

On the limit set in the law of the iterated logarithm for U -statistics of order two

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Abstract. We find the cluster set in the Law of the Iterated Logarithm for U -statistics of order 2 in some interesting special cases. The limsup is an unusual function of the quantities that determine the Bounded LIL.

1. Introduction and Notation.

In [GKLZ] necessary and sufficient conditions were obtained for the law of the iterated logarithm for canonical U -statistics of order 2 to hold. Here we continue the investigation of the LIL for U -statistics of order 2 by describing the cluster (or limit) set for the examples in [GZ], which helped motivate [GKLZ]. Namely, let X_1, X_2, \dots denote a sequence of iid r.v.'s with values in some measurable space (S, \mathcal{S}) . In the general case for a measurable kernel h on S^2 we define symmetrized U -statistics by the formula

$$U_n = \sum_{1 \leq i < j \leq n} \varepsilon_i \varepsilon_j h(X_i, X_j),$$

where (ε_i) is a Rademacher sequence (i.e. a sequence of independent symmetric ± 1 valued r.v.'s) independent of (X_i) . In our case we will assume that each X_i has a uniform distribution on $[0, 1]$ and

$$h(x, y) = \sum_{k=1}^{\infty} a_k h_k(x) h_k(y), \quad (1)$$

where

$$h_k(x) = I_{A_k}(x) \text{ and } A_k = (2^{-k}, 2^{-k+1}], k = 1, 2, \dots$$

We also assume that $0 \leq a_k \leq k^{-1/2} 2^k$ (this assumption seems not to be necessary but makes the calculations easier).

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The (nas) conditions for the (bounded) LIL for $\{U_n\}$ that were obtained in [GKLZ] imply that the conditions for the LIL in our case be in terms of

$$A = \sup\{\mathbf{E}h(X_1, X_2)f(X_1)f(X_2) : \mathbf{E}f^2(X_i) \leq 1\} = \sup_k |a_k 2^{-k}| \quad (2)$$

and

$$B = \limsup_{u \rightarrow \infty} \frac{\mathbf{E}(h^2(X_1, X_2) \wedge u)}{L_2 u}. \quad (3)$$

However, what is not so clear is the form of the function of A and B that determines the limsup. It turns out that the limsup is

$$C = \phi(A, B) = \begin{cases} A + \frac{B^2}{4A} & \text{if } B \leq 2A \\ B & \text{if } B \geq 2A. \end{cases}$$

In the sequel letters like K, K_1 , etc., will denote universal constants that may change from line to line, but do not depend on any parameters. To simplify the notation we define $Lx = \log(x \vee e)$ and $L_2 x = LLx$. We also write \log_2 for the logarithm to the base 2.

Now, a few comments about the organization of the paper. After presenting in Section 2 some known results, we present in Section 3 a few results for general U-statistics. Finally, in Section 4 we concentrate on the types of kernels of the form (1) that are the main focus of this paper.

2. Preliminary results.

In this section we gather a few inequalities proven elsewhere that we will use in the sequel.

Lemma 1. ([KW], Theorem 6.2.1) *There exists a universal constant K such that for any $t > 0$ and any sequence of real numbers $(a_{ij})_{1 \leq i < j \leq n}$ we have*

$$\mathbf{P}\left(\max_{1 \leq k \leq n} \left| \sum_{1 \leq i < j \leq k} a_{ij} \varepsilon_i \varepsilon_j \right| > t\right) \leq K \mathbf{P}\left(\left| \sum_{1 \leq i < j \leq n} a_{ij} \varepsilon_i \varepsilon_j \right| > t\right).$$

Lemma 2 (Bernstein inequality).

([de la P,G] Lemma 4.1.9 and Remark 4.1.10, [D] Th. 1.3.2) *If Z_i are independent r.v's with $\mathbf{E}Z_i = 0$, $\|Z_i\|_\infty \leq a$ and $b^2 = \sum \mathbf{E}Z_i^2$, then for all $t \geq 0$*

$$\mathbf{P}\left(\left| \sum_i Z_i \right| \geq t\right) \leq 2 \exp\left(-\frac{t^2}{2b^2 + \frac{2}{3}at}\right).$$

Lemma 3 (Kolmogorov's converse exponential inequality).

([S] Th. 5.2.2) *For any $\gamma > 0$ there exist numbers $\tilde{K}(\gamma) < \infty$ and $\tilde{\varepsilon}(\gamma) > 0$ such that if Z_i are independent r.v's with $\mathbf{E}Z_i = 0$, $\|Z_i\|_\infty \leq a$, $b^2 = \sum \mathbf{E}Z_i^2$ satisfying $t \geq \tilde{K}(\gamma)b$ and $ta \leq \tilde{\varepsilon}(\gamma)b^2$ for some $t > 0$, then*

$$\mathbf{P}\left(\sum_i Z_i \geq t\right) \geq \exp\left(-\frac{(1+\gamma)t^2}{2b^2}\right).$$

We will, however, prefer to use the following simple corollary of Kolmogorov's converse exponential inequality (one may take below $\varepsilon(\gamma) = (1 + \gamma)\tilde{\varepsilon}(\gamma)^2/2$ and $K(\gamma) = \exp((1 + \gamma)\tilde{K}(\gamma)^2/2)$.)

Corollary 1. *For any $\gamma > 0$ there exist numbers $K(\gamma) < \infty$ and $\varepsilon(\gamma) > 0$ such that if Z_i are independent r.v.'s with $\mathbf{E}Z_i = 0$, $\|Z_i\|_\infty \leq a$, $b^2 = \sum \mathbf{E}Z_i^2$, then for all $t > 0$*

$$\mathbf{P}\left(\sum_i Z_i \geq t\right) \geq \frac{1}{K(\gamma)} \exp\left(-\frac{(1 + \gamma)t^2}{2b^2}\right) - \exp\left(-\varepsilon(\gamma)\frac{b^2}{a^2}\right).$$

Lemma 4. ([GLZ], Corollary 3.4) *There exists a universal constant $K < \infty$ such that for all $t > 0$*

$$\mathbf{P}(|U_n| \geq t) \leq K \exp\left[-\frac{1}{K} \min\left(\frac{t^2}{n^2 E h^2}, \frac{t}{n \|h\|_{L_2 \rightarrow L_2}}, \frac{t^{2/3}}{[n(\|E_Y h^2\|_\infty + \|E_X h^2\|_\infty)]^{1/3}}, \frac{t^{1/2}}{\|h\|_\infty^{1/2}}\right)\right].$$

3. Technical Lemmas. General Kernels.

In this section we present few technical lemmas that do not require additional assumptions on the form of the kernel h .

Lemma 5. *We have*

$$\mathbf{E} \exp\left(\lambda \left(\sum_{i=1}^n \varepsilon_i\right)^2\right) \leq \frac{1}{\sqrt{1 - 2\lambda n}} \text{ for all } 0 \leq \lambda < \frac{1}{2n}. \quad (4)$$

Moreover, for each $\gamma > 0$, there exist positive numbers $K(\gamma)$ and $\delta(\gamma)$ such that for any n

$$\mathbf{P}\left(\sum_{i=1}^n \varepsilon_i \geq t\sqrt{n}\right) \geq \frac{1}{K(\gamma)} \exp\left(-\frac{(1 + \gamma)t^2}{2}\right) - \exp(-\delta(\gamma)n). \quad (5)$$

Proof. Notice that for any t

$$\mathbf{E} \exp\left(t \sum_{i=1}^n \varepsilon_i\right) = \left(\frac{1}{2}e^t + \frac{1}{2}e^{-t}\right)^n \leq e^{\frac{nt^2}{2}}.$$

So if g is $\mathcal{N}(0, 1)$ r.v. independent of ε_i , then

$$\begin{aligned} \mathbf{E} \exp\left(\lambda \left(\sum_{i=1}^n \varepsilon_i\right)^2\right) &= \mathbf{E}_\varepsilon \mathbf{E}_g e^{\sqrt{2\lambda}(\sum_{i=1}^n \varepsilon_i)g} \\ &= \mathbf{E}_g E_\varepsilon e^{\sqrt{2\lambda}g \sum_{i=1}^n \varepsilon_i} \leq \mathbf{E} e^{n\lambda g^2} = \frac{1}{\sqrt{1 - 2\lambda n}}. \end{aligned}$$

Inequality (5) is an immediate consequence of Kolmogorov's converse exponential inequality (Corollary 1). \square

Lemma 6. Suppose that $a_{ij}^{(n)}$ is a tripley indexed sequence of numbers such that

$$\limsup_{n \rightarrow \infty} \left| \sum_{i,j=1}^n a_{ij}^{(n)} \varepsilon_i \varepsilon_j \right| \leq C \text{ a.s.}$$

Then,

$$\limsup_{n \rightarrow \infty} \left| \sum_{i=1}^n a_{ii}^{(n)} \right| \leq C.$$

Proof. Let $t > C$, then $I(\sum_{i,j=1}^n a_{ij}^{(n)} \varepsilon_i \varepsilon_j \geq t) \rightarrow 0$ a.s. so in particular

$$\mathbf{P}\left(\sum_{i,j=1}^n a_{ij}^{(n)} \varepsilon_i \varepsilon_j \geq t\right) \rightarrow 0.$$

However

$$\mathbf{P}\left(\sum_{i,j=1}^n a_{ij}^{(n)} \varepsilon_i \varepsilon_j \geq \sum_{i=1}^n a_{ii}^{(n)}\right) = \mathbf{P}\left(\sum_{1 \leq i \neq j \leq n} a_{ij}^{(n)} \varepsilon_i \varepsilon_j \geq 0\right) \geq \frac{1}{K}$$

for some universal K ([de la P,G] Proposition 3.3.7 combined with Theorem 3.2.2).

This implies $\sum_{i=1}^n a_{ii}^{(n)} \leq t$ for large enough n , so $\limsup_{n \rightarrow \infty} \sum_{i=1}^n a_{ii}^{(n)} \leq C$. In a similar way we prove that $\limsup_{n \rightarrow \infty} (-\sum_{i=1}^n a_{ii}^{(n)}) \leq C$. \square

Lemma 7. a) If $C < \infty$ is a number such that

$$\forall \varepsilon > 0 \exists K, N \forall n \geq N \mathbf{P}(|U_n| \geq C(1 + \varepsilon)nL_2n) \leq \frac{K}{\log n(L_2n)^{1+\varepsilon}}, \quad (6)$$

then

$$\limsup_{n \rightarrow \infty} \frac{|U_n|}{nL_2n} \leq C \text{ a.s.}$$

b) If $C < \infty$ is a number such that

$$\forall \varepsilon > 0, n_0 \exists K, N > n_0 \forall N \leq n \leq N^2 \mathbf{P}(|U_n| \geq C(1 + \varepsilon)nL_2n) \geq \frac{1}{K \log n}, \quad (7)$$

then

$$\limsup_{n \rightarrow \infty} \frac{|U_n|}{nL_2n} \geq C \text{ a.s.}$$

Proof. We start with the proof of part a). Let $\alpha > 1$, in this part of the proof we will denote $U_a = U_{\lfloor a \rfloor}$ for all $a \geq 0$. Let $\varepsilon > 0$ and K, N be given by formula (6). Let us choose k_0 such that $\alpha^{k_0} \geq N$. Then, we have for all $t > 0$

$$\begin{aligned} \mathbf{P}\left(\max_{n \geq \alpha^{k_0}} \frac{|U_n|}{nL_2n} \geq t\right) &\leq \sum_{k=k_0}^{\infty} \mathbf{P}\left(\max_{\alpha^k \leq n \leq \alpha^{k+1}} \frac{|U_n|}{nL_2n} \geq t\right) \\ &\leq \sum_{k=k_0}^{\infty} \mathbf{P}\left(\max_{1 \leq n \leq \alpha^{k+1}} |U_n| \geq t\alpha^k L_2(\alpha^k)\right) \leq \sum_{k=k_0}^{\infty} K \mathbf{P}(|U_{\alpha^{k+1}}| \geq t\alpha^k L_2(\alpha^k)), \end{aligned}$$

where in the last line we used the maximal inequality (Lemma 1). Since for large enough k we have $L_2(\alpha^k) \geq \alpha^{-1}L_2(\alpha^{k+1})$ we get that for sufficiently large k_0

$$\begin{aligned} \mathbf{P}\left(\max_{n \geq \alpha^{k_0}} \frac{|U_n|}{nL_2n} \geq C\alpha^2(1+\varepsilon)\right) &\leq \sum_{k=k_0}^{\infty} K\mathbf{P}(|U_{\alpha^{k+1}}| \geq C(1+\varepsilon)\alpha^{k+1}L_2(\alpha^{k+1})) \\ &\leq \sum_{k=k_0}^{\infty} \frac{K}{\log[\alpha^{k+1}](L_2[\alpha^{k+1}])^{1+\varepsilon}}. \end{aligned}$$

This implies that

$$\lim_{k \rightarrow \infty} \mathbf{P}\left(\max_{n \geq \alpha^k} \frac{|U_n|}{nL_2n} \geq C\alpha^2(1+\varepsilon)\right) = 0,$$

so $\limsup_{n \rightarrow \infty} \frac{|U_n|}{nL_2n} \leq C\alpha^2(1+\varepsilon)$ a.s. and part a) follows, when $\alpha \rightarrow 1^+$ and $\varepsilon \rightarrow 0^+$.

To prove part b) suppose that

$$\limsup_{n \rightarrow \infty} \frac{|U_n|}{nL_2n} \leq C_1 < C \text{ a.s.}$$

(By the 0-1 Law we know that the limsup is constant a.s.). Let $m > 1$ be an integer (to be chosen later) and $\tilde{\varepsilon}_i$ be another Rademacher sequence independent of ε_i and X_i . Since for any choice of signs $\eta_i = \pm 1$ the sequence $\eta_i\varepsilon_i$ has the same distribution as ε_i we get that

$$\limsup_{n \rightarrow \infty} \frac{|\sum_{k,l=1}^n \tilde{\varepsilon}_k \tilde{\varepsilon}_l \sum_{m^{k-1} \leq i < m^k, m^{l-1} \leq j < m^l, i < j} \varepsilon_i \varepsilon_j h(X_i, X_j)|}{m^n L_2(m^n)} \leq C_1 \text{ a.s.}$$

So

$$\mathbf{P}_{\varepsilon, X} \left(\limsup_{n \rightarrow \infty} \frac{|\sum_{k,l=1}^n \tilde{\varepsilon}_k \tilde{\varepsilon}_l \sum_{m^{k-1} \leq i < m^k, m^{l-1} \leq j < m^l, i < j} \varepsilon_i \varepsilon_j h(X_i, X_j)|}{m^n L_2(m^n)} \leq C_1 \tilde{\varepsilon}\text{-a.s.} \right) = 1.$$

However by Lemma 6 it implies

$$\mathbf{P} \left(\limsup_{n \rightarrow \infty} \frac{|\sum_{k=1}^n \sum_{m^{k-1} \leq i < j < m^k} \varepsilon_i \varepsilon_j h(X_i, X_j)|}{m^n L_2(m^n)} \leq C_1 \right) = 1.$$

Let $1/2 > \delta > 0$ to be chosen later and $C_1 < C_2 < C$, then

$$\mathbf{P} \left(\max_{n \geq n_0} \frac{|\sum_{k=1}^n \sum_{m^{k-1} \leq i < j < m^k} \varepsilon_i \varepsilon_j h(X_i, X_j)|}{m^n L_2(m^n)} > C_2 \right) < \delta$$

for sufficiently large n_0 . Notice that if $|s_n| \leq C_2 m^n L_2(m^n)$ for $n \geq n_0$, then $|s_n - s_{n-1}| \leq C_2(m^n + m^{n-1})L_2(m^n)$ for $n > n_0$. Therefore

$$\mathbf{P} \left(\max_{n > n_0} \frac{|\sum_{m^{n-1} \leq i < j < m^n} \varepsilon_i \varepsilon_j h(X_i, X_j)|}{m^n L_2(m^n)} > C_2 \left(1 + \frac{1}{m}\right) \right) < \delta.$$

Thus by the independence (since $\mathbf{P}(\bigcup A_i) \geq 1/2 \sum \mathbf{P}(A_i)$ if A_i are independent and $\mathbf{P}(\bigcup A_i) \leq 1/2$)

$$\begin{aligned} & \sum_{n > n_0} \mathbf{P}\left(\left| \sum_{1 \leq i < j \leq m^n - m^{n-1}} \varepsilon_i \varepsilon_j h(X_i, X_j) \right| \geq C_2 m^n \left(1 + \frac{1}{m}\right) L_2(m^n)\right) \\ &= \sum_{n > n_0} \mathbf{P}\left(\left| \sum_{m^{n-1} \leq i < j < m^n} \varepsilon_i \varepsilon_j h(X_i, X_j) \right| \geq C_2 m^n \left(1 + \frac{1}{m}\right) L_2(m^n)\right) < 2\delta. \end{aligned}$$

Now choose m and increase n_0 , if necessary, in such a way that

$$C_2 m^n \left(1 + \frac{1}{m}\right) L_2(m^n) \leq C(1 + \varepsilon)(m^n - m^{n-1}) L_2(m^n - m^{n-1})$$

for $n > n_0$. By our assumption (7) we can find $N > m^{n_0}$ such that

$$\begin{aligned} \mathbf{P}(|U_{m^n - m^{n-1}}| \geq C(1 + \varepsilon)(m^n - m^{n-1}) L_2(m^n - m^{n-1})) \\ \geq \frac{1}{K \log(m^n - m^{n-1})} \geq \frac{1}{Kn \log m} \end{aligned}$$

for all n such that $N \leq m^n - m^{n-1} \leq N^2$. However

$$\sum_{n: N \leq m^n - m^{n-1} \leq N^2} \frac{1}{Kn \log m} \gtrsim \frac{\log 2}{K \log m} > 2\delta$$

if we choose δ small enough. \square

The next Lemma shows why the LIL-limit depends on two quantities in a very non-obvious way.

Lemma 8. *Suppose that S_1, S_2 are independent r.v.'s, $A, B > 0$ and*

$$C = \begin{cases} A + \frac{B^2}{4A} & \text{if } B \leq 2A \\ B & \text{if } B \geq 2A \end{cases}$$

a) *If for some $K \geq 1$ and $\varepsilon > 0$*

$$\mathbf{P}(S_1 \geq sAn) \geq \frac{1}{K} e^{-s(1+\varepsilon)} - \frac{1}{(\log n)^{1+\varepsilon}} \text{ for all } s \geq 0$$

and

$$\mathbf{P}(S_2 \geq sBn\sqrt{L_2 n}) \geq \frac{1}{K} e^{-s^2(1+\varepsilon)^2} - \frac{1}{(\log n)^{1+\varepsilon}} \text{ for all } s \geq 0,$$

then for sufficiently large n

$$\mathbf{P}(S_1 + S_2 \geq (1 + \varepsilon)^{-1} CnL_2 n) \geq \frac{1}{K^2} \frac{1}{\log n} - \frac{2}{(\log n)^{1+\varepsilon}}.$$

b) *On the other hand if for some $K, \varepsilon > 0$*

$$\mathbf{P}(S_1 \geq sAn) \leq K e^{-\frac{s}{1+\varepsilon}} + \frac{1}{(\log n)^{1+\varepsilon}} \text{ for all } s \geq 0$$

and

$$\mathbf{P}(S_2 \geq sBn\sqrt{L_2 n}) \leq K e^{-\frac{s^2}{(1+\varepsilon)^2}} + \frac{1}{(\log n)^{1+\varepsilon}} \text{ for all } s \geq 0,$$

then

$$\mathbf{P}(S_1 + S_2 \geq (1 + \varepsilon)^3 CnL_2n) \leq \left(\frac{1}{\varepsilon} + 1\right) \frac{(K + 2)^2}{(\log n)^{1+\varepsilon}}.$$

Proof. For the first part of the statement it is enough to notice that in the case when $B \geq 2A$ we get for sufficiently large n

$$\begin{aligned} \mathbf{P}(S_1 + S_2 \geq (1 + \varepsilon)^{-1} CnL_2n) &\geq \mathbf{P}(S_1 \geq 0) \mathbf{P}(S_2 \geq (1 + \varepsilon)^{-1} BnL_2n) \\ &\geq \left(\frac{1}{K} - \frac{1}{(\log n)^{1+\varepsilon}}\right) \left(\frac{1}{K} e^{-L_2n} - \frac{1}{(\log n)^{1+\varepsilon}}\right) \geq \frac{1}{K^2 \log n} - \frac{2}{(\log n)^{1+\varepsilon}}. \end{aligned}$$

In the case when $B \leq 2A$ we have for large enough n

$$\begin{aligned} &\mathbf{P}(S_1 + S_2 \geq (1 + \varepsilon)^{-1} CnL_2n) \\ &\geq \mathbf{P}(S_1 \geq (1 + \varepsilon)^{-1} (A - \frac{B^2}{4A})nL_2n) \mathbf{P}(S_2 \geq (1 + \varepsilon)^{-1} \frac{B^2}{2A}nL_2n) \\ &\geq \left(\frac{1}{K} \exp\left(-\left(1 - \frac{B^2}{4A^2}\right)L_2n\right) - \frac{1}{(\log n)^{1+\varepsilon}}\right) \left(\frac{1}{K} \exp\left(-\frac{B^2}{4A^2}L_2n\right) - \frac{1}{(\log n)^{1+\varepsilon}}\right) \\ &\geq \frac{1}{K^2 \log n} - \frac{2}{(\log n)^{1+\varepsilon}}. \end{aligned}$$

To prove part b) first notice that for all $x \in [0, C]$

$$\frac{x}{A} + \frac{(C - x)^2}{B^2} \geq 1.$$

Hence, for such x

$$\begin{aligned} &\mathbf{P}(S_1 \geq (1 + \varepsilon)^2 xnL_2n, S_2 \geq (1 + \varepsilon)^2 (C - x)nL_2n) \\ &\leq \left(K \exp\left(-\left(1 + \varepsilon\right)\frac{x}{A}L_2n\right) + \frac{1}{(\log n)^{1+\varepsilon}}\right) \cdot \\ &\left(K \exp\left(-\left(1 + \varepsilon\right)\frac{(C - x)^2}{B^2}L_2n\right) + \frac{1}{(\log n)^{1+\varepsilon}}\right) \leq \frac{(K + 1)^2}{(\log n)^{1+\varepsilon}}. \end{aligned}$$

Moreover,

$$\begin{aligned} \mathbf{P}(S_1 \leq 0, S_1 + S_2 \geq (1 + \varepsilon)^2 CnL_2n) &\leq \mathbf{P}(S_2 \geq (1 + \varepsilon)^2 CnL_2n) \\ &\leq K \exp\left(-\left(1 + \varepsilon\right)\frac{C^2}{B^2}L_2n\right) + \frac{1}{(\log n)^{1+\varepsilon}} \leq \frac{K + 1}{(\log n)^{1+\varepsilon}} \end{aligned}$$

and

$$\mathbf{P}(S_1 \geq (1 + \varepsilon)^2 CnL_2n) \leq \frac{K + 1}{(\log n)^{1+\varepsilon}}.$$

Let $k_0 = \lfloor \varepsilon^{-1} \rfloor$. Then,

$$\begin{aligned} & \mathbf{P}(S_1 + S_2 \geq (1 + \varepsilon)^3 C n L_2 n) \\ & \leq \mathbf{P}(S_1 \leq 0, \frac{S_1 + S_2}{(1 + \varepsilon)^2 n L_2 n} \geq C) + \mathbf{P}(\frac{S_1}{(1 + \varepsilon)^2 n L_2 n} \geq C) \\ & + \sum_{k=0}^{k_0} \mathbf{P}(\frac{S_1}{(1 + \varepsilon)^2 n L_2 n} \in [k\varepsilon C, (k + 1)\varepsilon C], \frac{S_2}{(1 + \varepsilon)^2 n L_2 n} \geq C - k\varepsilon C) \\ & \leq \frac{2K + 2}{(\log n)^{1+\varepsilon}} + (k_0 + 1) \frac{(K + 1)^2}{(\log n)^{1+\varepsilon}} \leq (\frac{1}{\varepsilon} + 1) \frac{(K + 2)^2}{(\log n)^{1+\varepsilon}}. \end{aligned}$$

□

4. Special Kernels

From this point on we will assume that our kernel is of the form (1). We consider the following (uncoupled) U -statistics Let

$$\tilde{U}_n = \sum_{k=1}^{\infty} a_k \sum_{1 \leq i < j \leq N_k} \varepsilon_i^k \varepsilon_j^k = \sum_{k=1}^{\infty} \frac{a_k}{2} \left(\sum_{i=1}^{N_k} \varepsilon_i^k \right)^2 - N_k,$$

where

$$N_k = \#\{1 \leq i \leq n : X_i \in (2^{-k}, 2^{-k+1}]\}, k = 1, 2, \dots$$

Notice that

$$\mathcal{L}(U_n | \sigma(X_1, X_2, \dots)) = \mathcal{L}(\tilde{U}_n | \sigma(X_1, X_2, \dots)),$$

so U_n and \tilde{U}_n have the same distribution.

Lemma 9. *We have for all $\delta > 0$*

$$\mathbf{P}(\exists_{k \leq m} |N_k - n2^{-k}| \geq \delta n 2^{-k}) \leq \frac{2^{m+1}}{\delta^2 n}.$$

Proof. Notice that

$$\begin{aligned} \mathbf{P}(\exists_{k \leq m} |N_k - n2^{-k}| \geq \delta n 2^{-k}) & \leq \sum_{k=1}^m \mathbf{P}(|N_k - \mathbf{E}N_k| \geq \delta n 2^{-k}) \\ & \leq \sum_{k=1}^m \frac{2^{2k}}{\delta^2 n^2} \text{Var}(N_k) \leq \frac{1}{\delta^2 n} \sum_{k=1}^m 2^k \leq \frac{2^{m+1}}{\delta^2 n}. \end{aligned}$$

□

Lemma 10. *Suppose that $s > 0$ and $|n_k - n2^{-k}| \leq \varepsilon n 2^{-k-1}$ for $k = 1, \dots, m$. Let $\alpha = \max\{2^{-k} |a_k| : 1 \leq k \leq m\}$, then*

$$\mathbf{P}\left(\left|\sum_{k=1}^m \frac{a_k}{2} \left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 - n_k\right| \geq \alpha s n\right) \leq \left(\frac{2e(1 + \varepsilon)}{\varepsilon}\right)^{m/2} e^{-\frac{s}{1+\varepsilon}} \quad (8)$$

On the other hand, if $\alpha_1 = \max\{2^{-k}a_k : 1 \leq k \leq m\} > 0$, then

$$\mathbf{P}\left(\sum_{k=1}^m \frac{a_k}{2} \left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 - n_k\right) \geq \alpha_1 sn \geq \frac{1}{K(\varepsilon)} e^{-(1+\varepsilon)s} - \exp(-\delta(\varepsilon)2^{-m}n), \quad (9)$$

and if $\alpha_2 = \max\{-2^{-k}a_k : 1 \leq k \leq m\}$, then

$$\mathbf{P}\left(-\sum_{k=1}^m \frac{a_k}{2} \left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 - n_k\right) \geq \alpha_2 sn \geq \frac{1}{K(\varepsilon)} e^{-(1+\varepsilon)s} - \exp(-\delta(\varepsilon)2^{-m}n), \quad (10)$$

where $K(\varepsilon)$ and $\delta(\varepsilon)$ depend only on ε .

Proof. Let $S = \sum_{i=1}^m \frac{|a_k|}{2} (\sum_{i=1}^{n_k} \varepsilon_i^k)^2$, then by (4) we have

$$\mathbf{E}e^{\lambda S} \leq \prod_{i=1}^m \frac{1}{\sqrt{1 - \lambda|a_k|n_k}}.$$

But by our assumptions $|a_k|n_k \leq (1 + \frac{\varepsilon}{2})\alpha n$, so

$$\mathbf{E} \exp\left(\frac{1}{\alpha n(1 + \varepsilon)} S\right) \leq \left(1 - \frac{1 + \frac{\varepsilon}{2}}{1 + \varepsilon}\right)^{-m/2} = \left(\frac{2(1 + \varepsilon)}{\varepsilon}\right)^{m/2}.$$

Notice that

$$\left|\sum_{k=1}^m \frac{a_k}{2} \left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 - n_k\right| \leq S + \frac{1}{2} \sum_{k=1}^m |a_k|n_k \leq S + \frac{1}{2}(1 + \varepsilon)\alpha nm,$$

so (8) immediately follows, since

$$\mathbf{P}\left(\left|\sum_{k=1}^m \frac{a_k}{2} \left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 - n_k\right| \geq \alpha sn\right) \leq \mathbf{P}\left(S \geq \alpha n\left(s - \frac{1}{2}(1 + \varepsilon)m\right)\right).$$

To get (9) let k_0 be such that $a_{k_0} = \alpha_1 2^{k_0}$, then

$$\begin{aligned} & \mathbf{P}\left(\sum_{k=1}^m \frac{a_k}{2} \left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 - n_k\right) \geq \alpha_1 sn \\ & \geq \mathbf{P}\left(\frac{a_{k_0}}{2} \left(\sum_{i=1}^{n_{k_0}} \varepsilon_i\right)^2 \geq \alpha_1 sn\right) \mathbf{P}\left(\sum_{k \neq k_0} a_k \sum_{1 \leq i < j \leq n_k} \varepsilon_i^k \varepsilon_j^k \geq 0\right) \\ & \geq \frac{1}{K} \mathbf{P}\left(\left(\sum_{i=1}^{n_{k_0}} \varepsilon_i\right)^2 \geq 2^{-k_0+1} sn\right), \end{aligned}$$

where in the last inequality we used the same properties of Rademacher chaoses as in the proof of Lemma 6 (see [de la P,G], Proposition 3.3.7). Thus (9) follows by (5). The proof of (10) is similar. \square

Lemma 11. *Suppose that $0 < \delta < 1, k_1 \geq 1, |n_k - n2^{-k}| \leq \delta n2^{-k}$ for $k_1 \leq k \leq k_2$ and*

$$a = \sup\{|a_k|2^{-k} : k_1 \leq k \leq k_2\}, b^2 = \sum_{k=k_1}^{k_2} a_k^2 2^{-2k}.$$

Then, for any $s > 0$ and $t > 0$, we have

$$\begin{aligned} & \mathbf{P}\left(\frac{1}{2} \sum_{k=k_1}^{k_2} a_k \left(\left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 - n_k\right) \geq t + 4k_2 n a e^{-s/8}\right) \\ & \leq \exp\left(-\frac{t^2}{(1+\delta)^2 n^2 b^2 (1+50e^{-s/8}) + 2tsan}\right) + 2k_2 e^{-s/4} \end{aligned} \quad (11)$$

and

$$\begin{aligned} & \mathbf{P}\left(\frac{1}{2} \sum_{k=k_1}^{k_2} a_k \left(\left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 - n_k\right) \geq t - 4k_2 n a e^{-s/8}\right) \\ & \geq \frac{1}{K(\delta)} \exp\left(-\frac{(1+\delta)t^2}{(1-\delta)^2 n^2 b^2 (1-50e^{-s/8}) - 2^{k_2+1} n b^2}\right) \\ & \quad - \exp\left(-\frac{\varepsilon(\delta)b^2[(1-\delta)^2(1-50e^{-s/8}) - n^{-1}2^{k_2+1}]}{s^2 a^2}\right) - 2k_2 e^{-s/4}, \end{aligned} \quad (12)$$

where positive constants $K(\delta)$ and $\varepsilon(\delta)$ depend only on δ .

Proof. Let

$$S_k = \left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 I_{(\sum_{i=1}^{n_k} \varepsilon_i^k)^2 \leq sn_k},$$

then

$$\|a_k(S_k - \mathbf{E}S_k)\|_\infty \leq sa_k n_k \leq 2san.$$

Notice that by (4) we have

$$\mathbf{P}\left(\left|\sum_{i=1}^{n_k} \varepsilon_i^k\right| \geq \sqrt{sn_k}\right) \leq 2e^{-s/4},$$

so

$$\begin{aligned} |n_k - \mathbf{E}S_k| &= \mathbf{E}\left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 I_{(\sum_{i=1}^{n_k} \varepsilon_i^k)^2 > sn_k} \\ &\leq \sqrt{\mathbf{E}\left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^4} \sqrt{\mathbf{P}\left(\left|\sum_{i=1}^{n_k} \varepsilon_i^k\right| \geq \sqrt{sn_k}\right)} \leq 4n_k e^{-s/8}. \end{aligned}$$

Therefore

$$\sum_{k=k_1}^{k_2} |a_k(\mathbf{E}S_k - n_k)| \leq 8n \sum_{k=k_1}^{k_2} |a_k| 2^{-k} e^{-s/8} \leq 8k_2 n a e^{-s/8} \quad (13)$$

and

$$\mathbf{P}\left(\sum_{k=k_1}^{k_2} a_k \left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 \neq \sum_{k=k_1}^{k_2} a_k S_k\right) \leq \sum_{k=k_1}^{k_2} \mathbf{P}\left(S_k \neq \left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2\right) \leq 2k_2 e^{-s/4}. \quad (14)$$

We have

$$\begin{aligned} & \left| \mathbf{E}\left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^4 - \mathbf{E}S_k^2 \right| = \mathbf{E}\left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^4 I_{\left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 > sn_k} \\ & \leq \sqrt{\mathbf{E}\left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^8} \sqrt{\mathbf{P}\left(\left|\sum_{i=1}^{n_k} \varepsilon_i^k\right| \geq \sqrt{sn_k}\right)} \leq 80n_k^2 e^{-s/8} \end{aligned}$$

by the Khinchine inequality. Moreover,

$$\begin{aligned} \left| (\mathbf{E}S_k)^2 - \left(\mathbf{E}\left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2\right)^2 \right| &= \left| (\mathbf{E}S_k)^2 - n_k^2 \right| = |\mathbf{E}S_k + n_k| \cdot |\mathbf{E}S_k - n_k| \\ &\leq 2n_k \cdot 4n_k e^{-s/8} = 8n_k^2 e^{-s/8}, \end{aligned}$$

so

$$\left| \text{Var}(S_k) - \text{Var}\left(\left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2\right) \right| \leq 100n_k^2 e^{-s/8}.$$

Therefore

$$\begin{aligned} \text{Var}\left(\frac{1}{2} \sum_{k=k_1}^{k_2} a_k S_k\right) &\leq \sum_{k=k_1}^{k_2} a_k^2 \left(\frac{1}{2} n_k (n_k - 1) + 25n_k^2 e^{-s/8}\right) \\ &\leq \frac{1}{2} (1 + \delta)^2 n^2 b^2 (1 + 50e^{-s/8}) \end{aligned}$$

and by the Bernstein inequality (Lemma 2) we have

$$\mathbf{P}\left(\frac{1}{2} \sum_{k=k_1}^{k_2} a_k (S_k - \mathbf{E}S_k) \geq t\right) \leq \exp\left(-\frac{t^2}{(1 + \delta)^2 n^2 b^2 (1 + 50e^{-s/8}) + 2stan}\right). \quad (15)$$

Inequality (11) follows by (13), (14) and (15). To get the other estimate notice that

$$\begin{aligned} 2\text{Var}\left(\frac{1}{2} \sum_{k=k_1}^{k_2} a_k S_k\right) &\geq \sum_{k=k_1}^{k_2} a_k^2 (n_k (n_k - 1) - 50n_k^2 e^{-s/8}) \\ &\geq (1 - \delta)^2 n^2 b^2 (1 - 50e^{-s/8}) - \sum_{k=k_1}^{k_2} a_k^2 n_k \geq (1 - \delta)^2 n^2 b^2 (1 - 50e^{-s/8}) - 2^{k_2+1} n b^2. \end{aligned}$$

So by Kolmogorov's converse exponential inequality (Corollary 1) we get

$$\begin{aligned} \mathbf{P}\left(\frac{1}{2} \sum_{k=k_1}^{k_2} a_k (S_k - ES_k) \geq t\right) \\ \geq \frac{1}{K(\delta)} \exp\left(-\frac{(1+\delta)t^2}{(1-\delta)^2 n^2 b^2 (1-50e^{-s/8}) - 2^{k_2+1} n b^2}\right) \\ - \exp\left(-\frac{\varepsilon(\delta)b^2[(1-\delta)^2(1-50e^{-s/8}) - n^{-1}2^{k_2+1}]}{s^2 a^2}\right). \end{aligned} \quad (16)$$

Inequality (12) follows by (13), (14) and (16). \square

Lemma 12. *Suppose that $|n_k - n2^{-k}| \leq \delta n2^{-k}$, $|a_k| \leq k^{-1/2}2^k$ for $k \leq k_2$ and*

$$k_0 = \sqrt{L_2 n}, k_1 = (L_2 n)^{10}, k_2 = \log_2 n - 10L_2 n.$$

Let, moreover,

$$A_n = \sup\{|a_k|2^{-k} : k \leq k_0\}, B_n^2 = \frac{1}{L_2 n} \sum_{k=k_1}^{k_2} a_k^2 2^{-2k}$$

and

$$C_n = \begin{cases} A_n + \frac{B_n^2}{4A_n} & \text{if } B_n \leq 2A_n \\ B_n & \text{if } B_n \geq 2A_n \end{cases}.$$

Then, for any $\varepsilon > 0$, there exists $K(\varepsilon)$ such that for sufficiently large n and sufficiently small δ we have

$$\mathbf{P}\left(\left|\left(\sum_{k \leq k_0} + \sum_{k=k_1}^{k_2}\right) \frac{a_k}{2} \left(\left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 - n_k\right)\right| \geq (1+\varepsilon)C_n n L_2 n\right) \leq \frac{1}{(\log n)^{1+\varepsilon}}$$

and

$$\mathbf{P}\left(\left|\left(\sum_{k \leq k_0} + \sum_{k=k_1}^{k_2}\right) \frac{a_k}{2} \left(\left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 - n_k\right)\right| \geq (1-\varepsilon)C_n n L_2 n\right) \geq \frac{1}{K(\varepsilon) \log n}.$$

Proof. Let

$$S_1 = \sum_{k \leq k_0} \frac{a_k}{2} \left(\left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 - n_k\right) \text{ and } S_2 = \sum_{k=k_1}^{k_2} \frac{a_k}{2} \left(\left(\sum_{i=1}^{n_k} \varepsilon_i^k\right)^2 - n_k\right).$$

We will show that for sufficiently small δ and sufficiently large n

$$\mathbf{P}(|S_2| \geq un(B_n \sqrt{L_2 n} + 1)) \leq 2 \exp\left(-\frac{u^2}{(1+\varepsilon/10)^2}\right) + \frac{1}{(\log n)^2}. \quad (17)$$

Obviously we may assume $0 < \varepsilon < 1$. It is enough to show that

$$\mathbf{P}(\pm S_2 \geq un(B_n \sqrt{L_2 n} + 1)) \leq \exp\left(-\frac{u^2}{(1+\varepsilon/10)^2}\right) + \frac{1}{4(\log n)^2} \quad (18)$$

for $u \in [1/2, 4\sqrt{L_2n}]$. Indeed, for $u < 1/2$ the right hand side of (17) is greater than 1 and for $u = 4\sqrt{L_2n}$ the right hand side of (18) is less than $(2 \log n)^{-2}$. Now apply Lemma 11 with $s = 20L_2n$, $t = un(B_n\sqrt{L_2n} + 1/2)$ and $b^2 = \max(B_n^2L_2n, 1/4)$ (notice that then $t^2/(n^2b^2) \geq u^2$ and that part (11) of Lemma 11 holds also under the assumption $b^2 \geq \sum_{k=k_1}^{k_2} a_k^2 2^{-2k}$ - the estimates are monotone in b^2). Since

$$a = \sup\{|a_k|2^{-k} : k_1 \leq k \leq k_2\} \leq k_1^{-1/2} \leq (L_2n)^{-5}$$

we have

$$\begin{aligned} 2tsan &\leq 2 \cdot 4\sqrt{L_2n} \cdot n(B_n\sqrt{L_2n} + 1) \cdot 20L_2n \cdot (L_2n)^{-5} \cdot n \\ &\leq 160(L_2n)^{-3}n^2(B_n\sqrt{L_2n} + 1) \leq \delta n^2b^2 \end{aligned}$$

for sufficiently large n . Also

$$\begin{aligned} 2k_2e^{-s/4} &\leq 2(\log_2 n)(\log n)^{-5} < (4 \log n)^{-1}, \\ 4k_2nae^{-s/8} &\leq 4(\log_2 n)ne^{-s/8} \leq n/4 \leq un/2 \end{aligned}$$

and $50e^{-s/8} < \delta$ for sufficiently large n . Now it is enough to choose sufficiently small δ (which will depend on ε). Lemma easily follows by Lemmas 8 and 10. \square

Lemma 13. *If $\varepsilon > 0$, $|a_k| \leq k^{-1/2}2^k$ for all k and*

$$k_0 = \sqrt{L_2n}, k_1 = (L_2n)^{10}, k_2 = \log_2 n - 10L_2n,$$

then for sufficiently large n

$$\mathbf{P}\left(\left|\sum_{k=k_0}^{k_1} + \sum_{k=k_2}^{\infty} \frac{a_k}{2} \left(\sum_{i=1}^{N_k} \varepsilon_i^k\right)^2 - N_k\right| \geq \varepsilon n L_2n\right) \leq \frac{5}{\log n (L_2n)^{3/2}}$$

Proof. In this proof K denotes a universal constant that may change from line to line. Let us additionally define

$$k_3 = \log_2 n, k_4 = \log_2 n + \frac{1}{4} \log_2 \log n \text{ and } k_5 = \log_2 n + \frac{1}{2} \log_2 \log n + \frac{3}{4} \log_2(L_2n)$$

Notice that

$$\begin{aligned} \mathbf{P}\left(\sum_{k=k_5}^{\infty} \frac{a_k}{2} \left(\sum_{i=1}^{N_k} \varepsilon_i^k\right)^2 - N_k \neq 0\right) &\leq \mathbf{P}(\exists_{k \geq k_5} N_k > 1) \\ &\leq \mathbf{P}(\exists_{i,j \leq n} |X_i|, |X_j| \leq 2^{-k_5+1}) \leq n^2 2^{-2k_5+1} \leq \frac{2}{\log n (L_2n)^{3/2}}. \end{aligned} \quad (19)$$

For $k \leq k_5$ we have $|a_k| \leq k_5^{-1/2} 2^{k_5} \leq Kn(L_2n)^{3/4}$, therefore

$$\begin{aligned} \left|\sum_{k=k_4}^{k_5-1} \frac{a_k}{2} \left(\sum_{i=1}^{N_k} \varepsilon_i^k\right)^2 - N_k\right| &\leq Kn(L_2n)^{3/4} \left(\sum_{i=k_4}^{k_5-1} N_k\right)^2 \\ &\leq Kn(L_2n)^{3/4} (\#\{i \leq n : |X_i| \leq 2^{-k_4+1}\})^2. \end{aligned}$$

Thus for fixed ε and sufficiently large n

$$\begin{aligned} \mathbf{P}\left(\left|\sum_{k=k_4}^{k_5-1} \frac{a_k}{2} \left(\sum_{i=1}^{N_k} \varepsilon_i^k\right)^2 - N_k\right| \geq \frac{\varepsilon}{2} n L_2 n\right) \\ \leq \mathbf{P}(\#\{i \leq n : |X_i| \geq 2^{-k_4+1}\} \geq (L_2 n)^{1/8}) \\ \leq \left(\frac{en2^{-k_4+1}}{(L_2 n)^{1/8}}\right)^{(L_2 n)^{1/8}} \leq \frac{1}{\log n (L_2 n)^{3/2}}. \end{aligned} \quad (20)$$

Here we used the fact that

$$\mathbf{P}(X \geq k) \leq \binom{n}{k} p^k \leq \left(\frac{enp}{k}\right)^k \text{ if } X \sim \text{Bin}(n, p).$$

Similarly, for $k \leq k_4$, $|a_k| \leq k_4^{-1/2} 2^{k_4} \leq Kn(\log n)^{-1/4}$, so

$$\begin{aligned} \left|\sum_{k=k_3}^{k_4-1} \frac{a_k}{2} \left(\sum_{i=1}^{N_k} \varepsilon_i^k\right)^2 - N_k\right| &\leq Kn(\log n)^{-1/4} \left(\sum_{i=k_3}^{k_4-1} N_k\right)^2 \\ &\leq Kn(\log n)^{-1/4} (\#\{i \leq n : |X_i| \leq 2^{-k_3+1}\})^2. \end{aligned}$$

Therefore, for sufficiently large n

$$\begin{aligned} \mathbf{P}\left(\left|\sum_{k=k_3}^{k_4-1} \frac{a_k}{2} \left(\sum_{i=1}^{N_k} \varepsilon_i^k\right)^2 - N_k\right| \geq \frac{\varepsilon}{2} n L_2 n\right) \\ \leq \mathbf{P}(\#\{i \leq n : |X_i| \geq 2^{-k_3+1}\} \geq (\log n)^{1/8}) \end{aligned} \quad (21)$$

$$\leq \left(\frac{en2^{-k_3+1}}{(\log n)^{1/8}}\right)^{(\log n)^{1/8}} \leq \frac{1}{\log n (L_2 n)^{3/2}}. \quad (22)$$

Finally

$$\mathcal{L}\left(\sum_{k=k_0}^{k_1} + \sum_{k=k_2}^{k_3-1}\right) \frac{a_k}{2} \left(\sum_{i=1}^{N_k} \varepsilon_i^k\right)^2 - N_k = \mathcal{L}\left(\sum_{i,j=1}^n \varepsilon_i \varepsilon_j \tilde{h}(X_i, X_j)\right),$$

where

$$\tilde{h}(x, y) = \left(\sum_{k=k_0}^{k_1} + \sum_{k=k_2}^{k_3-1}\right) a_k h_k(x) h_k(y).$$

Let $A = [k_0, k_1] \cup [k_2, k_3 - 1]$, notice that

$$\|\tilde{h}\|_{L^2 \rightarrow L^2} = \max_{k \in A} |a_k 2^k| \leq \frac{1}{\sqrt{k_0}} \leq \frac{1}{(L_2 n)^{1/4}},$$

$$E\tilde{h}^2 = \sum_{k \in A} a_k^2 2^{2k} \leq \sum_{k \in A} \frac{1}{k} \leq CL_3 n,$$

$$\|E_X \tilde{h}^2\|_\infty = \|E_Y h^2\|_\infty = \max_{k \in A} a_k^2 2^{-k} \leq \max_{k \in A} \frac{2^k}{k} \leq \frac{2^{k_3}}{k_3} \leq \frac{n}{\log_2 n}$$

and

$$\|\tilde{h}\|_\infty = \max_{k \in A} |a_k| \leq \frac{2^{k_3}}{\sqrt{k_3}} \leq \frac{n}{\sqrt{\log_2 n}}.$$

So by Lemma 4 it easily follows that

$$\mathbf{P}\left(\left|\left(\sum_{k=k_0}^{k_1} + \sum_{k=k_2}^{k_3}\right) \frac{a_k}{2} \left(\sum_{i=1}^{N_k} \varepsilon_i^k\right)^2 - N_k\right| \geq \frac{\varepsilon}{2} n L_2 n\right) \leq \frac{1}{\log n (L_2 n)^{3/2}}. \quad (23)$$

The lemma follows by (19)–(23). \square

Theorem 1. *If $|a_k| \leq \frac{2^k}{\sqrt{k}}$ and A and B are given by (2) and (3), then*

$$\limsup_{n \rightarrow \infty} \frac{|U_n|}{n L_2 n} = \begin{cases} A + \frac{B^2}{4A} & \text{if } B \leq 2A \\ B & \text{if } B \geq 2A \end{cases} \text{ a.s.}$$

Proof. Let A_n, B_n be as in Lemma 12 notice that $\lim_{n \rightarrow \infty} A_n = A$ and

$$\begin{aligned} (L_2 n) B_n^2 &\leq \mathbf{E}(h^2 \wedge n) \leq (L_2 n) B_n^2 + \left(\sum_{k \leq k_1} + \sum_{k=k_2}^{\log_2 n}\right) a_k^2 2^{-2k} + n \sum_{k \geq \log_2 n} 2^{-2k} \\ &\leq (L_2 n) B_n^2 + CL_3 n. \end{aligned}$$

Since $L_2 n / L_2(n^2) \rightarrow 1$ as $n \rightarrow \infty$ we get that $\limsup_{n \rightarrow \infty} B_n \leq B$ and

$$\forall \varepsilon > 0 \forall n_0 \exists n \geq n_0 \forall N \leq n \leq N^2 B_n \geq B - \varepsilon.$$

So the theorem follows by Lemmas 7, 9, 12 and 13. \square

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