Tail and moment estimates for sums of independent random vectors with logarithmically concave tails*

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Abstract

Let X_i be a sequence of independent symmetric real random variables with logarithmically concave tails. We consider a variable $X = \sum v_i X_i$, where v_i are vectors of some Banach space. We derive approximate formulas for the tail and moments of $\|X\|$. The estimates are exact up to some universal constant and they extend results of S. J. Dilworth and S. J. Montgomery-Smith [1] for Rademacher sequence and E. D. Gluskin and S. Kwapień [2] for real coefficients.

Definitions and Notation. Let X_i be a sequence of independent real symmetric random variables such that functions

$$N_i(t) = -\ln P(|X_i| \geq t), \quad t \geq 0$$

are convex. Since it is only matter of normalization we may and will assume that $N_i(1) = 1$.

Let us define the functions $\hat{N}_i$ by the formula

$$\hat{N}_i(t) = \begin{cases} t^2 & \text{for} \quad |t| \leq 1 \\ N_i(|t|) & \text{for} \quad |t| \geq 1 \end{cases}$$

For sequences (a_i) of real numbers and (v_i) of vectors in some Banach space F and $u > 0$ we define

$$\|(a_i)\|_{\mathcal{N},u} = \sup\{\sum a_i b_i : \sum \hat{N}_i(b_i) \leq u\}$$

*1991 *Mathematics Subject Classification* 60G50, 60B11

†Partially supported by Foundation for Polish Science and KBN Grant 2 P301 022 07

and

$$\|(v_i)\|^w_{\mathcal{N},u} = \sup\{\|(v^*(v_i))\|_{\mathcal{N},u} : v^* \in F^*, \|v^*\| \leq 1\}.$$

We denote by ε_i the Bernoulli sequence, i.e. a sequence of i.i.d. symmetric random variables taking on values ± 1.

For a random vector X and $p \geq 1$ we denote $\|X\|_p = (E\|X\|^p)^{1/p}$ and for a sequence $a = (a_i)$ of real numbers $\|a\|_p = (\sum |a_i|^p)^{1/p}$

Theorem 1 *Let v_i be vectors of some Banach space F such that the series $X = \sum v_i X_i$ is almost surely convergent. Then for each $p \geq 1$ we have*

$$\frac{1}{15}(\|X\|_1 + \|(v_i)\|^w_{\mathcal{N},p}) \leq \|X\|_p \leq K(\|X\|_1 + \|(v_i)\|^w_{\mathcal{N},p}),$$

where K is an universal constant ($K \leq 300$).

First we will prove estimation from below, by the same method as in [2]. Since $\|X\|_1 \leq \|X\|_p$ by the definition of $\|(v_i)\|^w_{\mathcal{N},p}$ it is enough to show that

$$\sum a_i b_i \leq 14 \|\sum a_i X_i\|_p$$

for every sequences (a_i) and (b_i) of real numbers such that $\sum \hat{N}_i(b_i) \leq p$. By symmetry we may assume that $a_i, b_i \geq 0$. Let $I = \{i : b_i \geq 1\}$, then card$(I) \leq p$. Since $E|X_i| \geq 1/e$ we obtain by the contraction principle and estimations of moments of Rademacher series [3] (Theorem 1 and Remark 1)

$$\|\sum a_i X_i\|_p \geq \frac{1}{e}\|\sum a_i \varepsilon_i\|_p \geq \frac{1}{2\sqrt{2}e}\inf\{\|a'\|_1 + \sqrt{p}\|a''\|_2 : a_i = a_i' + a_i''\} \geq$$

$$\geq \frac{1}{2\sqrt{2}e}\sup\{\sum a_i c_i : \sum c_i^2 \leq p, |c_i| \leq 1\} \geq \frac{1}{2\sqrt{2}e}\sum_{i \notin I} a_i b_i$$

we also have

$$\|\sum a_i X_i\|_p \geq \|\sum_{i \in I} a_i X_i\|_p \geq (\sum_{i \in I} a_i b_i)(P(X_i \geq b_i : i \in I))^{1/p} \geq$$

$$\geq \frac{1}{2}(\sum_{i \in I} a_i b_i)\exp(-\frac{1}{p}\sum_{i \in I} N_i(b_i)) \geq \frac{1}{2e}\sum_{i \in I} a_i b_i.$$

So $\sum a_i b_i \leq (2\sqrt{2}e + 2e)\|\sum a_i X_i\|_p \leq 14\|\sum a_i X_i\|_p$.

To prove the second inequality let us first observe that $X_i = Y_i + Z_i$ for some symmetric random variables Y_i and Z_i such that

$$P(|Y_i| \geq t) = e^{-\tilde{N}_i(t)}, \text{ where}$$

$$\tilde{N}_i(t) = \begin{cases} t & \text{for} \quad t \leq 1 \\ N_i(t) & \text{for} \quad t \geq 1 \end{cases}$$

and $|Z_i| \leq 1$ a.e., we will also assume that variables Y_i are independent and variables Z_i are independent. By the contraction principle

$$\|\sum v_i Z_i\|_p \leq \|\sum v_i \varepsilon_i\|_p \leq e\|\sum v_i Y_i\|_p,$$

$$\|\sum v_i Y_i\|_1 \leq \|\sum v_i X_i\|_1 + \|\sum v_i Z_i\|_1 \leq (1+e)\|\sum v_i X_i\|_1$$

and

$$\|\sum v_i X_i\|_p \leq \|\sum v_i Y_i\|_p + \|\sum v_i Z_i\|_p \leq (1+e)\|\sum v_i Y_i\|_p,$$

hence it is enough to proof the following inequality

$$\|\sum v_i Y_i\|_p \leq 2\|\sum v_i Y_i\|_1 + 74\|(v_i)\|^w_{\mathcal{N},p} \tag{1}$$

We may obviously assume that the above sum is finite. Let $M_i : R \to R$ be an odd function, which restricted to R^+ is the inverse of N_i. Then the variable Y_i has the distribution $M_i(\mu_1)$, where μ_1 is the measure on R with the density $\frac{1}{2}e^{-|x|}$. This means in particular that

$$P(\|\sum_{i=1}^n v_i Y_i\| > t) = \mu_1^n(x \in R^n : \|\sum_{i=1}^n v_i M_i(x_i)\| > t),$$

where μ_1^n is the product measure $\mu_1 \otimes \mu_1 \otimes \ldots \otimes \mu_1$ on R^n. Let M be a median of $\|\sum_{i=1}^n v_i Y_i\|$ and

$$A = \{x \in R^n : \|\sum_{i=1}^n v_i M_i(x_i)\| \leq M\}.$$

Then $\mu_1^n(A) \geq 1/2$ and by a result of Talagrand ([5] and [4] for a simpler proof)

$$\mu_1^n(A + V_s) \geq 1 - 2e^{-s},$$

where

$$V_s = \{x \in R^n : \sum_{i=1}^n \min(|x_i|, x_i^2) \leq 36s\}.$$

Let $x = y + z, y \in A, z \in V_s$, by the convexity of $\tilde{N}_i$ we have $|M_i(x_i) - M_i(y_i)| \leq 2M_i(|x_i - y_i|)$, so for some $v^* \in F^*$, $\|v^*\| \leq 1$we obtain

$$\|\sum_{i=1}^n v_i M_i(x_i)\| = v^*(\sum_{i=1}^n v_i M_i(x_i)) \leq M + \sum_{i=1}^n v^*(v_i)(M_i(x_i) - M_i(y_i)) \leq$$

$$M + 2\sum_{i=1}^n |v^*(v_i)| M_i(|z_i|) \leq M + 2\sup\{\sum_{i=1}^n |v^*(v_i)| b_i : \sum_{i=1}^n \hat{N}_i(b_i) \leq 36s\} \leq$$

$$M + 2\|(v_i)\|^w_{\mathcal{N},36s}.$$

So

$$P(\|\sum_{i=1}^n v_i Y_i\| > M + 2\|(v_i)\|^w_{\mathcal{N},36s}) \leq 2e^{-s}$$

and since $\|(v_i)\|^w_{\mathcal{N},\lambda u} \leq \lambda \|(v_i)\|^w_{\mathcal{N},u}$ for $\lambda \geq 1$ we have for $t \geq 2$

$$P(\|\sum_{i=1}^n v_i Y_i\| > M + t\|(v_i)\|^w_{\mathcal{N},u}) \leq 2e^{-\frac{tu}{72}}.$$

Therefore integrating by parts

$$\|\sum_{i=1}^n v_i Y_i\|_p \leq M + 2\|(v_i)\|^w_{\mathcal{N},p} +$$

$$\|(v_i)\|^w_{\mathcal{N},p}(\int_0^\infty pt^{p-1} P(\|\sum_{i=1}^n v_i Y_i\| > M + (2+t)\|(v_i)\|^w_{\mathcal{N},p}) dt)^{1/p} \leq$$

$$M + \|(v_i)\|^w_{\mathcal{N},p}(2 + (\int_0^\infty 2pt^{p-1} e^{-\frac{tp}{72}} dt)^{1/p}) =$$

$$M + \|(v_i)\|^w_{\mathcal{N},p}(2 + 72(2\frac{\Gamma(p+1)}{p^p})^{1/p}) \leq M + 74\|(v_i)\|^w_{\mathcal{N},p}.$$

Since $M \leq 2\|\sum_{i=1}^n v_i Y_i\|_1$ the proof of inequality (1) is now completed.

¿From Theorem 1 and Paley-Zygmund inequalities as in [1] and [2] follows the following

Corollary 1 *There exist universal constants $O < c < C < \infty$ such that under the assumptions of Theorem 1 for each $t > 0$ following inequalities holds*

$$P(\|X\| > C(\|X\|_1 + \|(v_i)\|^w_{\mathcal{N},t})) \leq e^{-t}$$

$$P(\|X\| > c(\|X\|_1 + \|(v_i)\|^w_{\mathcal{N},t})) \geq \min(c, e^{-t})$$

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