Studia Math. 169 (2005), 305–314.

Small ball probability estimate in terms of width

by

RAFAŁ LATAŁA and KRZYSZTOF OLESZKIEWICZ (Warszawa)

Abstract. A certain inequality conjectured by Vershynin is studied. It is proved that for any symmetric convex body $K \subseteq \mathbb{R}^n$ with inradius w and $\gamma_n(K) \le 1/2$ we have $\gamma_n(sK) \le (2s)^{w^2/4}\gamma_n(K)$ for any $s \in [0,1]$, where γ_n is the stanadard Gaussian probability measure. Some natural corollaries are deduced. Another conjecture of Vershynin is proved to be false.

1. Introduction. In his lecture at Snowbird'2004 AMS Conference Roman Vershynin posed two conjectures relating to the rate of decay of the Gaussian measure of convex symmetric sets under homothetic shrinking. The first conjecture concerned some bounds in terms of the width of a convex symmetric set. The second one stated that among all convex symmetric bodies with fixed both Gaussian measure and width the cylinders are the ones that have the slowest decay of Gaussian measure under homothetic shrinking. Both conjectures will be described more precisely below. In this paper we prove some version of the first conjecture (Section 2) and we demonstrate that the second conjecture cannot hold in general (Section 4). We also sketch some natural direct applications that motivated Vershynin's questions. More sophisticated geometric consequences related to the Dvoretzky theorem were recently proved by Klartag and Vershynin [4].

Let us introduce some notation and results that will be used. By γ_n we will denote the standard Gaussian probability measure on $\mathbb{R}^n$, with $\gamma_n(dx) = (2\pi)^{-n/2}e^{-|x|^2/2}dx$. For a set A in $\mathbb{R}^n$ we will write $A_t = \{x \in \mathbb{R}^n : d(x,A) < t\}$. *Gaussian isoperimetry* [1, 8] states that if $\gamma_n(A) = \Phi(x)$, then $\gamma_n(A_t) \ge \Phi(x+t)$ for all $t > 0$, where $\Phi(x) := \gamma_1((-\infty, x))$.

The *S-inequality* [6] says that if $K \subset \mathbb{R}^n$ is convex symmetric with $\gamma_n(K) = \gamma_1([-a,a])$ then $\gamma_n(tK) \le \gamma_1([-ta,ta])$ for all $0 < t < 1$. This easily implies that for some universal constant C,

$$\forall_{0\le t\le 1}\ \gamma_n(tK) \le Ct\gamma_n(K) \text{ if } \gamma_n(K) \le 1/2. \tag{1}$$

2000 *Mathematics Subject Classification*: Primary 60G15; Secondary 60E15.

Research partially supported by KBN Grant 2 PO3A 027 22.

The inequality (1) was first proved in [3] (see also its generalization to log-concave measures in [5]). Although in general one cannot improve (1), Vershynin conjectured that a stronger inequality can hold for sets of large width.

For a convex symmetric set K in $\mathbb{R}^n$ let the *inradius of* K be defined as

$$w(K) = \sup\{r > 0\colon B(0,r) \subset K\}$$

Notice also that $w(K)$ is half of the *width of* K.

The *B-inequality* proved recently by Cordero, Fradelizi and Maurey [2] implies that for any symmetric convex set K in $\mathbb{R}^n$ the function $t \mapsto \ln \gamma_n(e^t K)$ is concave. In particular

$$\forall_{0<s\le t\le 1} \quad \frac{\gamma_n(sK)}{\gamma_n(K)} \le \left(\frac{\gamma_n(tK)}{\gamma_n(K)}\right)^{\frac{\ln s}{\ln t}}. \tag{2}$$

2. Main results

Theorem 1. *For any convex symmetric set K in $\mathbb{R}^n$ with $\gamma_n(K) \le 1/2$ we have*

$$\forall_{0\le t\le 1/2} \quad \gamma_n(tK) \le t^{\frac{w^2(K)}{8\ln 2}} \gamma_n(K). \tag{3}$$

Proof. Let $s \ge 1$ be such that $\gamma_n(sK) = 1/2$. By the concavity of $t \mapsto \gamma_n(e^t K)$, we get $\gamma_n(tK)/\gamma_n(K) \le \gamma_n(tsK)/\gamma_n(sK)$ for $t \le 1$. Since $w(sK) \ge w(K)$, we may and will assume that $\gamma_n(K) = 1/2$.

Notice that $\frac{1}{2}K + \frac{1}{2}B(0,w(K)) \subset \frac{1}{2}K + \frac{1}{2}K = K$, so $(K^c)_{w(K)/2} \cap \frac{1}{2}K = \emptyset$. Thus by Gaussian isoperimetry,

$$\begin{aligned}\gamma_n\left(\frac{1}{2}K\right) &\le 1 - \gamma_n((K^c)_{w(K)/2}) \le 1 - \Phi(w(K)/2)\\ &\le \frac{1}{2}e^{-w^2(K)/8} = \gamma_n(K)\left(\frac{1}{2}\right)^{\frac{w^2(K)}{8\ln 2}}.\end{aligned}$$

We get (3) by applying (2). ∎

The following related conjecture seems reasonable.

Conjecture 1. *For any $\kappa \in (0,1)$ there exist positive constants $C = C(\kappa)$ and $w_0 = w_0(\kappa)$ such that for any convex symmetric set K in $\mathbb{R}^n$ with $\gamma_n(K) \le 1/2$ and $w(K) \ge w_0$ we have*

$$\forall_{0\le t\le 1}\ \gamma_n(tK) \le (Ct)^{\kappa w^2(K)} \gamma_n(K). \tag{4}$$

Vershynin's first conjecture (originally formulated in the language of our Theorem 4) was that the above inequality is true for some fixed $\kappa \in (0,1)$. Inequality (3) shows that (4) holds for $\kappa = 1/(8\ln 2) > \frac{1}{6}$; we will now present some more elaborate argument for $\kappa = 1/4$.

LEMMA 1. *For any $u, v \geq 0$ we have*

$$\gamma_1((u+v,\infty)) \leq e^{-uv}\gamma_1((u,\infty)).$$

Proof. We have

$$\int_{u+v}^{\infty} e^{-s^2/2}ds = \int_{u}^{\infty} e^{-(s+v)^2/2}ds \leq \int_{u}^{\infty} e^{-sv}e^{-s^2/2}ds \leq e^{-uv}\int_{u}^{\infty} e^{-s^2/2}ds. \ \blacksquare$$

THEOREM 2. *For any convex symmetric set K in $\mathbb{R}^n$ with $w = w(K)$ and $\gamma_n(K) \leq 1/2$ we have*

$$\gamma_n(sK) \leq (2s)^{w^2/4}\gamma_n(K) \quad \text{for } s \in [0,1].$$

Proof. Let us notice that $\gamma_n(K^c) \geq \frac{1}{2}$ and $\frac{1}{2}K \cap (K^c)_{w/2} = \emptyset$, hence by isoperimetry

$$\gamma_n\left(\frac{1}{2}K\right) \leq 1 - \gamma_n((K^c)_{w/2}) \leq \gamma_1((w/2,\infty)).$$

Let us define $u \geq w/2$ by the formula

$$\gamma_n\left(\frac{1}{2}K\right) = \gamma_1((u,\infty)).$$

We also have $\frac{t}{2}K \cap ((\frac{1}{2}K)^c)_{\frac{1-t}{2}w} = \emptyset$ for $0 < t < 1$, so again by isoperimetry and Lemma 1,

$$\gamma_n\left(\frac{t}{2}K\right) \leq \gamma_1\left(\left(u + \frac{1-t}{2}w, \infty\right)\right) \leq e^{-\frac{1-t}{2}wu}\gamma_1((u,\infty))$$
$$\leq e^{-\frac{1-t}{4}w^2}\gamma_n\left(\frac{1}{2}K\right).$$

Thus

$$\frac{\gamma_n(\frac{t}{2}K)}{\gamma_n(\frac{1}{2}K)} \leq \left(\frac{t/2}{1/2}\right)^{\frac{w^2}{4}\frac{t-1}{\ln t}}.$$

Hence, by the B-inequality for any $s \leq t/2$ we obtain

$$\frac{\gamma_n(sK)}{\gamma_n(\frac{1}{2}K)} \leq (2s)^{\frac{w^2}{4}\frac{t-1}{\ln t}}.$$

Taking the limit $t \to 1^-$ we get, for $s \in [0, 1/2]$,

$$\gamma_n(sK) \leq \gamma_n\left(\frac{1}{2}K\right)(2s)^{w^2/4}. \ \blacksquare$$

Before stating the next result let us introduce some notation. By σ_{n-1} we will denote the (normalized) Haar measure on $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. For a set A in $\mathbb{R}^n$ to simplify the notation we will write $\sigma_{n-1}(A)$ instead of $\sigma_{n-1}(A \cap S^{n-1})$.

THEOREM 3. *For any convex symmetric set K in $\mathbb{R}^n$ with $\sigma_{n-1}(K) \leq 1/2$ we have*

$$\forall_{0\leq t\leq 1}\ \sigma_{n-1}(tK) \leq (12t)^{\frac{1}{4}(\sqrt{n}w(K)-6)_+^2}. \tag{5}$$

The proof is based on the following simple lemma.

LEMMA 2. *There exists a universal constant $\alpha > 1/60$ such that for any star body $K \subseteq \mathbb{R}^n$ we have*

$$\gamma_n(\sqrt{n}K) \geq \alpha\sigma_{n-1}(K), \tag{6}$$

$$\gamma_n\big((\sqrt{n}K)^c\big) \geq \alpha\sigma_{n-1}(K^c). \tag{7}$$

Proof. Let

$$L = \big\{\xi \in \mathbb{R}^n\colon |\xi| \leq \sqrt{n}, \xi/|\xi| \in K \cap S^{n-1}\big\}.$$

Notice that $L \subseteq \sqrt{n}K$ and by the rotational invariance of γ_n,

$$\gamma_n(\sqrt{n}K) \geq \gamma_n(L) = \gamma_n(B(0,\sqrt{n}))\sigma_{n-1}(K).$$

Let $X = \sum_{i=1}^n (g_i^2 - 1)$, where g_i are i.i.d. $\mathcal{N}(0,1)$ r.v.'s. Since $\mathbf{E}|X|/2 = \mathbf{E}X_- \leq (\mathbf{E}X^2)^{1/2}\mathbf{P}(X \leq 0)^{1/2}$ we get

$$\gamma_n(B(0,\sqrt{n})) = \mathbf{P}(X \leq 0) \geq \frac{(\mathbf{E}|X|)^2}{4\mathbf{E}X^2} \geq \frac{(\mathbf{E}X^2)^2}{4\mathbf{E}X^4} \geq \frac{1}{60},$$

where the last inequality follows by an easy calculation, since $\mathbf{E}(g_i^2-1) = 0$, $\mathbf{E}(g_i^2-1)^2 = 2$ and $\mathbf{E}(g_i^2-1)^4 = 60$.

In a similar way we show that for

$$\widetilde{L} = \big\{\xi \in \mathbb{R}^n\colon |\xi| \geq \sqrt{n}, \xi/|\xi| \notin K \cap S^{n-1}\big\}$$

we have

$$\gamma_n\big((\sqrt{n}K)^c\big) \geq \gamma_n(\widetilde{L}) = \mathbf{P}(X \geq 0)\sigma_{n-1}(K^c) \geq \frac{1}{60}\sigma_{n-1}(K^c). \ \blacksquare$$

Proof of Theorem 3. Obviously we may assume that $\sqrt{n}w(K) > 6$ and $t < 1/12$. Let α be the constant given by the preceding lemma. If $\sigma_{n-1}(K) \leq 1/2$, then by (7), $\gamma_n((\sqrt{n}K)^c) \geq \alpha/2 > \Phi(-2.5)$. Hence by Gaussian isoperimetry (since $\operatorname{dist}((\sqrt{n}-s)K, (\sqrt{n}K)^c) \geq sw(K)$) we get

$$\begin{aligned}\gamma_n\bigg(\bigg(\sqrt{n} - \frac{5}{w(K)}\bigg)K\bigg) &\leq 1 - \gamma_n(((\sqrt{n}K)^c)_5) \leq 1 - \Phi(\Phi^{-1}(\alpha/2)+5)\\ &\leq 1 - \Phi(2.5) \leq \min(\alpha, 1/2).\end{aligned}$$

Let $t_0 = \sqrt{n} - 5(w(K))^{-1} \geq \sqrt{n}/6 > t\sqrt{n}$. Then by (6) and Theorem 2 we get

$$\sigma_{n-1}(tK) \leq \frac{1}{\alpha}\gamma_n(\sqrt{n}tK) \leq \frac{1}{\alpha}\bigg(\frac{2t\sqrt{n}}{t_0}\bigg)^{\frac{w^2(t_0K)}{4}}\gamma_n(t_0K) \leq (12t)^{\frac{1}{4}(\sqrt{n}w(K)-5)_+^2}. \blacksquare$$

3. Applications to Gaussian processes. Theorem 2 can be easily translated to the "small ball" inequality for Gaussian processes in the same way as the concentration of Gaussian measures yields the large deviation inequality for Gaussian processes (cf. [7, Theorem 7.1]).

THEOREM 4. *Let $G = (G_t)_{t\in T}$ be a centered Gaussian process indexed by a countable set T such that $\sup_{t\in T}|G_t| < \infty$ almost surely. Then for any $s \in [0,1]$ we have*

$$\mathbf{P}(\sup_{t\in T}|G_t| \le sM) \le \frac{1}{2}(2s)^{\frac{M^2}{4\sigma^2}},$$

where $M = \mathrm{Med}(\sup_{t\in T}|G_t|)$ and $\sigma = \sup_{t\in T}(\mathbf{E}G_t^2)^{1/2}$.

Proof. A standard argument (cf. [7, sect. 7.1]) shows that we may assume that T is a finite subset of $\mathbb{R}^n$ and $G_t = \sum_{i=1}^n t_i g_i$, where g_i are i.i.d. $\mathcal{N}(0,1)$ r.v.'s. Let

$$K := \Big\{\xi \in \mathbb{R}^n\colon \sup_{t\in T}\Big|\sum_{i=1}^n t_i\xi_i\Big| \le M\Big\}.$$

Obviously $\gamma_n(K) = \mathbf{P}(\sup_{t\in T}|G_t| \le M) = 1/2$, moreover if $\xi \notin K$ then for some $t \in T$ we have

$$M < \Big|\sum_{i=1}^n t_i\xi_i\Big| \le \Big(\sum_{i=1}^n t_i^2\Big)^{1/2}|\xi| \le \sigma|\xi|.$$

Thus $w(K) \ge M/\sigma$ and by Theorem 2,

$$\mathbf{P}\Big(\sup_{t\in T}|G_t| \le sM\Big) = \gamma_n(sK) \le \frac{1}{2}(2s)^{\frac{M^2}{4\sigma^2}}. \ \blacksquare$$

REMARK. The Gaussian isoperimetric inequality gives, for $s \in [0,1]$,

$$\mathbf{P}(\sup_{t\in T}|G_t| \le sM) = \mathbf{P}(\sup_{t\in T}|G_t| \le M - (1-s)M) \le 1 - \Phi((1-s)M/\sigma).$$

In particular one cannot get this way a better bound than $1 - \Phi(M/\sigma)$, and Theorem 4 obviously provides a sharper estimate for sufficiently small s.

Before formulating the next result let us recall that we define for $p \ne 0$ the pth moment of a random vector S as $\|S\|_p = (\mathbf{E}\|S\|^p)^{1/p}$ and for $p = 0$ as $\|S\|_0 = \exp(\mathbf{E}\ln\|S\|)$.

COROLLARY 1. *Under the assumptions of Theorem 4 for any $p > q > r := \min(-1, -\frac{M^2}{4\sigma^2})$ there exists a constant $C_{p,q,r}$ that depends on p, q and r only such that*

$$\big\|\sup_{t\in T}|G_t|\big\|_p \le C_{p,q,r}\big\|\sup_{t\in T}|G_t|\big\|_q.$$

Proof. Let $S := \sup_{t\in T}|G_t|$. It is well known that $\|S\|_p \le C_p M$, so it is enough to show that for each $q > r$, $\|X\|_q \ge c_{q,r}M$. However, by (1) and Theorem 4 we get $\mathbf{P}(\|X\| \leq tM) \leq (Ct)^{-r}$ and the desired estimate immediately follows. ∎

4. Counterexample to Vershynin's conjecture. Vershynin conjectured that cylinders have the slowest decay of the Gaussian measure under homothethic shrinking among all centrally symmetric convex bodies with fixed width and Gaussian measure. Namely, he conjectured that if $C = B_2^k(0,w) \times \mathbb{R}^l$ (where $B_2^k(0,w) = \{x \in \mathbb{R}^k : |x| \leq w\}$) and K is a centrally symmetric convex body such that $\gamma(K) = \gamma_{k+l}(C)$ and $w(K) = w = w(C)$ then $\gamma(tK) \leq \gamma_{k+l}(tC)$ for any $t \in (0,1)$. Certainly, the dimension parameter l is a bit artificial here and one can easily reduce the problem to the case $l = 0$. This conjecture seemed naturally related to Conjecture 1. Indeed, if a cylinder C has large width and $\gamma_{k+l}(C) = 1/2$ then $\gamma_k(B_2^k(0,w)) = 1/2$ and w must be close to $\sqrt{k}$. On the other hand, $\log_t \gamma_{k+l}(tC) \to k$ as $t \to 0+$. Although Conjecture 1 still seems open, we prove that Vershynin's cylinder conjecture cannot hold in general.

For simplicity the counterexample will be produced for $k = 2$ but one can easily extend our construction to any $k > 2$.

Let us recall that for a set A in $\mathbb{R}^n$

$$\gamma_n^+(A) := \liminf_{t\to 0+} \frac{\gamma_n(A_t) - \gamma_n(A)}{t}.$$

We begin with two simple and quite standard lemmas.

LEMMA 3. *There exist positive numbers w and a such that $\gamma_2(B_2^2(0,w)) = \gamma_1((-a,a))$ and $\gamma_2^+(B_2^2(0,w)) > \gamma_1^+((-a,a))$.*

Proof. Let w and a be positive numbers such that $\gamma_2(B_2^2(0,w)) = \gamma_1((-a,a))$. We will prove that if a is large enough then $\gamma_2^+(B_2^2(0,w)) > \gamma_1^+((-a,a))$. Let us recall a standard estimate for the Gaussian tails:

$$a^{-1}e^{-a^2/2} - \int_a^\infty e^{-s^2/2}ds = \int_a^\infty \frac{d}{dx}\Big(\int_x^\infty e^{-s^2/2}ds - x^{-1}e^{-x^2/2}\Big)dx = \int_a^\infty x^{-2}e^{-x^2/2}dx,$$

so that

$$\int_a^\infty e^{-s^2/2}ds \leq a^{-1}e^{-a^2/2}$$

and

$$\int_a^\infty e^{-s^2/2}ds \geq a^{-1}e^{-a^2/2} - a^{-3}\int_a^\infty xe^{-x^2/2}dx = (a^{-1} - a^{-3})e^{-a^2/2}.$$

Since

$$e^{-w^2/2} = 1 - \gamma_2\big(B_2^2(0,w)\big) = 1 - \gamma_1((-a,a)) = \frac{2}{\sqrt{2\pi}} \int_a^\infty e^{-s^2/2} ds$$

$$\le \frac{2}{\sqrt{2\pi}} a^{-1} e^{-a^2/2}$$

we have $w \ge a + a^{-1}\ln a + o(a^{-1}\ln a)$ as $a \to \infty$. Therefore for sufficiently large a we have

$$\begin{aligned}\gamma_2^+(B_2^2(0,w)) &= we^{-w^2/2} > (a+2a^{-1})e^{-w^2/2} \\ &= (a+2a^{-1})\frac{2}{\sqrt{2\pi}}\int_a^\infty e^{-s^2/2}ds \\ &\ge \frac{2}{\sqrt{2\pi}}(a+2a^{-1})(a^{-1}-a^{-3})e^{-a^2/2} > \frac{2}{\sqrt{2\pi}}e^{-a^2/2} \\ &= \gamma_1^+((-a,a)). \ \blacksquare\end{aligned}$$

LEMMA 4. *If a sequence of positive numbers $u(n)$ satisfies*

$$\liminf_{n\to\infty} \gamma_n\big(B_2^n(0,u(n))\big) > 0$$

then $\liminf_{n\to\infty} n^{-1/2}u(n) \ge 1$. *Conversely, if a sequence of positive numbers $v(n)$ satisfies* $\liminf_{n\to\infty} n^{-1/2}v(n) > 1$ *then*

$$\lim_{n\to\infty} \gamma_n\big(B_2^n(0,v(n))\big) = 1.$$

Proof. Let $g_1, g_2, \ldots$ be i.i.d. $\mathcal{N}(0,1)$ random variables. The assertion easily follows from the observation that

$$\gamma_n(B_2^n(0,r)) = \mathbf{P}\Big(\sum_{i=1}^n g_i^2 < r^2\Big)$$

and the Law of Large Numbers. ∎

Contruction of the counterexample. Let $w, a>0$ be such that $\gamma_2(B_2^2(0,w)) = \gamma_1((-a,a))$ and $\gamma_2^+(B_2^2(0,w)) > \gamma_1^+((-a,a))$ and let $t \in (0,1)$. For $x \in (0,w)$ let

$$y = y(x) = \frac{x}{2} + \frac{w^2}{2x} \quad \text{and} \quad s = s(x) = \frac{2w^2x}{w^2+x^2}.$$

Note that $x < s < y$. We define a continuous function $f : [0,\infty) \to [0,w]$ by

$$f(r) := \begin{cases} w & \text{for } r \in [0,x], \\ \frac{(y-r)w}{y-x} & \text{for } r \in (x,y), \\ 0 & \text{for } r > y. \end{cases}$$

Let $K_n = K_n(x)$ be a *flying saucer* body defined by

$$K_n = \Big\{\xi \in \mathbb{R}^n : f\Big(\Big(\sum_{i=1}^{n-1} \xi_i^2\Big)^{1/2}\Big) > |\xi_n|\Big\}.$$

Clearly, K_n is a convex body contained in the symmetric strip $S = \{\xi \in \mathbb{R}^n : |\xi_n| \leq w\}$. The line $\{(\xi_1, \xi_2)\colon \xi_2 = (y - \xi_1)w/(y-x)\}$ is tangent to the ball $B_2^2(0,w)$ and the point of tangency is $(s, f(s))$. Thus K_n contains the inscribed Euclidean ball $B_2^n(0,w)$. Hence the inradius $w(K_n)$ equals w. For sufficiently large n one can choose $x \in (0,w)$ such that $\gamma_n(K_n) = \gamma_2(B_2^2(0,w))$. Indeed, for fixed n, the body K_n tends to S as $x \to 0^+$ and it tends to $C_n = B_2^{n-1}(0,w) \times (-w,w)$ as $x \to w^-$. Since $\gamma_n(S) > \gamma_2(B_2^2(0,w))$ and $\gamma_n(C_n) \to 0$ as $n \to \infty$, our claim follows by continuity. Moreover, $x = x(n) \to 0$ as $n \to \infty$. Indeed, it suffices to note that $y(n) = y(x(n)) \to \infty$ since

$$\begin{aligned}\liminf_{n\to\infty} \gamma_{n-1}\big(B_2^{n-1}(0, y(n))\big) &\geq \liminf_{n\to\infty} \gamma_n(B_2^{n-1}(0,y(n)) \times (-w,w)) \\ &\geq \liminf \gamma_n(K_n) = \gamma_2(B_2^2(0,w)) > 0.\end{aligned}$$

From now on we assume that n is large enough and $x = x(n)$ is such that $\gamma_n(K_n) = \gamma_2(B_2^2(0,w))$. Since $\gamma_1((-a,a)) = \gamma_2(B_2^2(0,w)) < \gamma_1((-w,w))$, we have $w > a$. Let b, c and d be positive numbers such that $a > b > c > d$. Let $u, v, z \in (x,y)$ be such that $f(u) = b, f(v) = c$ and $f(z) = d$. Simple calculations show that

$$\begin{aligned} u &= x + \frac{1}{2}\Big(\frac{w}{x} - \frac{x}{w}\Big)(w-b), \\ v &= x + \frac{1}{2}\Big(\frac{w}{x} - \frac{x}{w}\Big)(w-c) \\ z &= x + \frac{1}{2}\Big(\frac{w}{x} - \frac{x}{w}\Big)(w-d). \end{aligned}$$

Hence $v/u \to \frac{w-c}{w-b} > 1$ and $v/z \to \frac{w-c}{w-d} < 1$ as $n \to \infty$. Since

$$K_n \subset (\mathbb{R}^{n-1} \times (-b,b)) \cup (B_2^{n-1}(0,u) \times ([b,w) \cup (-w,-b]))$$

we have

$$\begin{aligned}\gamma_1((-a,a)) &= \gamma_2(B_2^2(0,w)) = \gamma_n(K_n) \\ &\leq \gamma_1(-b,b) + 2\gamma_1((b,w))\gamma_{n-1}(B_2^{n-1}(0,u))\end{aligned}$$

and therefore

$$\liminf_{n\to\infty} \gamma_{n-1}(B_2^{n-1}(0,u)) > 0,$$

so that $\liminf_{n\to\infty} n^{-1/2}u \geq 1$. Hence $\liminf_{n\to\infty} n^{-1/2}v \geq \frac{w-c}{w-b} > 1$ and consequently

$$\lim_{n\to\infty} \gamma_{n-1}(B_2^{n-1}(0,v)) = 1.$$

If $t > \frac{w-c}{w-d}$ then

$$\liminf_{n\to\infty} \frac{tz}{v} \geq t\frac{w-d}{w-c} > 1,$$

so that $tz \geq v$ for n large enough. Moreover

$$B_2^{n-1}(0,z) \times (-d,d) \subset K_n,$$

so

$$B_2^{n-1}(0,tz) \times (-td,td) \subset tK_n$$

and (assuming Vershynin's cylinder conjecture is true) we deduce that for any $\frac{w-c}{w-d} < t < 1$ (and sufficiently large n),

$$\begin{aligned}\gamma_2(B_2^2(0,tw)) = \gamma_2(tB_2^2(0,w)) \geq \gamma_n(tK_n) \geq \gamma_{n-1}(B_2^{n-1}(0,tz))\gamma_1((-td,td)) \\ \geq \gamma_{n-1}(B_2^{n-1}(0,v))\gamma_1((-td,td)) \to \gamma_1((-td,td))\end{aligned}$$

as $n \to \infty$. We have proved $\gamma_2(B_2(0,tw)) \geq \gamma_1((-td,td))$ if only $\frac{w-c}{w-d} < t < 1$. Note however that for a fixed $t \in (0,1)$ one can set $b, c \to a$ and $d \to (a-(1-t)w)/t$ in such way that $t > \frac{w-c}{w-d}$. Then $td \to a-(1-t)w$ and we deduce that

$$\gamma_2\big(B_2^2(0,tw)\big) \geq \gamma_1((-(a-(1-t)w), a-(1-t)w)) \text{ for any } t \in (0,1).$$

The above inequality becomes an equality for $t = 1$ so by differentiating at $t = 1$ we obtain

$$\gamma_2^+\left(B_2^2(0,w)\right) \leq \gamma_1^+((-a,a)),$$

contrary to the choice w and a at the beginning. This proves that Vershynin's cylinder conjecture cannot be true in general. □

Remark. Note that one really needs some "extra" dimensions in the construction of the counterexample. If we assume that $K \subset \mathbb{R}^k$ is a convex symmetric body with inradius w and $\gamma_k(K) = \gamma_k(B_2^k(0,w))$ then obviously K must be equal to $B_2^k(0,w)$ up to some boundary points, so that also $\gamma_k(tK) = \gamma_k(B_2^k(0,tw))$ for $t \in (0,1)$ even though the Euclidean ball has *the fastest* decay of the Gaussian measure under homothetic shrinking among all bodies of a fixed Gaussian measure.

Acknowledgments. Part of this research was done during the Snowbird AMS-IMS-SIAM Conference "Gaussian Measure and Geometric Convexity" in July 2004. We would like to thank the organizers for creating excellent research conditions.

References

[1] C. Borell, *The Brunn-Minkowski inequality in Gauss space*, Invent. Math. 30 (1975), 207–216.

[2] D. Cordero-Erausquin, M. Fradelizi and B. Maurey, *The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems*, J. Func. Anal. 114 (2004), 410–427.

[3] P. Hitczenko, S. Kwapień, V. Li, G. Schechtman, T. Schlumprecht and J. Zinn, *Hypercontractivity and comparison of moments of iterated maxima and minima of independent random variables*, Electron. J. Probab. 3 (1998), No. 2, 26 pp.

[4] B. Klartag and R. Vershynin, *Small ball probability and Dvoretzky Theorem*, to appear.

[5] R. Latała, *On the equivalence between geometric and arithmetic means for log-concave measures*, in: Convex Geometric Analysis (Berkeley, CA, 1996), Math. Sci. Res. Inst. Publ. 34, Cambridge Univ. Press, Cambridge 1999, 123–127.

[6] R. Latała and K. Oleszkiewicz, *Gaussian measures of dilatations of convex symmetric sets*, Ann. Probab. 27 (1999), 1922–1938.

[7] M. Ledoux, *The Concentration of Measure Phenomenon*, Math. Surveys Monogr. 89, Amer. Math. Soc., Providence, RI, 2001.

[8] V. N. Sudakov and B. S. Tsirel'son, *Extremal properties of half-spaces for spherically invariant measures*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 41 (1974), 14–24 (in Russian).

Institute of Mathematics
Warsaw University
Banacha 2
02-097 Warsaw, Poland
E-mail: rlatala@mimuw.edu.pl,
koles@mimuw.edu.pl