Gaussian measure on real line, $\mathcal{E}(f) = E_\mu(f')^2$ and \mathcal{C} is a class of non-negative smooth functions then (μ, \mathcal{E}) satisfies $I(1)$ with constant 1.*

Proof. If μ is a $\mathcal{N}(0, 1)$ Gaussian measure and operator L is defined as before then

$$E_\mu f L f = E_\mu (f')^2.$$

The assertion follows from Lemma 2 and well known fact ([G]) that Gaussian measures satisfy the logarithmic Sobolev inequality with constant 2. \square

Remark 1 *Method used in Lemma 2 seems applicable also in more general situation (see [O] for possible directions of generalization). Let us mention just one interesting application. If $\Omega = \{-1, 1\}$, $\mu(\{-1\}) = \mu(\{1\}) = 1/2$ and $\mathcal{E}(f) = \left(\frac{f(1)-f(-1)}{2}\right)^2$ then (μ, \mathcal{E}) satisfies $I(1)$ with constant 1.*

Remark 2 *Let μ be a non-symmetric two-point distribution on $\{-1, 1\}$, $\mu(\{1\}) = 1 - \mu(\{-1\}) = \alpha$ with $\alpha \in (0, 1/2) \cup (1/2, 1)$. Then for any $p \in [1, 2)$ and any $f : \{-1, 1\} \rightarrow \mathbb{R}_+$ the inequality*

$$E_\mu f^2 - (E_\mu f^p)^{2/p} \leq C_\alpha(p)(f(1) - f(-1))^2$$

holds with

$$C_\alpha(p) = \frac{\alpha^{1-2/p} - (1-\alpha)^{1-2/p}}{\alpha^{-2/p} - (1-\alpha)^{-2/p}}$$

and the constant cannot be improved.

Proof (sketch). To check the optimality of $C_\alpha(p)$ put $f(-1) = \alpha^{2/p}$ and $f(1) = (1-\alpha)^{2/p}$. To prove the inequality observe that for $p \in (1, 2)$, $\varphi(y) = ((1+\sqrt{y})^p + (1-\sqrt{y})^p)^{2/p}$ is a strictly convex function of $y \in (0, 1)$, since

$$\begin{aligned} \varphi'(y) &= [(1+\sqrt{y})^p + (1-\sqrt{y})^p]^{\frac{2}{p}-1} \frac{(1+\sqrt{y})^{p-1} - (1-\sqrt{y})^{p-1}}{\sqrt{y}} \\ &= \left(2 \sum_{k=0}^{\infty} \binom{p}{2k} y^k\right)^{\frac{2}{p}-1} 2 \sum_{k=0}^{\infty} \binom{p-1}{2k+1} y^k \end{aligned}$$

is clearly increasing (note that $\binom{p}{2k}$ and $\binom{p-1}{2k+1}$ are positive for $k = 0, 1, \dots$). Hence for each $y_0 \in (0, 1)$ and $p \in (1, 2)$ there exist unique real numbers A and B such that

$$\varphi(y^2) = ((1+y)^p + (1-y)^p)^{2/p} \geq A + B y^2 \text{ for all } y \in (-1, 1)$$

with equality holding for $|y| = y_0$ only. By the homogeneity we may assume that $f(-1) = (1-\alpha)^{-1/p}(1+y)$ and $f(1) = \alpha^{-1/p}(1-y)$. Putting $y_0 = \frac{(1-\alpha)^{1/p} - \alpha^{1/p}}{(1-\alpha)^{1/p} + \alpha^{1/p}}$, using the above inequality after some elementary, but a little involved computations one proves the assertion. \square

Definition 4 Let us denote by Φ the class of all continuous functions $\varphi : [0, \infty) \rightarrow \mathbb{R}$ having strictly positive second derivative and such that $1/\varphi''$ is a concave function. Let us additionally include in Φ all functions φ of the form $\varphi(x) = ax + b$, where a and b are some real constants.

Although it is not obvious, functions belonging to Φ form a convex cone. There are some interesting questions connected with the class Φ and its generalizations but we postpone them till the end of the note.

Lemma 3 For any $\varphi \in \Phi$ and $t \in [0, 1]$ the function $F_t : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ defined by

$$F_t(x, y) = t\varphi(x) + (1-t)\varphi(y) - \varphi(tx + (1-t)y)$$

is non-negative and convex.

Proof. Non-negativity of F_t is an easy consequence of convexity of φ . Obviously F_t is continuous on $[0, \infty) \times [0, \infty)$ and twice differentiable on $(0, \infty) \times (0, \infty)$. Therefore it suffices to prove that $Hess F_t$ (second derivative matrix) is positive definite on $(0, \infty) \times (0, \infty)$. We skip the trivial case of φ being an affine function. Note that from the positivity of φ'' and the concavity of $1/\varphi''$ it follows that

$$\frac{1}{\varphi''(tx + (1-t)y)} \geq \frac{t}{\varphi''(x)} + \frac{1-t}{\varphi''(y)} \geq \frac{t}{\varphi''(x)}.$$

Therefore

$$\frac{\partial^2 F_t}{\partial x^2}(x, y) = t\varphi''(x) - t^2\varphi''(tx + (1-t)y) \geq 0.$$

In a similar way we prove that $\frac{\partial^2 F_t}{\partial y^2}(x, y) \geq 0$. Now it is enough to prove that $\det(Hess F_t) \geq 0$ i.e. that

$$\frac{\partial^2 F_t}{\partial x^2}(x, y) \cdot \frac{\partial^2 F_t}{\partial y^2}(x, y) \geq \left(\frac{\partial^2 F_t}{\partial x \partial y}(x, y)\right)^2$$

which is equivalent to

$$\begin{aligned} & (t\varphi''(x) - t^2\varphi''(tx + (1-t)y))((1-t)\varphi''(y) - (1-t)^2\varphi''(tx + (1-t)y)) \\ & \geq (-t(1-t)\varphi''(tx + (1-t)y))^2 \end{aligned}$$

or

$$\varphi''(x)\varphi''(y) \geq t\varphi''(y)\varphi''(tx + (1-t)y) + (1-t)\varphi''(x)\varphi''(tx + (1-t)y).$$

After dividing by $\varphi''(x)\varphi''(y)\varphi''(tx + (1-t)y)$ the last inequality follows from concavity of $1/\varphi''$ and the proof is complete. \square

Lemma 4 For a non-negative real random variable Z defined on probability space (Ω, μ) and having finite first moment, and for $\varphi \in \Phi$ let

$$\Psi_\varphi(Z) = E_\mu\varphi(Z) - \varphi(E_\mu Z).$$

Then for any non-negative real random variables X and Y defined on (Ω, μ) and having finite first moment, and for any $t \in [0, 1]$ the following inequality holds:

$$\Psi_\varphi(tX + (1-t)Y) \geq t\Psi_\varphi(X) + (1-t)\Psi_\varphi(Y);$$

in other words Ψ_φ is a convex functional on the convex cone of integrable non-negative real random variables defined on (Ω, μ) .

Proof. Let us note that (under notation of Lemma 3)

$$\begin{aligned} & \Psi_\varphi(tX + (1-t)Y) - t\Psi_\varphi(X) - (1-t)\Psi_\varphi(Y) = \\ & (E_\mu\varphi(tX + (1-t)Y) - tE_\mu\varphi(X) - (1-t)E_\mu\varphi(Y)) - \\ & (\varphi(tE_\mu X + (1-t)E_\mu Y) - t\varphi(E_\mu X) - (1-t)\varphi(E_\mu Y)) \\ & = E_\mu F_t(X, Y) - F_t(E_\mu X, E_\mu Y) = E_\mu F_t(X, Y) - F_t(E_\mu(X, Y)). \end{aligned}$$

We are to prove that it is a non-negative expression and this follows easily from Jensen inequality. For the sake of clarity we present a detailed argument.

Let $x_0 = E_\mu X$ and $y_0 = E_\mu Y$. Lemma 3 yields that F_t is convex, so that there exist constants $a, b, c \in R$ such that

$$F_t(x, y) \geq ax + by + c$$

for any $x, y \in [0, \infty)$ and

$$F_t(x_0, y_0) = ax_0 + by_0 + c.$$

Therefore

$$E_\mu F_t(X, Y) \geq E_\mu(ax + by + c) = ax_0 + by_0 + c = F_t(x_0, y_0) = F_t(E_\mu X, E_\mu Y)$$

and the proof is finished. \square

Lemma 5 Let (Ω_1, μ_1) and (Ω_2, μ_2) be probability spaces and let $(\Omega, \mu) = (\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$ be their product probability space. For any non-negative random variable Z defined on (Ω, μ) and having finite first moment and for any $\varphi \in \Phi$ the following inequality holds true:

$$E_\mu\varphi(Z) - \varphi(E_\mu Z) \leq E_\mu([E_{\mu_1}\varphi(Z) - \varphi(E_{\mu_1}Z)] + [E_{\mu_2}\varphi(Z) - \varphi(E_{\mu_2}Z)]).$$

Proof. For $\omega_2 \in \Omega_2$ let $Z_{(\omega_2)}$ be a non-negative random variable defined on (Ω_1, μ_1) by the formula

$$Z_{[\omega_2]}(\omega_1) = Z(\omega_1, \omega_2).$$

By Lemma 4 used for the probability space (Ω_1, μ_1) and Jensen inequality used for the family of random variables $(Z_{[\omega_2]})_{\omega_2 \in \Omega_2}$ (this time we skip the detailed argument which the reader can easily repeat after the proof of Lemma 4) we get

$$E_{\mu_2}(E_{\mu_1}\varphi(Z) - \varphi(E_{\mu_1}Z)) \geq E_{\mu_1}\varphi(E_{\mu_2}Z) - \varphi(E_{\mu_1}(E_{\mu_2}Z))$$

which is equivalent to the assertion of Lemma 5. \square

By an easy induction argument we obtain

Corollary 3 *Let $(\Omega_1, \mu_1), (\Omega_2, \mu_2), \dots, (\Omega_n, \mu_n)$ be probability spaces and let $(\Omega, \mu) = (\Omega_1 \times \Omega_2 \times \dots \times \Omega_n, \mu_1 \otimes \mu_2 \otimes \dots \otimes \mu_n)$ be their product probability space. Let Z be any integrable non-negative real random variable defined on (Ω, μ) . Then for any $\varphi \in \Phi$ the following inequality holds:*

$$E_{\mu}\varphi(Z) - \varphi(E_{\mu}Z) \leq \sum_{k=1}^n E_{\mu}(E_{\mu_k}\varphi(Z) - \varphi(E_{\mu_k}Z)).$$

Let us observe that the function φ defined by $\varphi(x) = x^{2/p}$ belongs to the class Φ if $p \in [1, 2]$. Therefore by applying Corollary 3 to the random variable $Z = f^p$, where $f \in L_+^2(\Omega, \mu)$, we obtain

Corollary 4 *Under the notation of Corollary 3 for any $f \in L_+^2(\Omega, \mu)$ we have*

$$E_{\mu}f^2 - (E_{\mu}f^p)^{2/p} \leq \sum_{k=1}^n E_{\mu}(E_{\mu_k}f^2 - (E_{\mu_k}f^p)^{2/p}).$$

This sub-additivity property of functional $Var(p)_{\mu}$ immediately yields the following

Corollary 5 *Assume that pairs $(\mu_1, \mathcal{E}_1), (\mu_2, \mathcal{E}_2), \dots, (\mu_n, \mathcal{E}_n)$ satisfy the inequality $I(a)$ with some constant C . Let $\mu = \mu_1 \otimes \mu_2 \otimes \dots \otimes \mu_n$ and $\mathcal{E}(f) = E_{\mu}(\mathcal{E}_1(f_1) + \mathcal{E}_2(f_2) + \dots + \mathcal{E}_n(f_n))$, where*

$$f_i(x) = f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)$$

for given $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$. Class \mathcal{C} can be chosen in any way which assures that $f \in \mathcal{C}$ implies $f_i \in \mathcal{C}_i$, for example $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \dots \otimes \mathcal{C}_n$. Then the pair (μ, \mathcal{E}) also satisfies the inequality $I(a)$ with constant C .

The case we will concentrate on is $\mathcal{E}(f) = E_{\mu}\|\nabla f\|^2$.

Proposition 1 Let $\mu_1, \mu_2, \dots, \mu_n$ be probability measures on R . Let $C > 0$ and $a \in [0, 1]$. Assume that for any smooth function $f : R \rightarrow [0, \infty)$ the inequality

$$E_{\mu_i} f^2 - (E_{\mu_i} f^p)^{2/p} \leq C(2-p)^a E_{\mu_i} (f')^2$$

holds true for $p \in [1, 2)$ and $i = 1, 2, \dots, n$. Then for $\mu = \mu_1 \otimes \mu_2 \otimes \dots \otimes \mu_n$ the inequality

$$E_{\mu} f^2 - (E_{\mu} f^p)^{2/p} \leq C(2-p)^a E_{\mu} \|\nabla f\|^2,$$

where $\|\cdot\|$ denotes standard Euclidean norm, is satisfied for $p \in [1, 2)$ and any smooth function $f : R^n \rightarrow [0, \infty)$.

Proof. Use Corollary 5 and note that

$$\begin{aligned} E_{\mu} \|\nabla f\|^2 &= E_{\mu} \left[\left(\frac{\partial f}{\partial x_1} \right)^2 + \dots + \left(\frac{\partial f}{\partial x_n} \right)^2 \right] = E_{\mu} [(f'_1)^2 + \dots + (f'_n)^2] \\ &= E_{\mu} [E_{\mu_1} (f'_1)^2 + \dots + E_{\mu_n} (f'_n)^2]. \quad \square \end{aligned}$$

Now let us demonstrate that the inequality $I(a)$ for the $\mathcal{E}(f) = E_{\mu} \|\nabla f\|^2$ functional implies concentration of Lipschitz functions.

Theorem 1 Let μ be a probability measure on R^n . Assume that there exist constants $C > 0$ and $a \in [0, 1]$ such that the inequality

$$E_{\mu} f^2 - (E_{\mu} f^p)^{2/p} \leq C(2-p)^a E_{\mu} \|\nabla f\|^2$$

is satisfied for any smooth function $f : R^n \rightarrow [0, \infty)$ and $p \in [1, 2)$. Let $h : R^n \rightarrow R$ be a Lipschitz function with Lipschitz constant 1, i.e. $|h(x) - h(y)| \leq \|x - y\|$ for any $x, y \in R^n$, where $\|\cdot\|$ denotes a standard Euclidean norm. Then $E_{\mu} |h| < \infty$ and

- for any $t \in [0, 1]$

$$\mu(h - E_{\mu} h \geq t\sqrt{C}) \leq e^{-Kt^2}$$

- for any $t \geq 1$

$$\mu(h - E_{\mu} h \geq t\sqrt{C}) \leq e^{-Kt^{\frac{2}{2-a}}}$$

where K is some universal constant.

Proof. Our proof will work for $K = 1/3$ but we do not know optimal constants (it is also interesting what the optimal K is for given value of parameter a). Note that it is essential part of the assumptions that we study the limit behaviour when $p \rightarrow 2$. For any fixed $p \in (1, 2)$ the inequality

$$E_{\mu} f^2 - (E_{\mu} f^p)^{2/p} \leq C(2-p)^a E_{\mu} \|\nabla f\|^2$$

is weaker than the Poincaré inequality with constant $C(2-p)^a$ and therefore it cannot imply anything stronger than the exponential concentration.

We will follow the approach of [AS]. Assume first that h is bounded and smooth. Then $\|\nabla h\| \leq 1$. Define $H(\lambda) = E_\mu e^{\lambda h}$ for $\lambda \geq 0$. Assumptions of Theorem 1 for $f = e^{\lambda h/2}$ give

$$H(\lambda) - H\left(\frac{p}{2}\lambda\right)^{2/p} \leq \frac{C\lambda^2}{4}(2-p)^a E_\mu \|\nabla h\|^2 e^{\lambda h} \leq \frac{C\lambda^2}{4}(2-p)^a H(\lambda).$$

Hence

$$H(\lambda) \leq \frac{H\left(\frac{p}{2}\lambda\right)^{2/p}}{1 - \frac{C}{4}(2-p)^a \lambda^2}$$

for any $p \in [1, 2)$ and $0 \leq \lambda \leq \frac{2}{\sqrt{C}}(2-p)^{-a/2}$. Applying the same inequality for $\frac{p}{2}\lambda$ and iterating, after m steps we get

$$H(\lambda) \leq \frac{H\left(\left(\frac{p}{2}\right)^m \lambda\right)^{(2/p)^m}}{\prod_{k=0}^{m-1} \left(1 - \frac{C\lambda^2}{4}(2-p)^a \cdot \left(\frac{p}{2}\right)^{2k}\right)^{(2/p)^k}}.$$

Note that

$$1 - \frac{C\lambda^2}{4}(2-p)^a \cdot \left(\frac{p}{2}\right)^{2k} \geq \left(1 - \frac{C\lambda^2}{4}(2-p)^a\right)^{(p/2)^{2k}}$$

since $\left(\frac{p}{2}\right)^{2k} < 1$. Hence

$$H(\lambda) \leq H\left(\left(\frac{p}{2}\right)^m \lambda\right)^{(2/p)^m} \left(1 - \frac{C\lambda^2}{4}(2-p)^a\right)^{-\sum_{k=0}^{m-1} (p/2)^{2k}}.$$

As $\lim_{m \rightarrow \infty} \left(\frac{p}{2}\right)^m = 0$ we get

$$\lim_{m \rightarrow \infty} H\left(\left(\frac{p}{2}\right)^m \lambda\right)^{(2/p)^m} = e^{\lambda E_\mu h}.$$

Therefore

$$E_\mu e^{\lambda(h - E_\mu h)} \leq \left(1 - \frac{C\lambda^2}{4}(2-p)^a\right)^{-\frac{2}{2-p}}$$

and

$$\mu(h - E_\mu h \geq t\sqrt{C}) \leq e^{-\lambda t\sqrt{C}} \cdot \left(1 - \frac{C\lambda^2}{4}(2-p)^a\right)^{-\frac{2}{2-p}}.$$

- Putting $p = 1$ and $\lambda = \frac{t}{\sqrt{C}}$ we get for any $t \in [0, 2)$

$$\mu(h - E_\mu h \geq t\sqrt{C}) \leq e^{-t^2} \cdot \left(1 - \frac{t^2}{4}\right)^{-2}.$$

In particular for $t \in [0, 1]$ we have $1 - \frac{t^2}{4} > e^{-t^2/3}$ and

$$\mu(h - E_\mu h \geq t\sqrt{C}) \leq e^{-t^2/3}.$$

- If $t \geq 1$, let us put $p = 2 - t^{-\frac{2}{2-a}}$ and $\lambda = t^{\frac{a}{2-a}}/\sqrt{C}$. Then we arrive at

$$\mu(h - E_\mu h \geq t\sqrt{C}) \leq e^{-t^{\frac{2}{2-a}}} \cdot \left(1 - \frac{1}{4}\right)^{-2t^{\frac{2}{2-a}}} = \left(\frac{16}{9e}\right)t^{\frac{2}{2-a}}$$

which completes the proof (if h is bounded and smooth) since $\frac{16}{9e} \leq e^{-1/3}$.

Therefore by a standard approximation argument we prove the assertion for any bounded h which satisfies assumptions of Theorem 1. Finally for general h define its bounded truncations $(h_N)_{N=1}^\infty$ putting $h_N(x) = h(x)$ if $|x| \leq N$ and $h_N(x) = N \cdot \text{sgn}(x)$ if $|x| \geq N$. One can easily check that if h satisfies the assumptions of Theorem 1 then $|h_N|$ is also a Lipschitz function with a Lipschitz constant 1 and therefore using Theorem 1 for a bounded function $|h_N|$ we arrive at

$$\mu(|h_N| - E_\mu |h_N| \geq 4\sqrt{C}) \leq \left(\frac{16}{9e}\right)^{4\frac{2}{2-a}} \leq \left(\frac{16}{9e}\right)^4 \leq \frac{1}{5}.$$

Similarly

$$\mu(|h_N| - E_\mu |h_N| \leq -4\sqrt{C}) = \mu(-|h_N| - E_\mu(-|h_N|) \geq 4\sqrt{C}) \leq \frac{1}{5}.$$

Hence

$$\mu(| |h_N| - E_\mu |h_N| | \geq 4\sqrt{C}) \leq \frac{2}{5}$$

and

$$\mu(| |h| - E_\mu |h_N| | \geq 4\sqrt{C}) \leq \frac{2}{5} + \mu(|h| > N).$$

Therefore

$$\begin{aligned} \mu(|E_\mu |h_N| - E_\mu |h_M| | \geq 8\sqrt{C}) &\leq \\ \mu(| |h| - E_\mu |h_N| | \geq 4\sqrt{C}) + \mu(| |h| - E_\mu |h_M| | \geq 4\sqrt{C}) &\leq \\ \frac{4}{5} + \mu(|h| > N) + \mu(|h| > M) &\longrightarrow \frac{4}{5} < 1 \end{aligned}$$

as $\min(N, M) \longrightarrow \infty$, which means that the sequence $(E_\mu |h_N|)_{N=1}^\infty$ is bounded. As $|h_N|$ grows monotonically to $|h|$, by Lebesgue Lemma we get $E_\mu |h| < \infty$ and $E_\mu h_N \longrightarrow E_\mu h$ as $N \longrightarrow \infty$. Now an easy approximation argument completes the proof. \square

In order to prove that the order of concentration implied by Theorem 1 cannot be improved we will need the following

Theorem 2 *Let $a \in [0, 1]$ and $r \in [1, 2]$ satisfy $r = 2/(2-a)$. Put $c_r = \frac{1}{2\Gamma(1+1/r)} = \frac{r}{2\Gamma(1/r)}$. Then $\mu_r(dx) = c_r^n \exp(-(|x_1|^r + |x_2|^r + \dots + |x_n|^r)) dx_1 dx_2 \dots dx_n$ is a probability measure on \mathbb{R}^n and there exists a universal constant $C > 0$ (not depending on a or n) such that*

$$E_{\mu_r} f^2 - (E_{\mu_r} f^p)^{2/p} \leq C(2-p)^a E_{\mu_r} \|\nabla f\|^2$$

for any smooth non-negative function f on \mathbb{R}^n and any $p \in [1, 2)$.

Proof. Proposition 1 shows that it is enough to prove Theorem 2 in the case $n = 1$. Therefore the assertion easily follows from the two following propositions. \square

Proposition 2 *Let $a \in [0, 1]$ and $r \in [1, 2]$ satisfy $r = 2/(2 - a)$. Put $c_r = \frac{1}{2\Gamma(1+1/r)}$, so that $\mu_r(dx) = c_r \exp(-|x|)^r dx$ is a probability measure on R . Let $\lambda(dx) = \frac{1}{2}e^{-|x|}$ be a symmetric exponential probability measure on R . Under these assumptions the following implications hold true:*

- *If $C > 0$ is a constant such that for any smooth function $f : R \rightarrow [0, \infty)$ and any $p \in [1, 2)$ there is*

$$E_{\mu_r} f^2 - (E_{\mu_r} f^p)^{2/p} \leq C(2-p)^a E_{\mu_r} (f')^2$$

then for any smooth function $g : R \rightarrow [0, \infty)$ and any $p \in [1, 2)$ there is

$$\int_R g(x)^2 \lambda(dx) - \left(\int_R g(x)^p \lambda(dx) \right)^{2/p} \leq 600C(2-p)^a \int_R \max(1, |x|^a) g'(x)^2 \lambda(dx).$$

- *Conversely, if $C > 0$ is such a constant that for any smooth function $g : R \rightarrow [0, \infty)$ and any $p \in [1, 2)$ there is*

$$\int_R g(x)^2 \lambda(dx) - \left(\int_R g(x)^p \lambda(dx) \right)^{2/p} \leq C(2-p)^a \int_R \max(1, |x|^a) g'(x)^2 \lambda(dx)$$

then for any smooth function $f : R \rightarrow [0, \infty)$ and any $p \in [1, 2)$ there is

$$E_{\mu_r} f^2 - (E_{\mu_r} f^p)^{2/p} \leq 50C(2-p)^a E_{\mu_r} (f')^2.$$

Proposition 3 *There exists a universal constant C such that for any $a \in [0, 1]$, any $p \in [1, 2)$ and any smooth function $g : R \rightarrow [0, \infty)$ the following inequality holds*

$$\int_R g(x)^2 \lambda(dx) - \left(\int_R g(x)^p \lambda(dx) \right)^{2/p} \leq C(2-p)^a \int_R \max(1, |x|^a) g'(x)^2 \lambda(dx).$$

We will start with proof of Proposition 2. The proof of Proposition 3 will be postponed to the end of the paper.

Proof of Proposition 2. Let us define the function $z_r : R \rightarrow R$ by

$$\frac{1}{2} \int_{z_r(x)}^{\infty} e^{-|t|} dt = c_r \int_x^{\infty} e^{-|t|^r} dt,$$

where $c_r = \frac{r}{2\Gamma(1/r)} = \frac{1}{2\Gamma(1+1/r)}$. It is easy to see that z_r is a homeomorphism of R onto itself and

$$z'_r(x) = 2c_r e^{|z_r(x)| - |x|^r}.$$

Therefore z_r is a C^1 -diffeomorphism of R onto itself. Binding f and g by relation $f(x) = g(z_r(x))$ and using standard change of variables formula we reduce the proof of Proposition 2 to the following lemma. \square

Lemma 6 *Under notation introduced above*

$$\frac{1}{50} \max(1, |x|^a) \leq (z'_r(z_r^{-1}(x)))^2 \leq 600 \max(1, |x|^a)$$

for any $x \in R$.

Proof. First let us note that $1/3 \leq c_r \leq e/2$. Indeed,

$$\Gamma(1/r) = \int_0^\infty x^{\frac{1}{r}-1} e^{-x} dx \leq \int_0^1 x^{\frac{1}{r}-1} dx + \int_1^\infty e^{-x} dx = r + 1/e.$$

Hence $c_r \geq \frac{r}{2r+2/e} \geq 1/3$. On the other hand

$$\Gamma(1/r) = \int_0^\infty x^{\frac{1}{r}-1} e^{-x} dx \geq \frac{1}{e} \int_0^1 x^{\frac{1}{r}-1} dx = r/e.$$

Therefore $c_r \leq e/2$. Let us also notice that by obvious symmetry we can consider only the case $x > 0$. Now let us estimate from below $z_r^{-1}(1)$. We have

$$\frac{e}{2} z_r^{-1}(1) \geq c_r z_r^{-1}(1) \geq c_r \int_0^{z_r^{-1}(1)} e^{-t^r} dt = \frac{1}{2} \int_0^1 e^{-t} dt = \frac{1}{2}(1 - 1/e)$$

and therefore $z_r^{-1}(1) \geq \frac{e-1}{e^2} \geq 1/5$. Note that by definition of $z_r(x)$ for $x > 0$ we have

$$\frac{1}{2} e^{-z_r(x)} = c_r \int_x^\infty e^{-t^r} dt \leq c_r \int_x^\infty \frac{rt^{r-1}}{rx^{r-1}} e^{-t^r} dt = \frac{c_r e^{-x^r}}{rx^{r-1}}$$

and therefore

$$z'_r(x) = 2c_r e^{z_r(x)-x^r} \geq rx^{r-1}.$$

Hence also $z_r(x) \geq x^r$ and $z_r^{-1}(x) \leq x^{1/r}$ for all positive x . If $x \geq 1/5$ then

$$\begin{aligned} \int_x^\infty e^{-t^r} dt &\geq \int_x^{6x} e^{-t^r} dt \geq \frac{1}{r(6x)^{r-1}} \int_x^{6x} rt^{r-1} e^{-t^r} dt = \\ &6^{1-r} \frac{e^{-x^r} - e^{-6^r x^r}}{rx^{r-1}} \geq \frac{1}{12} \frac{e^{-x^r}}{rx^{r-1}}, \end{aligned}$$

since $6^r x^r \geq x^r + 1$ for $x \geq 1/5$ and $r \in [1, 2]$. Therefore for $x \geq z_r^{-1}(1) \geq 1/5$ we have

$$z'_r(x) \leq 12rx^{r-1} \leq 24x^{r-1}$$

and

$$z_r(x) \leq z_r(z_r^{-1}(1)) + 12 \int_{z_r^{-1}(1)}^x rt^{r-1} dt = 1 + 12(x^r - [z_r^{-1}(1)]^r) \leq 1 + 12x^r \leq 37x^r.$$

Hence $z_r^{-1}(x) \geq (x/37)^{1/r}$ for $x \geq z_r^{-1}(1)$. If $x \geq 1$ then $z_r^{-1}(x) \geq 1/5$ and therefore

$$z'_r(z_r^{-1}(x)) \leq 24[z_r^{-1}(x)]^{r-1} \leq 24x^{\frac{r-1}{r}} = 24x^{a/2}.$$

Also if $x \geq 1$ then $z_r^{-1}(x) \geq z_r^{-1}(1)$ and

$$z'_r(z_r^{-1}(x)) \geq r[z_r^{-1}(x)]^{r-1} \geq (x/37)^{\frac{r-1}{r}} \geq 37^{\frac{1}{r}-1}x^{a/2} \geq \frac{1}{7}x^{a/2}.$$

This proves Lemma 6 for $|x| \geq 1$. For any $x \geq 0$ we have

$$z'_r(z_r^{-1}(x)) = 2c_r e^{x-z_r^{-1}(x)r} \geq 2c_r \geq 2/3.$$

We used the previously proved fact that $z_r^{-1}(x) \leq x^{1/r}$. Now it remains only to establish upper estimate on $z'_r(z_r^{-1}(x))$ for $x \in [0, 1]$. Note that if $x \leq z_r^{-1}(1)$ then

$$c_r \int_x^\infty e^{-tr} dt = \frac{1}{2} \int_{z_r(x)}^\infty e^{-t} dt \geq \frac{1}{2} \int_1^\infty e^{-t} dt = \frac{1}{2e}$$

and therefore

$$z'_r(x) = \frac{2c_r e^{-x^r}}{2c_r \int_x^\infty e^{-tr} dt} \leq \frac{c_r}{c_r \int_x^\infty e^{-tr} dt} \leq 2ec_r \leq e^2 \leq 8.$$

Hence $z'_r(z_r^{-1}(x)) \leq 8$ for any $|x| \leq 1$ and the proof is finished. \square

Lemma 7 For $s \in (1, 2]$ and $x, y \geq 0$ put

$$\rho_s(x, y) = \left(\frac{x^s + y^s}{2} - \left(\frac{x+y}{2} \right)^s \right)^{1/2}.$$

Then ρ_s is a metric on $[0, \infty)$.

Proof. Since $k_t(a, b) = e^{-(a+b)t}$ is obviously positive definite integral kernel and $K(a, b) = s(s-1)(a+b)^{s-2} = \frac{s(s-1)}{\Gamma(2-s)} \int_0^\infty t^{1-s} k_t(a, b) dt$ we get, by Schwartz inequality (applied to a scalar product defined by the kernel $K(a, b)$), that for any $y \geq x \geq 0$ and $z \geq t \geq 0$ the following inequality is true:

$$\begin{aligned} & \int_{x/2}^{y/2} \int_{t/2}^{z/2} K(a, b) da db \\ & \leq \left(\int_{x/2}^{y/2} \int_{x/2}^{y/2} K(a, b) da db \right)^{1/2} \left(\int_{t/2}^{z/2} \int_{t/2}^{z/2} K(a, b) da db \right)^{1/2}. \end{aligned}$$

Now, as

$$K(a, b) = \frac{\partial^2}{\partial a \partial b} (a+b)^s,$$

we get by integration by parts

$$\begin{aligned} & \left(\frac{y+z}{2}\right)^s + \left(\frac{x+t}{2}\right)^s - \left(\frac{x+z}{2}\right)^s - \left(\frac{y+t}{2}\right)^s \leq \\ & (x^s + y^s - 2\left(\frac{x+y}{2}\right)^s)^{1/2} (z^s + t^s - 2\left(\frac{z+t}{2}\right)^s)^{1/2} \end{aligned}$$

Putting $t = y$ we arrive at

$$\left(\frac{x+y}{2}\right)^s + \left(\frac{y+z}{2}\right)^s - \left(\frac{x+z}{2}\right)^s - y^s \leq 2\rho_s(x, y)\rho_s(y, z)$$

which is equivalent to

$$\rho_s(x, z)^2 - \rho_s(x, y)^2 - \rho_s(y, z)^2 \leq 2\rho_s(x, y)\rho_s(y, z).$$

Hence $\rho_s(x, z) \leq \rho_s(x, y) + \rho_s(y, z)$. For $x \leq y \leq z$ we have also easily $\rho_s(x, z) \geq \rho_s(x, y)$ and $\rho_s(x, z) \geq \rho_s(y, z)$, so that $\rho_s(x, y) \leq \rho_s(x, z) + \rho_s(z, y)$ and $\rho_s(y, z) \leq \rho_s(y, x) + \rho_s(x, z)$. This completes the proof of triangle inequality for $s < 2$. Other metric properties of ρ_s as well as the case $s = 2$ are trivial. \square

Remark 3 In a similar way one can prove that $\rho_s(x, y) = \left|\frac{x^s+y^s}{2} - \left(\frac{x+y}{2}\right)^s\right|^{1/2}$ is a metric on $(0, \infty)$ for $s \in (-\infty, 0) \cup (0, 1)$. It was pointed out to the authors by B. Maurey that Lemma 7 follows also from isometrical immersion of $([0, \infty), \rho_s)$ into $L^2([0, \infty), \kappa_s^{-1}t^{-s-1}dt)$, where $x \in [0, \infty)$ is sent to the function $e^{-xt} - 1$ and $\kappa_s = 2^{s+1} \int_0^\infty (e^{-u} - 1 + u)u^{-s-1}du$.

Lemma 8 Let $s \in [1, 2]$, $t \in [0, 1]$ and c, d, x be nonnegative numbers. The following inequality holds

$$\begin{aligned} & (1-t)c^s + td^s - ((1-t)c + td)^s \leq \\ & K[(1-t)c^s + td^s + x^s - ((1-t)c + tx)^s - (td + (1-t)x)^s]. \end{aligned} \quad (1)$$

under anyone of the following additional assumptions

- x lies outside the open interval (c, d) and $K = 1$
- $t = \frac{1}{2}$ and $K = 2$
- $t \leq \frac{1}{2}$, $c \geq d$ and $K = 12$

Proof. Let us remind that

$$F_t(x, y) = tx^s + (1-t)y^s - (tx + (1-t)y)^s$$

is a convex function on $[0, \infty) \times [0, \infty)$. Note that the inequality of Lemma 8 is equivalent to

$$F_t(d, c) \leq K[F_t(d, x) + F_t(x, c)].$$

- As

$$\frac{\partial}{\partial x}[F_t(d, x) + F_t(x, c)]$$

$$= s[(1-t)(x^{s-1} - (td + (1-t)x)^{s-1}) + t(x^{s-1} - (tx + (1-t)c)^{s-1})],$$

we see that the right-hand side of the inequality as a function of x is increasing on $(\max(c, d), \infty)$ and decreasing on $[0, \min(c, d)]$. For $x = \max(c, d)$ and $x = \min(c, d)$ the inequality is trivially satisfied with $K = 1$. This completes the case of x which does not lie between c and d .

- The second part of Lemma 8 follows easily by Lemma 7, as

$$\begin{aligned} F_{1/2}(d, c) &= \rho_s(d, c)^2 \leq (\rho_s(d, x) + \rho_s(x, c))^2 \leq \\ &2[\rho_s(d, x)^2 + \rho_s(x, c)^2] = 2[F_{1/2}(d, x) + F_{1/2}(x, c)]. \end{aligned}$$

- To prove the last part of the statement we will use convexity of F_t . Since $F_t(d, x) + F_t(x, c) \geq F_t(\frac{d+x}{2}, \frac{x+c}{2})$, it suffices to prove that $F_t(d, c) \leq 12F_t(\frac{d+x}{2}, \frac{x+c}{2})$. Thanks to the first part of Lemma 8 we can restrict our considerations to the case $x \in [d, c]$. Note that

$$\frac{\partial}{\partial x} F_t\left(\frac{d+x}{2}, \frac{x+c}{2}\right)$$

$$= \frac{s}{2} \left[t \left(\frac{d+x}{2}\right)^{s-1} + (1-t) \left(\frac{x+c}{2}\right)^{s-1} - \left(t \left(\frac{d+x}{2}\right) + (1-t) \left(\frac{x+c}{2}\right)\right)^{s-1} \right] \leq 0,$$

since the function $\varphi(u) = u^{s-1}$ is concave. Therefore it is enough to prove that

$$F_t(d, c) \leq 12F_t\left(\frac{d+c}{2}, c\right).$$

Using the homogeneity of the above formula we can reduce our task to proving that

$$F_t(1-u, 1) \leq 12F_t(1-u/2, 1)$$

for any $u \in [0, 1]$ and $t \in [0, 1/2]$.

Using the Taylor expansion we get

$$F_t(1-u, 1) = t(1-u)^s + 1-t - (1-tu)^s =$$

$$s(s-1)u^2t(1-t) \cdot \left[\frac{1}{2} + \sum_{k=1}^{\infty} \frac{u^k}{(k+1)(k+2)} \sum_{m=0}^k t^m \cdot \prod_{l=1}^k \left(1 - \frac{s-1}{l}\right) \right].$$

Therefore

$$F_t(1-u/2, 1) \geq \frac{1}{2}s(s-1)(u/2)^2t(1-t)$$

and

$$\begin{aligned} F_t(1-u, 1) &\leq s(s-1)u^2t(1-t) \cdot \left[\frac{1}{2} + 2 \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)} \right] \\ &= \frac{3}{2}s(s-1)u^2t(1-t) \end{aligned}$$

because $\sum_{m=0}^{\infty} t^m \leq 2$. Hence

$$F_t(1-u, 1) \leq 12F_t(1-u/2, 1)$$

which completes the proof. \square

Lemma 9 *Let $a \in [0, 1]$, $0 \leq x_1 < x_2$ and g be a smooth function on $[x_1, x_2]$ such that $g(x_1) = y_1, g(x_2) = y_2$. Then*

$$\int_{x_1}^{x_2} \max(1, x^a) g'(x)^2 d\lambda(x) \geq \frac{(y_2 - y_1)^2}{4(e^{x_2} - e^{x_1})} \max(1, x_2^a). \quad (2)$$

Proof. By the Schwartz inequality

$$\begin{aligned} |y_2 - y_1| &\leq \int_{x_1}^{x_2} |g'(x)| dx \\ &\leq \left(\int_{x_1}^{x_2} \max(1, x^a) g'(x)^2 d\lambda(x) \right)^{1/2} \left(2 \int_{x_1}^{x_2} \min(1, x^{-a}) e^x dx \right)^{1/2}. \end{aligned}$$

Therefore to show (2) it is enough to prove that

$$f_1(x_2) = \int_{x_1}^{x_2} \min(1, x^{-a}) e^x dx \leq 2 \min(1, x_2^{-a}) (e^{x_2} - e^{x_1}) = f_2(x_2).$$

For $x_2 \leq 2$ this is obvious because for $0 < x < x_2 \leq 2$ we have $\min(1, x^{-a}) \leq 1 \leq 2 \min(1, x_2^{-a})$, and for $x \geq 2$ we have

$$f_2'(x) = 2x^{-a}(e^x - ax^{-1}(e^x - e^{x_1})) \geq x^{-a}e^x = f_1'(x). \square$$

Lemma 10 *Let $0 \leq y_1 < y_2$, $0 \leq x_1 < x_2$ and g is defined on $(-\infty, x_2)$ by the formula*

$$g(x) = \begin{cases} y_1 & \text{for } x \leq x_1 \\ y_1 + (e^x - e^{x_1}) \frac{y_2 - y_1}{e^{x_2} - e^{x_1}} & \text{for } x \in (x_1, x_2] \end{cases}.$$

Then

$$\int_{-\infty}^{x_2} g'(x)^2 d\lambda(x) = \frac{(y_2 - y_1)^2}{2(e^{x_2} - e^{x_1})}. \quad (3)$$

and for all $p \geq 1$

$$\int_{-\infty}^{x_2} g(x)^p d\lambda(x) \leq \lambda(-\infty, x_2) \left[\left(1 - \frac{x_2}{2} e^{-x_2}\right) y_1^p + \frac{x_2}{2} e^{-x_2} y_2^p \right]. \quad (4)$$

Proof. Equation (3) follows by direct calculations. It is easy to see that $g(x)$ is maximal (for fixed values of x_2, y_1 and y_2) when $x_1 = 0$, so to prove (4) we may and will assume that this is the case. To easy the notation we will denote x_2 by x . First we will consider $p = 1$. After some standard calculations (4) is equivalent in this case to

$$\frac{e^x(x-1+e^{-x})}{(2e^x-1)(e^x-1)} \leq \frac{1}{2}xe^{-x} \text{ for all } x > 0,$$

that is

$$2 + 3x \leq xe^{-x} + 2e^x \text{ for all } x > 0,$$

which immediately follows from well known estimates $e^{-x} \geq 1 - x$ and $e^x \geq 1 + x + x^2/2$.

Now, for arbitrary $p \geq 1$ notice that $g(x) = (1 - \theta(x))y_1 + \theta(x)y_2$ with $0 \leq \theta(x) \leq 1$. Therefore we have by the convexity of x^p

$$\begin{aligned} \int_{-\infty}^{x_2} g(x)^p d\lambda(x) &\leq \int_{-\infty}^{x_2} ((1 - \theta(x))y_1^p + \theta(x)y_2^p) d\lambda(x) \leq \\ &\lambda(-\infty, x_2) \left[\left(1 - \frac{x_2}{2}e^{-x_2}\right)y_1^p + \frac{x_2}{2}e^{-x_2}y_2^p \right], \end{aligned}$$

where the last inequality follows by the previously established case $p = 1$. \square

Lemma 11 Suppose that $s \in (1, 2]$, $t \in (0, 1)$, $u = \frac{s}{4(s-1)}e^{-s/2(s-1)}$ and positive numbers $a, b, c, d, \tilde{a}, \tilde{c}, x$ satisfy the following conditions

$$c < x < d, c^s \leq a, d^s \leq b, \tilde{c}^s \leq \tilde{a}, \tilde{c} \leq (1 - u)c + ux.$$

Then

$$\begin{aligned} &(1 - t)a + tb - ((1 - t)c + td)^s \leq \\ &8[(1 - t)\tilde{a} + t\tilde{c} - ((1 - t)\tilde{c} + t\tilde{d})^s + (1 - t)a + tx^s - ((1 - t)c + tx)^s]. \end{aligned} \quad (5)$$

Proof. Without loss of generality we may assume that $a = c^s, b = d^s, \tilde{a} = \tilde{c}^s$. Since the function $y \rightarrow (1 - t)y^s - ((1 - t)y + td)^s$ is nonincreasing on $[0, d]$, it is enough to show that

$$\begin{aligned} &(1 - t)c^s + td^s - ((1 - t)c + td)^s \leq \\ &3[(1 - t)((1 - u)c + ud)^s + td^s - ((1 - t)(1 - u)c + (t + (1 - t)u)d)^s]. \end{aligned}$$

By the homogeneity we may and will assume that $d = 1$. We are then to show that

$$f((1 - c)) \leq 8f((1 - u)(1 - c)), \quad (6)$$

where

$$f(x) = (1 - t)(1 - x)^s + t - (1 - (1 - t)x)^s = \sum_{i=2}^{\infty} (-1)^i \binom{s}{i} (1 - t)(1 - (1 - t)^{i-1})x^k.$$

We use the following simple observation: if a_i, b_i are two summable sequences of positive numbers such that for any $i > j$, $a_i/a_j \geq b_i/b_j$ then for any nondecreasing nonnegative sequence c_i

$$\frac{\sum a_i c_i}{\sum a_i} \geq \frac{\sum b_i c_i}{\sum b_i}.$$

We apply the above to the sequences $a_i = (-1)^i \binom{s}{i} (1-t)(1-(1-t)^{i-1})x^i$, $b_i = (i-1)(-1)^i \binom{s}{i}$ and $c_i = (1-u)^i$, $i = 2, 3, \dots$ and notice that

$$h(y) := \sum_{i=2}^{\infty} b_i y^i = 1 - (1-y)^{s-1}(1+(s-1)y) \text{ for } y \in [0, 1]$$

Therefore we get

$$f((1-u)x) \geq \frac{h(1-u)}{h(1)} = (1-u^{s-1}(1+(s-1)(1-u)))f(x)$$

Inequality (6) follows if we notice that

$$u^{s-1}(1+(s-1)(1-u)) \leq su^{s-1} = \frac{s^s}{4^{s-1}} e^{-s/2} \left(\frac{1}{s-1}\right)^{s-1} \leq 1e^{-1/2}e^{1/e} \leq \frac{7}{8} \square$$

Proposition 4 *Suppose that for all $p \in [1, 2)$ and all nonnegative smooth functions g we have*

$$\int_{\mathbb{R}} g^2 d\lambda - \left(\int_{\mathbb{R}} g^p d\lambda\right)^{2/p} \leq K_1(2-p)^i \int_{\mathbb{R}} (g'(x))^2 \max(1, |x|^i) d\lambda(x) \text{ for } i = 0, 1, \quad (7)$$

where K_1 is a universal constant. Then for all p and g as above we have

$$\int_{\mathbb{R}} g^2 d\lambda - \left(\int_{\mathbb{R}} g^p d\lambda\right)^{2/p} \leq K_2(2-p)^a \int_{\mathbb{R}} (g'(x))^2 \max(1, |x|^a) d\lambda(x) \text{ for } a \in (0, 1), \quad (8)$$

where $K_2 \leq 32K_1$ is some universal constant.

Proof. An easy approximation argument shows that (7) holds for any continuous function g , continuously differentiable everywhere except possibly finitely many points.

First we assume that g is constant on R^- or R^+ , without loss of generality say it is R^- , and we show that (8) holds with $K_2 = 16K_1$. Let us fix $p \in [1, 2)$ and define

$$x_p = (2-p)^{-1}, y = g(x_p), t = \lambda(x_p, \infty), s = \frac{2}{p},$$

$$a = \frac{1}{1-t} \int_{-\infty}^{x_p} g^2 d\lambda, b = \frac{1}{t} \int_{x_p}^{\infty} g^2 d\lambda$$

$$c = \frac{1}{1-t} \int_{-\infty}^{x_p} g^p d\lambda \text{ and } d = \frac{1}{t} \int_{x_p}^{\infty} g^p d\lambda.$$

Notice that by Hölder's inequality we have

$$a \geq c^s \text{ and } b \geq d^s. \quad (9)$$

We will consider two cases

Case 1. y^p lies outside (c, d) or $c > d$.

We first apply inequality (7) for $i = 1$ and a function $gI_{(-\infty, x_p)} + yI_{[x_p, \infty)}$ to get

$$(1-t)a + ty^2 - ((1-t)c + ty^p)^s \leq K_1(2-p) \int_0^{x_p} (g'(x))^2 \max(1, |x|) d\lambda(x) \leq$$

$$K_1(2-p)^a \int_0^{x_p} (g'(x))^2 \max(1, |x|^a) d\lambda(x).$$

In a similar way using the case of $i = 0$ for the function $yI_{(-\infty, x_p)} + gI_{[x_p, \infty)}$ we get

$$tb + (1-t)y^2 - (td + (1-t)y^p)^s \leq K_1 \int_{x_p}^{\infty} (g'(x))^2 d\lambda(x) \leq$$

$$K_1(2-p)^a \int_{x_p}^{\infty} (g'(x))^2 \max(1, |x|^a) d\lambda(x).$$

Notice also that

$$\int_R g^2 d\lambda - \left(\int_R g^p d\lambda \right)^{2/p} = (1-t)a + tb - ((1-t)c + td)^s \leq$$

$$12[(1-t)a + ty^2 - ((1-t)c + ty^p)^s + tb + (1-t)y^2 - (td + (1-t)y^p)^s] \leq$$

$$12K_1(2-p)^a \int_R (g'(x))^2 \max(1, |x|^a) d\lambda(x).$$

The middle inequality follows by Lemma 8 with $x = y^p$ together with estimates (9).

Case 2. $c < y^p < d$, we can then find $0 < x_0 < x_p$ such that $g(x_0) = c^{1/p}$. Define new function f by the formula

$$f(x) = \begin{cases} g(x) & \text{for } x > x_p \\ c^{1/p} + \frac{y-c^{1/p}}{e^{x_p}-e^{x_0}}(e^x - e^{x_0}) & \text{for } x \in [x_0, x_p] \\ c^{1/p} & \text{for } x < x_0. \end{cases}$$

Let

$$\tilde{a} = \frac{1}{1-t} \int_{-\infty}^{x_p} f^2 d\lambda \text{ and } \tilde{c} = \frac{1}{1-t} \int_{-\infty}^{x_p} f^p d\lambda.$$

By Lemma 9 and 10 we have

$$\int_R f'(x)^2 d\lambda(x) \leq 2(2-p)^a \int_R \max(1, |x|^a) g'(x)^2 d\lambda(x).$$

Therefore by (7) with $i = 0$, used for the function f , we have

$$(1-t)\tilde{a} + tb - ((1-t)\tilde{c} + td)^s \leq 2K_1(2-p)^a \int \max(1, |x|^a) g'(x)^2 d\lambda(x).$$

We conclude as in the previous case using Lemmas 10 and 11 instead of Lemma 8.

Finally suppose that g is arbitrary. A similar argument as in case 1 (but now with $x_p = 0$ and $t = 1/2$) together with the already proved case of g constant on R_- or R_+ proves the assertion in this case. \square

Proof of Proposition 3. We need only to prove that assumptions of Proposition 4 are satisfied. But in view of Proposition 2 they are equivalent to the Poincaré inequality for symmetric exponential probability measure ($i = 0$) and the logarithmic Sobolev inequality for the centered $\mathcal{N}(0, \sqrt{2}/2)$ Gaussian measure ($i = 1$) which are well known to hold with some universal constants. This completes the proof. \square

In the end of the paper we would like to come back to the class Φ introduced in Definition 4. It is easy to check that if Lemma 5 holds for some function $\varphi \in C^2((0, \infty)) \cap C([0, \infty))$ for any $(\Omega_1, \mu_1), (\Omega_2, \mu_2)$ and any Z then $\varphi \in \Phi$. Indeed, it is even true if we restrict our consideration to (Ω_1, μ_1) and (Ω_2, μ_2) being two-point probability spaces whose atoms have $1/2$ measures. This gives a natural characterization of the class Φ .

One can try to generalize the definition of Φ . Let U be an open, convex subset of R^d . We will say that a continuous function $f : U \rightarrow R$ belongs to the class $C_n(U)$ if for any probability spaces $(\Omega_1, \mu_1), \dots, (\Omega_n, \mu_n)$ and any integrable random variable Z with values in U , defined on $(\Omega, \mu) = (\Omega_1 \times \dots \times \Omega_n, \mu_1 \otimes \dots \otimes \mu_n)$ the following inequality is satisfied:

$$\sum_{K \subseteq \{1, 2, \dots, n\}} (-1)^{|K|} E_{K^c} f(E_K Z) \geq 0,$$

where E_K denotes expectation with respect to μ_k for all $k \in K$. One can easily see that $C_1(U)$ is just a set of all convex functions on U , while $C_2((0, \infty))$ is closely related to the class Φ . In fact $f \in C_2((0, \infty))$ if and only if it is an affine function or it has a continuous strictly positive second derivative such

that $1/f''$ is a concave function. One can prove that always $C_{n+1}(U) \subseteq C_n(U)$ and therefore it is natural to define $C_\infty(U)$ as an intersection of all $C_n(U)$. Then it appears that $f \in C_\infty(U)$ if and only if f is given by the formula $f(x) = Q(x, x) + x^*(x) + y$, where Q is a non-negative definite symmetric quadratic form, x^* is a linear functional on R^d and y is a constant. The above inclusions do not need to be strict. For example it is easy to see that $C_2(R) = C_\infty(R)$. It would be interesting to know some nice characterization of $C_2(U)$ for general U and $C_n((0, \infty))$ for $n > 2$. It is not clear what applications of C_n for $n > 2$ could be found but it is easy to see that this class has some tensorization property. By now, we do not know even the answer to the following question: For which $p \in [1, 2]$ does $f(x) = x^p$ belong to $C_n((0, \infty))$? We can only give some estimates. These problems will be discussed in a separate paper.

Remark 4 *Recently some new results were announced to the authors by F. Barthe (private communication) - he proved (using Theorem 2 above) that if a log-concave probability measure μ on the Euclidean space $(R^n, \|\cdot\|)$ satisfies inequality $\mu(\{x \in R^n; \|x\| > t\}) \leq ce^{-(t/c)^r}$ for some constants $c > 0, r \in [1, 2]$ and any $t > 0$ then it satisfies also inequality*

$$E_\mu f^2 - (E_\mu f^p)^{2/p} \leq C(c, n, r)(2-p)^a E_\mu \|\nabla f\|^2$$

for any non-negative smooth function f on R^n and $p \in [1, 2)$, where $C(c, n, r)$ is some positive constant depending on c, n and r only and $a = 2 - 2/r$.

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