

SOME ESTIMATES OF NORMS OF RANDOM MATRICES

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ABSTRACT. We show that for any random matrix (X_{ij}) with independent mean zero entries

$$\mathbf{E}\|(X_{ij})\| \leq C \left(\max_i \sqrt{\sum_j \mathbf{E}X_{ij}^2} + \max_j \sqrt{\sum_i \mathbf{E}X_{ij}^2} + \sqrt[4]{\sum_{ij} \mathbf{E}X_{ij}^4} \right),$$

where C is some universal constant.

1. INTRODUCTION

In this paper we consider the Euclidean operator norm

$$\|(a_{ij})\| = \|(a_{ij})\|_{l^2 \rightarrow l^2} := \sup \left\{ \sum_{i,j} a_{ij} x_i y_j : \sum x_i^2 \leq 1, \sum y_j^2 \leq 1 \right\}$$

of random matrices whose entries are independent random variables. The case of $n \times m$ matrices with iid entries is quite well understood - the operator norm is (under some mild assumptions on the underlying distribution) of order $\max(\sqrt{n}, \sqrt{m})$ (cf. [1, 6, 7] for the Gaussian case and [2, 8] for general iid entries).

Recently Y. Seginer [4] showed that for any $n \times m$ random matrix $(X_{ij})_{i \leq n, j \leq m}$ with iid mean zero entries

$$\mathbf{E}\|(X_{ij})\| \leq C \left(\mathbf{E} \max_{i \leq n} \sqrt{\sum_{j \leq m} X_{ij}^2} + \mathbf{E} \max_{j \leq m} \sqrt{\sum_{i \leq n} X_{ij}^2} \right),$$

where C is an universal constant.

However if we do not assume the identical distribution (or at least identical variances) of the entries very little seems to be known about the average operator norm of random matrices, even in the Gaussian case. In the paper we present some general estimates for norms of random matrices with independent mean zero entries, assuming only the existence of the 4th moments. In the iid case our estimates are of the right order $\max(\sqrt{n}, \sqrt{m})$.

By (g_{ij}) we denote the iid sequence of standard $\mathcal{N}(0, 1)$ normal r.v.'s. We use the letter C for universal constants that may change from line to line.

2. MAIN RESULTS

The purpose of the paper is to prove the following

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Theorem 1. For any finite matrix (a_{ij}) the following estimate holds

$$(1) \quad \mathbf{E}\|(a_{ij}g_{ij})\| \leq C \left(\max_i \sqrt{\sum_j a_{ij}^2} + \max_j \sqrt{\sum_i a_{ij}^2} + \sqrt[4]{\sum_{ij} a_{ij}^4} \right).$$

The next theorem is formally stronger, but it is almost immediate consequence of Theorem 1

Theorem 2. For any finite matrix of independent mean zero r.v.'s X_{ij} we have

$$(2) \quad \mathbf{E}\|(X_{ij})\| \leq C \left(\max_i \sqrt{\sum_j \mathbf{E}X_{ij}^2} + \max_j \sqrt{\sum_i \mathbf{E}X_{ij}^2} + \sqrt[4]{\sum_{ij} \mathbf{E}X_{ij}^4} \right).$$

Remark 1. In the case when (X_{ij}) is an $n \times m$ matrix with iid mean zero entries such that $\mathbf{E}|X_{ij}|^4 < \infty$ Theorem 2 gives the right order estimate $\mathbf{E}\|(X_{ij})\| \leq C(\mathbf{E}X_{ij}^2, \mathbf{E}X_{ij}^4) \max(\sqrt{n}, \sqrt{m})$.

Remark 2. The presence of fourth moments in Theorem 2 is not surprising. The result of Silverstein [5] states that if $X_{i,j}$ are iid mean zero random variables such that the norms of random matrices $n^{-1/2}\|(X_{i,j})_{1 \leq i,j \leq n}\|$ are stochastically bounded then $\mathbf{E}|X_{i,j}|^p < \infty$ for any $p < 4$.

Proof of Theorem 2. Let \tilde{X}_{ij} be an independent copy of r.v.'s X_{ij} and ε_{ij} be a sequence of independent symmetric Bernoulli r.v.'s (i.e. $\mathbf{P}(\varepsilon_{ij} = \pm 1) = 1/2$), independent of all other r.v.'s. Then by Jensen's inequality

$$\mathbf{E}\|(X_{ij})\| \leq \mathbf{E}\|(X_{ij} - \tilde{X}_{ij})\| = \mathbf{E}\|(\varepsilon_{ij}(X_{ij} - \tilde{X}_{ij}))\| \leq 2\mathbf{E}\|(\varepsilon_{ij}X_{ij})\|.$$

Thus it is enough to show (2) in the case of symmetric r.v.'s (X_{ij}) . We have

$$\mathbf{E}\|(a_{ij}g_{ij})\| = \mathbf{E}\|(a_{ij}\varepsilon_{ij}|g_{ij})\| \geq \mathbf{E}_\varepsilon\|(a_{ij}\varepsilon_{ij}\mathbf{E}|g_{ij}|)\| = \sqrt{\frac{2}{\pi}}\mathbf{E}\|(a_{ij}\varepsilon_{ij})\|.$$

Therefore Theorem 1 implies (2) for $X_{ij} = a_{ij}\varepsilon_{ij}$. By conditioning we get

$$\mathbf{E}\|(X_{ij})\|^2 = \mathbf{E}_X \mathbf{E}_\varepsilon\|(\varepsilon_{ij}X_{ij})\|^2 \leq C\mathbf{E}\left(\max_i \sum_j X_{ij}^2 + \max_j \sum_i X_{ij}^2 + \sqrt{\sum_{ij} X_{ij}^4} \right).$$

Obviously

$$\mathbf{E}\sqrt{\sum_{ij} X_{ij}^4} \leq \sqrt{\sum_{ij} \mathbf{E}X_{ij}^4}.$$

To estimate the other terms we notice first that

$$\begin{aligned} \mathbf{E} \max_i \sum_j (X_{ij}^2 - \mathbf{E}X_{ij}^2) &\leq \mathbf{E} \sqrt{\sum_i \left(\sum_j (X_{ij}^2 - \mathbf{E}X_{ij}^2) \right)^2} \\ &\leq \sqrt{\sum_i \mathbf{E} \left(\sum_j (X_{ij}^2 - \mathbf{E}X_{ij}^2) \right)^2} = \sqrt{\sum_{ij} \text{Var}(X_{ij}^2)} \leq \sqrt{\sum_{ij} \mathbf{E}X_{ij}^4}. \end{aligned}$$

Hence

$$\mathbf{E} \max_i \sum_j X_{ij}^2 \leq \mathbf{E} \max_i \sum_j (X_{ij}^2 - \mathbf{E}X_{ij}^2) + \max_i \sum_j \mathbf{E}X_{ij}^2 \leq \sqrt{\sum_{ij} \mathbf{E}X_{ij}^4} + \max_i \sum_j \mathbf{E}X_{ij}^2.$$

In similar way we show

$$\mathbf{E} \max_j \sum_i X_{ij}^2 \leq \sqrt{\sum_{ij} \mathbf{E} X_{ij}^4} + \max_j \sum_i \mathbf{E} X_{ij}^2. \square$$

3. PROOF OF THEOREM 1

Lemma 1 (Gaussian concentration). *Suppose that g_n are iid $\mathcal{N}(0, 1)$ r.v.'s and*

$$G = \sup_i (l_i + \sum_n k_{i,n} g_n).$$

Then

$$\mathbf{P}(G \geq (EG^2)^{1/2} + wt) \leq \mathbf{P}(G \geq EG + wt) \leq e^{-t^2/2},$$

where

$$w = \sup_i \sqrt{\sum_n k_{i,n}^2}.$$

Proof. We may treat G as a w -Lipschitz function of variables g_n and the statement is an immediate consequence of concentration of Gaussian measures (cf. [3, Chapter 5]). \square

Lemma 2. *Suppose that r.v. Y_x, A_x and numbers $b_x \geq 0$ satisfy the inequality*

$$\mathbf{P}(Y_x \geq A_x + tb_x) \leq e^{-t} \text{ for all } t > 0, x \in U.$$

Then for all $u \geq -\log \#U$

$$(3) \quad \mathbf{E} \max_{x \in U} [Y_x - A_x - b_x(u + \log \#U)] \leq e^{-u} \max_{x \in U} b_x.$$

In particular

$$\mathbf{E} \max_{x \in U} [Y_x - A_x - b_x \log \#U] \leq \max_{x \in U} b_x.$$

Proof. Let $Z_x = Y_x - A_x - b_x(u + \log \#U)$, then

$$\begin{aligned} \mathbf{E} \max_{x \in U} Z_x &\leq \int_0^\infty \mathbf{P}(\max_{x \in U} Z_x \geq s) ds \leq \sum_{x \in U} \int_0^\infty \mathbf{P}(Y_x \geq A_x + b_x(u + \log \#U + \frac{s}{b_x})) ds \\ &\leq \sum_{x \in U} e^{-u} \frac{1}{\#U} \int_0^\infty e^{-s/b_x} ds = e^{-u} \frac{1}{\#U} \sum_{x \in U} b_x \leq e^{-u} \max_{x \in U} b_x \end{aligned}$$

and (3) follows. \square

Lemma 3. *Suppose that $g_{i,j}$ are iid $\mathcal{N}(0, 1)$ r.v., U is a finite subset of l_2 and for each $x \in U$ there are given sets $I(x)$ and r.v.'s $(L_j(x))_j$ independent of $(g_{ij})_{i \in I(x), j}$. Then for any $c > 0$*

$$\begin{aligned} (4) \quad &\mathbf{E} \max_{x \in U} \left[\sum_j (L_j(x) + \sum_{i \in I(x)} a_{ij} g_{ij} x_i)^2 - (1+c) \sum_j (L_j(x))^2 + \sum_{i \in I(x)} a_{ij}^2 x_i^2 \right. \\ &\quad \left. - 2(1+c^{-1}) \max_j \sum_{i \in I(x)} a_{ij}^2 x_i^2 \log \#U \right] \\ &\leq 2(1+c^{-1}) \max_{x \in U} \max_j \sum_{i \in I(x)} a_{ij}^2 x_i^2. \end{aligned}$$

Proof. Let

$$G_x = \sqrt{\sum_j (L_j(x) + \sum_{i \in I(x)} a_{ij} g_{ij} x_i)^2} = \sup_{\|y\|_2 \leq 1} \sum_j (L_j(x) + \sum_{i \in I(x)} a_{ij} g_{ij} x_i) y_j.$$

Then

$$\mathbf{E}_g G_x^2 = \sum_j (L_j(x)^2 + \sum_{i \in I(x)} a_{ij}^2 x_i^2),$$

where \mathbf{E}_g denotes the expectation with respect to $(g_{ij})_{i \in I(x)}$. If we put

$$w(x) = \sup_{\|y\|_2 \leq 1} \sqrt{\sum_j \sum_{i \in I(x)} a_{ij}^2 x_i^2 y_j^2} = \max_j \sqrt{\sum_{i \in I(x)} a_{ij}^2 x_i^2},$$

then since

$$(\sqrt{\mathbf{E}_g G_x^2} + tw(x))^2 \leq (1+c)\mathbf{E}_g G_x^2 + (1+c^{-1})t^2 w^2(x)$$

we get by Lemma 1

$$\begin{aligned} \mathbf{P}(G_x^2 \geq (1+c)\mathbf{E}_g G_x^2 + (1+c^{-1})t^2 w^2(x)) \\ \leq \mathbf{P}(G_x \geq \sqrt{\mathbf{E}_g G_x^2} + tw(x)) \leq e^{-t^2/2}. \end{aligned}$$

Therefore Lemma 3 follows by Lemma 2 applied to

$$Y_x = G_x^2, A_x = (1+c) \sum_j (L_j(x)^2 + \sum_{i \in I(x)} a_{ij}^2 x_i^2)$$

and

$$b_x = 2(1+c^{-1})w^2(x). \square$$

Before stating the next lemma let us introduce some notation

$$D_2 = \{x \in l_2 : \|x\|_2 \leq 1, \forall_i \exists_{k \in \mathbf{Z}} x_i^2 = 2^{-k}\},$$

$$D_2^n = \{x \in l_2^{2^n} : \|x\|_2 \leq 1, \forall_i \exists_{k \in \mathbf{Z}} x_i^2 = 2^{-k}\}.$$

With a little abuse of notation we will treat $l_2^{2^n}$ as a subset of l_2 and D_2^n as a subset of D_2 . For $x \in D_2$, $k \in \mathbf{Z}_+$ let

$$A_k(x) = \{i : i \geq 2^k, x_i^2 \geq 2^{-k} \text{ or } 2^l \leq i < 2^{l+1}, x_i^2 \geq 2^{l-2k}, l = 0, 1, \dots, k-1\},$$

$$\pi_k(x) = (\pi_{k,i}(x))_i, \text{ where } \pi_{k,i}(x) = x_i I_{A_k(x)}(i).$$

Finally let

$$\Pi_k^n = \{\pi_k(x) : x \in D_2^n\}.$$

Lemma 4. For any $k, n \in \mathbf{Z}_+$ we have

$$\log \#\Pi_k^n \leq \begin{cases} C2^k(1+n-k) & \text{for } k \leq n \\ C2^n(1+k-n) & \text{for } k \geq n \end{cases}.$$

Proof. Let

$$U_l^n = \{x \in l_2^{2^n} : \forall_i x_i^2 = 2^{-l} I_{x_i \neq 0}\} \text{ and } V_l^n = \{x \in l_2^{2^n} : \forall_i \exists_{m=0,1,\dots,l} x_i^2 = 2^{-m} I_{x_i \neq 0}\}.$$

Notice that if $x \in U_l^n$ then $\#\{i : x_i \neq 0\} \leq 2^l$, so for $l \leq n$

$$\#U_l^n \leq \sum_{k=0}^{2^l} \binom{2^n}{k} 2^k \leq 1 + \sum_{k=1}^{2^l} \left(\frac{2e2^n}{k}\right)^k \leq 1 + 2^l (2e2^{n-l})^{2^l} \leq \exp(C2^l(1+n-l)),$$

where the second inequality follows by the classical estimate $\binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{en}{k}\right)^k$ and the third one by the monotonicity of the function $x \rightarrow a^x x^{-x}$ on $(0, a/e]$. For $l \geq n$ we get

$$\#U_l^n \leq \sum_{k=0}^{2^n} \binom{2^n}{k} 2^k = 3^{2^n}.$$

Now notice that for any $a, b > 0$ there exist finite constants $C_{a,b}$ and $C'_{a,b}$ depending only on a and b such that

$$(5) \quad \sum_{l=-\infty}^m 2^{al}(1+b(n-l)) \leq C_{a,b} 2^{am}(1+b(n-m)) \text{ for } m \leq n$$

and

$$(6) \quad \sum_{l=m}^{\infty} 2^{-al}(1+b(l-n)) \leq C'_{a,b} 2^{-am}(1+b(m-n)) \text{ for } m \geq n.$$

Indeed, observe that for $l \leq m \leq n$, $1+b(n-l) \leq (1+b(m-l))(1+b(n-m))$, so

$$\begin{aligned} \sum_{l=-\infty}^m 2^{al}(1+b(n-l)) &\leq 2^{am}(1+b(n-m)) \sum_{l=-\infty}^m 2^{a(l-m)}(1+b(m-l)) \\ &= 2^{am}(1+b(n-m)) \sum_{k=0}^{\infty} 2^{-ak}(1+bk) \end{aligned}$$

and (5) immediately follows. The proof of (6) is entirely similar.

Thus if we use (5) with $a = b = 1$ and the previously obtained estimates of $\#U_l^n$ we get that for any n, k

$$\log \#V_k^n \leq \sum_{l=0}^k \log \#U_l^n \leq \begin{cases} C2^k(1+n-k) & \text{for } k \leq n \\ C2^n(1+k-n) & \text{for } k \geq n \end{cases}.$$

Now, for $k \leq n$ we have

$$\#\Pi_k^n \leq \#V_k^n \prod_{l=0}^{k-1} \#V_{2k-l}^l,$$

so by (5) with $a = 1, b = 2$ we obtain

$$\log \#\Pi_k^n \leq C[2^k(1+n-k) + \sum_{l=0}^{k-1} 2^l(1+2k-2l)] \leq C2^k(1+n-k).$$

Simirarily for $k \geq n$

$$\#\Pi_k^n \leq \prod_{l=0}^n \#V_{2k-l}^l$$

and therefore

$$\log \#\Pi_k^n \leq C \sum_{l=0}^n 2^l(1+2k-2l) \leq C2^n(1+k-n). \square$$

Lemma 5. *For any $x \in l^2$ with $\|x\|_2 \leq 1$ we can find $y \in D_2$ such that $\|x-y\|_2 \leq 3/10$. Therefore for any linear operator T on l^2 we have*

$$\sup_{\|x\| \leq 1} \|Tx\| \leq \frac{10}{7} \sup_{x \in D_2} \|Tx\|.$$

Proof. To prove the first part of the statement it is enough to show

$$\forall_{a \in [-1,1]} \exists_b b^2 = 2^{-k} I_{b \neq 0}, b^2 \leq a^2, |a-b| \leq \frac{3}{10}|a|.$$

If $a = 0$ we take $b = 0$, so we may assume $2^{-k} \leq a^2 \leq 2^{-k+1}$ and put $b = \operatorname{sgn}(a)2^{-k/2}$, then

$$\frac{|a-b|}{|a|} = \left(1 - \frac{|b|}{|a|}\right) \leq \left(1 - \frac{1}{\sqrt{2}}\right) \leq \frac{3}{10}.$$

Therefore any $x \in l^2$ with $\|x\|_2 \leq 1$ can be represented as

$$x = \sum_{i \geq 0} \left(\frac{3}{10}\right)^i y_i, \text{ where } y_0, y_1, \dots \in D_2$$

and the last part of the lemma immediately follows. \square

Now we are ready to formulate the main technical lemma needed for the proof of Theorem 1.

Lemma 6. *Let $\Delta = \max_j (\sum_{i < 2^n} a_{ij}^4)^{1/4}$ then*

$$\begin{aligned} \mathbf{E} \sup_{x \in D_2} \left[\sum_j \left(\sum_{i < 2^n} a_{ij} g_{ij} x_i \right)^2 - C_1 \sum_{ij} a_{ij}^2 x_i^2 - C_1 \Delta^2 2^{\frac{n}{2}} \sum_{l=1}^n 2^{\frac{l-n}{8}} \sqrt{\sum_{2^{l-1} \leq i < 2^l} x_i^2} \right] \\ \leq C_1 \Delta^2 2^{\frac{n}{4}}. \end{aligned}$$

Proof. For $x \in D_2^n$, $l, n = 1, 2, \dots$ let us define

$$A_l^n(x) = \{i < 2^n : i \in A_l(x)\},$$

$$B_0^n(x) = A_0^n(x) = \{i < 2^n : x_i^2 = 1\}, b_0(x) = 2^{\frac{n}{4}} \log \#\Pi_0^n \max_j \sum_{i \in B_0^n(x)} a_{ij}^2 x_i^2,$$

$$\begin{aligned} B_l^n(x) &= A_l^n(x) \setminus A_{l-1}^n(x) = \{i < 2^n : i \geq 2^{l-1}, x_i^2 = 2^{-l} \\ &\text{or } 2^{m-1} \leq i < 2^m, x_i^2 \in \{2^{m-2l}, 2^{m-1-2l}\}, m = 1, 2, \dots, l\}, \end{aligned}$$

$$c_l(x) = \sqrt{\sum_{2^{l-1} \leq i < 2^l} x_i^2}, b_l(x) = 2^{\frac{l-n}{4}} \log \#\Pi_l^n \max_j \sum_{i \in B_l^n(x)} a_{ij}^2 x_i^2$$

and

$$d_k = \prod_{l=0}^k (1 + 2^{-\frac{l-n}{4}}).$$

We first prove by induction on $k = 0, 1, \dots$ that

$$(7) \quad \begin{aligned} & \mathbf{E} \sup_{x \in D_2^n} \left[\sum_j \left(\sum_{i \in A_k^n(x)} a_{ij} g_{ij} x_i \right)^2 - d_k \sum_{i \in A_k^n(x), j} a_{ij}^2 x_i^2 - 4d_k \sum_{l=0}^k b_l(x) \right] \\ & \leq 4d_k \sum_{l=0}^k 2^{\frac{|n-l|}{4}} \sup_{x \in D_2^n} \max_j \sum_{i \in B_l^n(x)} a_{ij}^2 x_i^2. \end{aligned}$$

For $k = 0$ inequality (7) follows by Lemma 3 applied to $U = \Pi_0^n$, $L_j(x) = 0$, $I(x) = B_0^n(x)$ and $c = 2^{-n/4}$.

To show the inductive step let us denote

$$S_k(x) = \sum_j \left(\sum_{i \in A_k^n(x)} a_{ij} g_{ij} x_i \right)^2 - d_k \sum_{i \in A_k^n(x), j} a_{ij}^2 x_i^2 - 4d_k \sum_{l=0}^k b_l(x).$$

Then

$$\begin{aligned} & S_k(x) - (1 + 2^{-\frac{|n-k|}{4}}) S_{k-1}(x) \\ & \leq \sum_j \left(\sum_{i \in A_k^n(x)} a_{ij} g_{ij} x_i \right)^2 - (1 + 2^{-\frac{|n-k|}{4}}) \sum_j \left(\sum_{i \in A_{k-1}^n(x)} a_{ij} g_{ij} x_i \right)^2 \\ & \quad - (1 + 2^{-\frac{|n-k|}{4}}) \sum_j \sum_{i \in B_k^n(x)} a_{ij}^2 x_i^2 - 2(1 + 2^{-\frac{|n-k|}{4}}) \log \# \Pi_k^n \max_j \sum_{i \in B_k^n(x)} a_{ij}^2 x_i^2. \end{aligned}$$

Lemma 3 applied to

$$U = \Pi_k^n, L_j(x) = \sum_{i \in A_{k-1}^n(x)} a_{ij} g_{ij} x_i, I(x) = B_k^n(x), c = 2^{-\frac{|n-k|}{4}}$$

implies

$$\mathbf{E} \sup_{x \in D_2^n} [S_k(x) - (1 + 2^{-\frac{|n-k|}{4}}) S_{k-1}(x)] \leq 2(1 + 2^{-\frac{|n-k|}{4}}) \sup_{x \in D_2^n} \max_j \sum_{i \in B_k^n(x)} a_{ij}^2 x_i^2.$$

The induction step from $k-1$ to k easily follows, since

$$\mathbf{E} \sup_{x \in D_2^n} S_k(x) \leq \mathbf{E} \sup_{x \in D_2^n} [S_k(x) - (1 + 2^{-\frac{|n-k|}{4}}) S_{k-1}(x)] + (1 + 2^{-\frac{|n-k|}{4}}) \mathbf{E} \sup_{x \in D_2^n} S_{k-1}(x).$$

When we take $k \rightarrow \infty$ in (7) we get

$$(8) \quad \begin{aligned} & \mathbf{E} \sup_{x \in D_2^n} \left[\sum_j \left(\sum_{i < 2^n} a_{ij} g_{ij} x_i \right)^2 - d_\infty \sum_{i, j} a_{ij}^2 x_i^2 - 4d_\infty \sum_{l=0}^{\infty} b_l(x) \right] \\ & \leq 4d_\infty \sum_{l \geq 0} 2^{\frac{|n-l|}{4}} \sup_{x \in D_2^n} \max_j \sum_{i \in B_l^n(x)} a_{ij}^2 x_i^2, \end{aligned}$$

where

$$d_\infty = \lim_{k \rightarrow \infty} d_k = \prod_{l \geq 0} (1 + 2^{-\frac{|n-l|}{4}}) \leq C.$$

Now notice that

$$(9) \quad \max_j \sum_{i < 2^n} a_{ij}^2 y_i^2 \leq \max_j \sqrt{\sum_{i < 2^n} a_{ij}^4} \sqrt{\sum_i y_i^4} \leq \Delta^2 \|y\|_2 \|y\|_\infty.$$

Thus

$$4 \sum_{l \geq 0} 2^{\frac{l-n}{4}} \sup_{x \in D_2^n} \max_j \sum_{i \in B_l^n(x)} a_{ij}^2 x_i^2 \leq 4\Delta^2 \left(\sum_{l=0}^n 2^{\frac{n-l}{4}} 2^{-\frac{l}{2}} + \sum_{l \geq n+1} 2^{\frac{l-n}{4}} 2^{\frac{n-2l}{2}} \right) \\ \leq C\Delta^2 2^{\frac{n}{4}}.$$

Using the inequality (9) we also get

$$\max_j \sum_{i \in B_l^n(x)} a_{ij}^2 x_i^2 \leq \begin{cases} \Delta^2 \left(\sum_{m=1}^l 2^{\frac{m-2l}{2}} c_m(x) + 2^{-\frac{l}{2}} \sum_{m=l+1}^n c_m(x) \right) & \text{for } l \leq n \\ \Delta^2 \sum_{m=1}^n 2^{\frac{m-2l}{2}} c_m(x) & \text{for } l > n \end{cases}.$$

Therefore by Lemma 4

$$\sum_{l \geq 0} b_l(x) = \sum_{l \geq 0} 2^{\frac{l-n}{4}} \log \#\Pi_l^n \max_j \sum_{i \in B_l^n(x)} a_{ij}^2 x_i^2 \leq C\Delta^2 \sum_{m=1}^n \alpha_m c_m(x),$$

where

$$\alpha_m = \sum_{l=0}^{m-1} 2^{\frac{n-l}{4}} 2^l (1+n-l) 2^{-\frac{l}{2}} + \sum_{l=m}^n 2^{\frac{n-l}{4}} 2^l (1+n-l) 2^{\frac{m-2l}{2}} + \sum_{l \geq n+1} 2^{\frac{l-n}{4}} 2^n (1+l-n) 2^{\frac{m-2l}{2}}.$$

To estimate the first and third term we use respectively (5) with $a = \frac{1}{4}$ and $b = 1$ and (6) with $a = \frac{1}{4}$ and $b = 1$. To bound the second term we notice that $1+n-l \leq 1+n-m$ for $m \leq l \leq n$. Hence we get

$$\alpha_m \leq C2^{\frac{n+m}{4}} (1+n-m) \leq C2^{\frac{n}{2}} 2^{\frac{m-n}{8}} \sup_{x \geq 0} ((1+x)2^{-\frac{x}{8}}) \leq C2^{\frac{n}{2}} 2^{\frac{m-n}{8}}.$$

Therefore

$$\sum_{l \geq 0} b_l(x) \leq C\Delta^2 2^{\frac{n}{2}} \sum_{l=1}^n 2^{\frac{l-n}{8}} c_l(x)$$

and Lemma follows by the inequality (8). \square

Proof of Theorem 1. We may and will assume that

$$\sum_i a_{i,j_1}^4 \geq \sum_i a_{i,j_2}^4 \text{ for } j_1 < j_2$$

and

$$\sum_j a_{i_1,j}^4 \geq \sum_j a_{i_2,j}^4 \text{ for } i_1 < i_2.$$

Notice first that

$$\sup_{\|x\|_2, \|y\|_2 \leq 1} \sum_{ij} a_{ij} g_{ij} x_i y_j \leq \sup_{\|x\|_2, \|y\|_2 \leq 1} \sum_{i \leq j} a_{ij} g_{ij} x_i y_j + \sup_{\|x\|_2, \|y\|_2 \leq 1} \sum_{j < i} a_{ij} g_{ij} x_i y_j \\ = \sup_{\|x\|_2 \leq 1} \sqrt{\sum_j \left(\sum_{i \leq j} a_{ij} g_{ij} x_i \right)^2} + \sup_{\|y\|_2 \leq 1} \sqrt{\sum_i \left(\sum_{j < i} a_{ij} g_{ij} y_j \right)^2}.$$

So to prove Theorem 1 it is enough to show

$$(10) \quad E \sup_{\|x\|_2 \leq 1} \sum_j \left(\sum_{i \leq j} a_{ij} g_{ij} x_i \right)^2 \leq C \left(\max_i \sum_j a_{ij}^2 + \sqrt{\sum_{ij} a_{ij}^4} \right)$$

and

$$(11) \quad E \sup_{\|y\|_2 \leq 1} \sum_i \left(\sum_{j < i} a_{ij} g_{ij} y_j \right)^2 \leq C \left(\max_j \sum_i a_{ij}^2 + \sqrt{\sum_{ij} a_{ij}^4} \right).$$

We will prove only (10), since (11) is basically the same inequality. By Lemma 5 it is enough to show

$$(12) \quad E \sup_{x \in D_2} \sum_j \left(\sum_{i \leq j} a_{ij} g_{ij} x_i \right)^2 \leq C \left(\max_i \sum_j a_{ij}^2 + \sqrt{\sum_{ij} a_{ij}^4} \right).$$

Let us introduce some additional notation for $n = 1, 2, \dots$ and $x \in D_2$

$$\Delta_n^4 = \sum_i a_{i, 2^{n-1}}^4 = \max_{j \geq 2^{n-1}} \sum_i a_{ij}^4 \geq \max_{j \geq 2^{n-1}} \sum_{i \leq j} a_{ij}^4$$

and

$$f_n(x) = \sum_{2^{n-1} \leq j < 2^n} \sum_{i \leq j} a_{ij}^2 x_i^2 + \Delta_n^2 2^{\frac{n}{2}} \sum_{l \leq n} 2^{\frac{l-n}{8}} c_l(x),$$

where as in the proof of Lemma 6

$$c_n(x) = \sqrt{\sum_{2^{n-1} \leq i < 2^n} x_i^2}.$$

Let C_1 be the same constant as in Lemma 6, then

$$(13) \quad \begin{aligned} & \mathbf{E} \sup_{x \in D_2} \left[\sum_j \left(\sum_{i \leq j} a_{ij} g_{ij} x_i \right)^2 - C_1 \sum_{n=1}^{\infty} f_n(x) \right] \\ & \leq \sum_{n=1}^{\infty} \mathbf{E} \sup_{x \in D_2} \left[\sum_{2^{n-1} \leq j < 2^n} \left(\sum_{i \leq j} a_{ij} g_{ij} x_i \right)^2 - C_1 f_n(x) \right] \leq \sum_{n=1}^{\infty} C_1 \Delta_n^2 2^{\frac{n}{4}}. \end{aligned}$$

Notice that $2^{n-1} \Delta_n^4 \leq \sum_{ij} a_{ij}^4$, so

$$(14) \quad \sum_{n \geq 1} \Delta_n^2 2^{\frac{n}{4}} \leq C \sqrt{\sum_{ij} a_{ij}^4}.$$

Obviously we have

$$(15) \quad \sup_{x \in D_2} \sum_{i \leq j} a_{ij}^2 x_i^2 \leq \max_i \sum_j a_{ij}^2.$$

For any $u, v \in l^2$ we have

$$\sum_{l \leq n} u_l v_l 2^{\frac{l-n}{8}} = \sum_{i \geq 0} 2^{-\frac{i}{8}} \sum_l u_l v_{l+i} \leq \sum_{i \geq 0} 2^{-\frac{i}{8}} \|u\|_2 \|v\|_2 \leq C \|u\|_2 \|v\|_2.$$

Since $\sum_l c_l(x)^2 = \|x\|_2^2$ and $\sum_n \Delta_n^4 2^n \leq C \sum_{ij} a_{ij}^4$ we get

$$(16) \quad \sum_{l \leq j} c_l(x) \Delta_j^2 2^{\frac{j}{2}} 2^{\frac{l-j}{8}} \leq C \|x\|_2 \sqrt{\sum_{ij} a_{ij}^4}.$$

By (15) and (16) we get

$$(17) \quad \sum_{n=1}^{\infty} f_n(x) \leq \max_i \sum_j a_{ij}^2 + C \sqrt{\sum_{ij} a_{ij}^4} \text{ for any } x \in D_2.$$

Inequalities (13), (14) and (17) imply (12) and the proof of Theorem 1 is completed. \square

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