Gaussian measures of dilatations of convex symmetric sets

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We prove that the inequality $\Psi^{-1}(\mu(tA)) \geq t\Psi^{-1}(\mu(A))$ holds for any centered Gaussian measure μ on a separable Banach space F, any convex, closed, symmetric set $A \subset F$ and $t \geq 1$, where $\Psi(x) = \gamma_1(-x,x) = (2\pi)^{-1/2}\int_{-x}^{x} e^{-y^2/2}dy$. As an application the best constants in comparison of moments of Gaussian vectors are calculated.

The main theorem we will prove in this paper is the following one previously known also as an S-conjecture

Theorem 1. *Let μ be a centered Gaussian measure on a separable Banach space F. If A is a symmetric, convex, closed subset of F and $P \subset F$ is a symmetric strip, i.e. $P = \{x \in F : |x^*(x)| \leq 1\}$ for some $x^* \in F^*$, such that $\mu(A) = \mu(P)$ then*

$$\mu(tA) \geq \mu(tP) \text{ for } t \geq 1$$

and

$$\mu(tA) \leq \mu(tP) \text{ for } 0 \leq t \leq 1.$$

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The question comes from an unpublished manuscript of L. A. Shepp (1969), later it was published by S. Szarek [6]. A simple approximation argument using representation of Gausssian measures, presented in details in [4], shows that it is enough to prove Theorem 1 for $F = R^n$ and $\mu = \gamma_n$ – canonical Gaussian measure in R^n (i.e. the measure with density $(2\pi)^{-n/2}e^{-|x|^2/2}$). The positive answer for $n \leq 3$ was given by V. N. Sudakov and V. A. Zalgaller [5]. In the special case of A in R^n symmetric with respect to each coordinate Theorem 1 was proved by S. Kwapień and J. Sawa [4].

Before formulating the next results which will lead to the proof of Theorem 1 let us state few definitions. We will below always assume that A is a subset of R^n unless we state it otherwise.

- $\Phi(x) = \gamma_1(-\infty, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$
- $\Psi(x) = \gamma_1(-x, x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-y^2/2} dy$
- $A^h = \{x \in R^n : \mathrm{dist}(x, A) \leq h\}$ - h-neighbourhood of A
- $\gamma_n^+(A) = \liminf_{h \to 0+} (\gamma_n(A^h) - \gamma_n(A))/h$ - Gaussian perimeter of A
- $w(A) = \sup\{r : B(0, r) \subset A\}$

Let us notice that for a symmetric strip P, $w(P)$ is equal to a half of width of P and for a symmetric convex set A

$$w(A) = \inf\{w(P) : A \subset P, P \text{ is a symmetric strip in } R^n\}.$$

Thus $2w(A)$ can be considered as the width of the set A.

THEOREM 2. *Suppose that* $\gamma_2(A) = \gamma_2(P)$ *where* P *is a symmetric strip with width* $2p$ *and* A *is a set in* R^2 *symmetric about* $y-$*axis, lying under the graph of some symmetric, smooth, concave function* $f : (-w, w) \to R$*, non-increasing on [0,w) with* $\lim_{x\to w-} f(x) = -\infty$*. Then*

$$w\gamma_2^+(A) \geq w(P)\gamma_2^+(P) = \sqrt{\frac{2}{\pi}}pe^{-p^2/2}.$$

We postpone the proof of Theorem 2 till the end of the paper and now show how it implies the main result and the following theorem.

THEOREM 3. *If* $\gamma_n(A) = \gamma_n(P)$ *where* P *is a symmetric strip and* A *is a convex symmetric set in* R^n *then*

$$w(A)\gamma_n^+(A) \geq w(P)\gamma_n^+(P).$$

PROOF OF THEOREM 3. For $n = 1$ there is nothing to prove, so we will assume that $n \geq 2$. Let $w = w(A)$, without loss of generality we may then assume that

$$A \subset \{x \in R^n : |x_1| \leq w\}.$$

For $x \in (-w, w)$ let

$$A_x = \{y \in R^{n-1} : (x, y) \in A\}$$

and

$$f(x) = \Phi^{-1}(\gamma_{n-1}(A_x)).$$

Then by the convexity of A and Ehrhard's inequality [1] f is concave, moreover it is symmetric and hence non-increasing on $[0, w)$. Let us define

$$B = \{(x, y) \in R^2 : |x| < w, y \leq f(x)\},$$

thus we have $\gamma_2(B) = \gamma_n(A) = \gamma_n(P)$. Let $h > 0, x \in (-w - h, w + h)$ and $y \in (B^h)_x$, then there exists a point $(x^{'}, y^{'}) \in B$ such that $|x - x^{'}| = h_1, |y - y^{'}| = h_2$ and $h_1^2 + h_2^2 \leq h^2$. Since $(A_{x'})^{h_2} \subset (A^h)_x$ we get by the isoperimetric inequality

$$\Phi^{-1}(\gamma_{n-1}((A^h)_x)) \geq \Phi^{-1}(\gamma_{n-1}((A_{x'})^{h_2}))$$

$$\geq \Phi^{-1}(\gamma_{n-1}(A_{x'})) + h_2 \geq y^{'} + h_2 \geq y.$$

Taking the supremum over all $y \in (B^h)_x$ we get that $\gamma_1((B^h)_x) \leq \gamma_{n-1}((A^h)_x)$ for any $h > 0$ and $x \in (-w - h, w + h)$. Thus $\gamma_2(B^h) \leq \gamma_n(A^h)$ and $\gamma_2^+(B) \leq \gamma_n^+(A)$. Therefore it is enough to prove that

$$w\gamma_2^+(B) \geq w(P)\gamma_n^+(P). \tag{1}$$

Easy approximation argument shows that we may assume that f is smooth and $\lim_{x \to w-} f(x) = -\infty$, so (1) follows by Theorem 2.

Proof of Theorem 1. Let us define for any measurable set B in R^n

$$\gamma_B(t) = \gamma_n(tB) \text{ for } t > 0.$$

Taking derivatives of both sides of inequalities in Theorem 1 (for details see [4]) one can see that it is enough to show that for any convex closed symmetric set A in R^n we have

$$\gamma_A'(1) \geq \gamma_P'(1), \tag{2}$$

where P is a strip $P = \{|x_1| \le p\}$ such that $\gamma_n(A) = \gamma_n(P)$. Let $w = w(A)$, so $B(0, w) \subset A$. Let us notice that for $t > 1$ if $x \in A$ then $B(t^{-1}x, (t-1)w/t) = t^{-1}x + (1 - t^{-1})B(0, w) \subset A$ so $B(x, (t-1)w) \subset tA$, hence

$$A^{(t-1)w} \subset tA.$$

Therefore

$$\gamma'_A(1) \ge w\gamma_n^+(A) = w(A)\gamma_n^+(A).$$

Moreover for the strip P

$$\gamma'_P(1) = \sqrt{\frac{2}{\pi}} p e^{-p^2/2} = w(P)\gamma_n^+(P)$$

and (2) follows by Theorem 3.

The following Corollary is just a reformulation of Theorem 1. The second part of it was proved in [2].

Corollary 1. *If μ is a centered Gaussian measure on a separable Banach space F and B is convex, symmetric, closed subset of F then*

$$\mu(rB) \ge \Psi(\frac{r}{s}\Psi^{-1}(\mu(sB))) \textit{ for } r \ge s > 0.$$

In particular for each $b < 1$ exists a constant $C_b < \infty$ depending only on b such that if $\mu(B) \le b$ then

$$\mu(tB) \le C_b t\mu(B) \textit{ for } t \in [0, 1].$$

The next Corollary can be considered as some kind of isoperimetric inequality for convex, symmetric sets.

Corollary 2. *For any convex, symmetric subset A of a symmetric strip P in R^n and any $h > 0$ the following inequality holds:*

$$\frac{\Psi^{-1}(\gamma_n(A^h))}{\Psi^{-1}(\gamma_n(A))} \geq \frac{\Psi^{-1}(\gamma_n(P^h))}{\Psi^{-1}(\gamma_n(P))}.$$

Proof. Notice that $w(A^h) = w(A) + h$. Consider the function $r(h) = \Psi^{-1}(\gamma_n(A^h))/w(A^h)$. From the definition of γ_n^+ we deduce that

$$\liminf_{\varepsilon\to 0^+} \frac{r(h+\varepsilon)-r(h)}{\varepsilon}$$

$$= \frac{1}{w(A^h)^2}\left(\sqrt{\frac{\pi}{2}}\gamma_n^+(A^h)w(A^h)e^{\Psi^{-1}(\gamma_n(A^h))^2/2} - \Psi^{-1}(\gamma_n(A^h))\right) \geq 0,$$

by Theorem 3 applied to the set A^h.

The function $r(h)$ is continuous, hence $r(h) \geq r(0)$ for any $h > 0$. Therefore

$$\Psi^{-1}(\gamma_n(A^h)) = (w(A^h))r(h) \geq (w(A)+h)r(0) = \Psi^{-1}(\gamma_n(A))(1+\frac{h}{w(A)})$$

$$\geq \Psi^{-1}(\gamma_n(A))(1+\frac{h}{w(P)}) = \Psi^{-1}(\gamma_n(A))\frac{\Psi^{-1}(\gamma_n(P^h))}{\Psi^{-1}(\gamma_n(P))},$$

which completes the proof.

Finally as a consequence of Theorem 1 let us state the following result which gives the best constants in comparison of moments of Gaussian vectors. The proof presented below is due to S. Szarek (private communication).

Corollary 3. *If g_i are independent standard normal r.v. and x_i are vectors in some separable Banach space $(E, \|.\|)$ such that the series $S = \sum x_i g_i$ is a.s. convergent then*

$$(E\|S\|^p)^{1/p} \leq \frac{\gamma_p}{\gamma_q}(E\|S\|^q)^{1/q} \text{ for any } p \geq q > 0, \tag{3}$$

where

$$\gamma_p = (E|g_1|^p)^{1/p} = \sqrt{2}(\frac{1}{\sqrt{\pi}}\Gamma(\frac{p+1}{2}))^{1/p}.$$

PROOF. Let $a \in R$ be such that $E\|S\|^p = E|ag_1|^p$. Then

$$\int_0^\infty t^{p-1}P(\|S\| > t)dt = \int_0^\infty t^{p-1}P(|ag_1| > t)dt.$$

So for some $t_0 > 0$ we have $P(\|S\| > t_0) = P(|ag_1| > t_0)$. Applying Theorem 1 we easily obtain that $P(\|S\| > t) \geq P(|ag_1| > t)$ for $0 \leq t \leq t_0$ and $P(\|S\| > t) \leq P(|ag_1| > t)$ for $t \geq t_0$. Therefore for $t > 0$ and $p \geq q > 0$ we get

$$(\frac{t}{t_0})^{p-1}(P(\|S\| > t) - P(|ag_1| > t)) \leq (\frac{t}{t_0})^{q-1}(P(\|S\| > t) - P(|ag_1| > t)).$$

This gives

$$\int_0^\infty t^{q-1}P(\|S\| > t)dt \geq \int_0^\infty t^{q-1}P(|ag_1| > t)dt,$$

that is $E\|S\|^q \geq E|ag_1|^q$ and proves (3).

Proof of Theorem 2.

During this section we will frequently use the following functions

- $T(y) = 1 - \Phi(y)$
- $h(y) = 2\pi T(y)^2 e^{y^2}$

LEMMA 1. *The function $h(y)$ is decreasing for $y \geq 0$.*

PROOF. We have to prove that $T(y)e^{y^2/2}$ is a decreasing function of y on $(0, \infty)$. To see this note that

$$\frac{d}{dy}(T(y)e^{y^2/2}) = \frac{1}{\sqrt{2\pi}}(ye^{y^2/2}\int_y^\infty e^{-s^2/2}ds - 1)$$

$$< \frac{1}{\sqrt{2\pi}}(e^{y^2/2}\int_y^\infty se^{-s^2/2}ds - 1) = 0.$$

LEMMA 2. *The function* $g(y) = h(y)^{-1} - y^2$ *is non-decreasing for* $y \geq 0$*, in particular*

$$h(y)^{-1} \geq y^2 + 1.5 \textit{ for } y > 1.5$$

and

$$\sqrt{2\pi}T(y) \geq \frac{1}{\sqrt{y^2+2}}e^{-y^2/2} \textit{ for } y > 0. \tag{4}$$

PROOF. First let us notice that the function $\varphi(y) = \sqrt{2\pi}T(y) - \frac{e^{-y^2/2}}{\sqrt{y^2+2}}$ is decreasing on $(0, \infty)$. Indeed,

$$(y^2+2)^{3/2}e^{y^2/2}\varphi'(y) = y^3 + 3y - (y^2+2)^{3/2}$$

$$= \frac{1}{y^3 + 3y + (y^2+2)^{3/2}}((y^3+3y)^2 - (y^2+2)^3)$$

$$= -\frac{3y^2+8}{y^3+3y+(y^2+2)^{3/2}} < 0.$$

As $\lim_{y \longrightarrow \infty} \varphi(y) = 0$, we obtain the inequality (4). We have also

$$T(y) = \frac{1}{\sqrt{2\pi}}\int_y^\infty e^{-s^2/2}ds \leq \frac{1}{\sqrt{2\pi}y}\int_y^\infty se^{-s^2/2}ds = \frac{1}{\sqrt{2\pi}}\frac{e^{-y^2/2}}{y}. \tag{5}$$

By (4) and (5) we deduce that $0 \leq g(y) \leq 2$ for $y \geq 0$. Fix $a \in [0, 2]$. We only need to prove that if $g(y_a) \geq a$ for some $y_a > 0$ then also $g(y) \geq a$ for all $y \geq y_a$.

Now, $g(y) \geq a$ is equivalent to $T(y) \leq \frac{1}{\sqrt{2\pi}} \frac{e^{-y^2/2}}{\sqrt{y^2+a}}$. Let us investigate behaviour of the function $\psi_a(y) = \frac{1}{\sqrt{2\pi}} \frac{e^{-y^2/2}}{\sqrt{y^2+a}} - T(y)$. We have

$$\sqrt{2\pi} e^{y^2/2} (y^2+a)^{3/2} \psi_a'(y) = (y^2+a)^{3/2} - y - y(y^2+a).$$

Hence $\psi_a'(y) \geq 0$ if and only if $(y^2+a)^3 \geq (y^3+(a+1)y)^2$, which is equivalent to $(2-a)y^4 + (1+2a-2a^2)y^2 - a^3 \leq 0$. The left-hand side of the last inequality is a second degree polynomial in y^2 with nonnegative leading coefficient $2-a$. Moreover, for $y = 0$ the last inequality is obviously satisfied. Therefore there exists a nonnegative number m_a such that ψ_a is non-decreasing on the interval $(0, m_a)$ and it is non-increasing on the interval (m_a, ∞). As $\lim_{y \longrightarrow \infty} \psi_a(y) = 0$ this proves that $\psi_a(y_a) \geq 0$ implies $\psi_a(y) \geq 0$ for all $y \geq y_a$, which completes the proof, since $h(1.5)^{-1} \geq 1.5^2 + 1.5$.

Lemma 3. *The function* $xT(x)e^{x^2/2}$ *is increasing on* $[0, \infty)$.

Proof. We have

$$(\sqrt{2\pi} x T(x) e^{x^2/2})' = (1+x^2)\sqrt{2\pi} T(x) e^{x^2/2} - x,$$

so it is enough to show that $\sqrt{h(x)} > x/(x^2+1)$. But by Lemma 2

$$\sqrt{h(x)} \geq \frac{1}{\sqrt{x^2+2}} > \frac{x}{x^2+1}.$$

Lemma 4. *The function* $F(x) = h(x)^{-1} + 2 \ln T(x)$ *is non-increasing on* $[0, \infty)$.

Proof. First let us note that, due to a well-known Komatsu's estimate (see

[3], p. 17),

$$T(x) \geq \frac{1}{\sqrt{2\pi}} \frac{2}{x+\sqrt{x^2+4}} e^{-x^2/2}.$$

Hence $\sqrt{h(x)} \geq 2/(x+\sqrt{x^2+4})$ and therefore

$$x\sqrt{h(x)} \geq \frac{2x}{x+\sqrt{x^2+4}} = 1-(\frac{2}{x+\sqrt{x^2+4}})^2 \geq 1-h(x).$$

So,

$$F'(x) = -\frac{h'(x)}{h(x)^2} + 2\frac{T'(x)}{T(x)}$$

$$= -\frac{1}{h(x)} \frac{(2\pi T(x)^2 e^{x^2})'}{2\pi T(x)^2 e^{x^2}} - \frac{2e^{-x^2/2}}{\sqrt{2\pi} T(x)}$$

$$= -\frac{2}{h(x)}(\frac{T'(x)}{T(x)} + x) - \frac{2}{\sqrt{h(x)}} = \frac{2}{h(x)^{3/2}}(1-h(x)-x\sqrt{h(x)}) \leq 0$$

and the proof is completed.

Lemma 5. *For any real y we have $\Phi^2(y)h(y) \leq \frac{\pi}{8}$.*

Proof. Note that

$$\frac{8}{\pi}\Phi(y)^2 h(y) e^{-y^2} = (4\Phi(y)T(y))^2$$

$$= (1-\gamma_2([-|y|,|y|]\times[-|y|,|y|]))^2 \leq (1-\gamma_2(B_2(0,|y|)))^2$$

$$= (1-\frac{1}{2\pi}\int_0^{|y|} e^{-r^2/2}\cdot 2\pi r dr)^2 = (e^{-y^2/2})^2 = e^{-y^2}$$

and the proof is completed.

Lemma 6. *Let f be a non-increasing integrable function on $(0,\infty)$ and μ any finite positive measure on $(0,\infty)$. Then for any $0 \leq a_1 < b_1 \leq \infty$, $0 \leq a_2 <$*

$b_2 \leq \infty$ *such that* $a_1 \leq a_2$ *and* $b_1 \leq b_2$ *we have*

$$\frac{\int_{a_1}^{b_1} f(x)d\mu(x)}{\mu(a_1,b_1)} \geq \frac{\int_{a_2}^{b_2} f(x)d\mu(x)}{\mu(a_2,b_2)}.$$

PROOF. This is obvious.

LEMMA 7. *For any* $0 \leq c_1 < d_1 \leq \infty$, $0 \leq c_2 < d_2 \leq \infty$ *such that* $c_1 \leq c_2$ *and* $d_1 \leq d_2$ *we have*

$$\frac{\Phi(d_1)-\Phi(c_1)}{e^{-c_1^2/2}-e^{-d_1^2/2}} \geq \frac{\Phi(d_2)-\Phi(c_2)}{e^{-c_2^2/2}-e^{-d_2^2/2}}.$$

PROOF. Let us notice that

$$\sqrt{2\pi}(\Phi(d)-\Phi(c)) = \int_{c^2}^{d^2} \frac{1}{2\sqrt{y}} e^{-y/2} dy$$

and we may apply Lemma 6 with $f(y) = 1/\sqrt{y}$.

LEMMA 8. *Suppose that* $s \geq u > 0$ *and* $p > 0$ *satisfy the inequality*

$$1-\Phi(u) \leq 1-\Phi(p) + \frac{1}{2}(1-\Phi(s)), \tag{6}$$

then

$$\frac{1}{2}e^{(u^2-s^2)/2} + e^{(u^2-p^2)/2} \geq 1. \tag{7}$$

PROOF. If $u \geq p$ then (7) is obvious so we may assume that $p > u$. Inequality (6) immediately implies that

$$\frac{\Phi(p)-\Phi(u)}{1-\Phi(s)} \leq \frac{1}{2}$$

and by Lemma 7

$$\frac{\Phi(p)-\Phi(u)}{1-\Phi(s)} \geq \frac{e^{-u^2/2}-e^{-p^2/2}}{e^{-s^2/2}}.$$

From the above two inequalities immediately follows (7).

LEMMA 9. *If $c > 0$ and $p_0 > 0$ are such that $e^{-cp_0} \leq 1 - p_0$ then $e^{-cp} \leq 1 - p$ for all $p \in [0, p_0]$. In particular*

$$(1-p)e^{4p/\pi} \geq 1 \textit{ for all } p \in [0, 1/3]$$

and

$$(1-p)e^{4p/(\pi-4/9)} \geq 1 \textit{ for all } p \in [0, 1/2].$$

PROOF. The function $e^{-cp} - 1 + p$ is a convex function of p and that implies first part. The last statements follow by the first one and inequalities $e^{4/3\pi} \geq 3/2$ and $e^{2/(\pi-4/9)} \geq 2$.

LEMMA 10. *For any $p \in (0, 1/2]$ and $z \geq 0$*

$$pe^{-\pi z^2/16p^2} + (1-p) \geq e^{-z/2}$$

PROOF. Using Taylor expansion we have

$$pe^{z/2-\pi z^2/16p^2} + (1-p)e^{z/2} \geq p(1 + \frac{z}{2} - \frac{\pi}{16p^2}z^2) + (1-p)(1 + \frac{z}{2} + \frac{z^2}{8})$$

$$= 1 + \frac{z}{2} - \frac{\pi + 2(p-1)p}{16p}z^2,$$

so inequality is satisfied for $z \leq 8p/(\pi + 2(p-1)p)$. It is enough to show that

$$f(p) = (1-p)e^{4p/(\pi+2(p-1)p)} \geq 1.$$

If $p \leq 1/3$ then $f(p) \geq (1-p)e^{4p/\pi} \geq 1$ by previous Lemma. If $p \in [1/3, 1/2]$ then $(1-p)p \geq 2/9$, so again

$$f(p) \geq (1-p)e^{4p/(\pi-4/9)} \geq 1.$$

Lemma 11. *If* $y \leq 1.5$ *and* $z \geq 0$ *or if* $0 \leq z \leq y^2 + 1.5$ *then*

$$\Phi(y)e^{-h(y)z^2/2} + 1 - \Phi(y) \geq e^{-z/2}. \tag{8}$$

Proof. If $y \leq 0$ then the lemma follows by Lemmas 10 and 5. For $y > 0$, put $R_y(z) = e^{-z/2} - \Phi(y)e^{-h(y)z^2/2}$ and $M(y) = \sup_{z>1/h(y)} R_y(z)$. First note that in view of Lemma 1, $R_y(z)$ is a decreasing function of positive argument y for any fixed z. As $1/h(y)$ is an increasing function for $y > 0$ we see that $\sup_{z>1/h(y)}$ is taken over decreasing set. Together these facts show that $M(y)$ is non-increasing for $y > 0$. We have

$$\frac{\partial}{\partial z} R_y(z) = R_y'(z) = -\frac{1}{2}e^{-z/2} + \Phi(y)h(y)ze^{-h(y)z^2/2}.$$

Therefore $R_y'(0) < 0$ and $R_y'(z) < 0$ for z large enough. Note that $R_y'(1/h(y)) = (\Phi(y) - 1/2)\exp(-\frac{1}{2h(y)}) > 0$. As $R_y'(z) = 0$ if and only if $\ln(2\Phi(y)h(y)z) = h(y)z^2/2 - z/2$ we deduce that the function R_y has for each fixed $y > 0$ at most two local extrema on $(0, \infty)$ because the left-hand side of the last equation is concave and the right-hand side is convex. Summarizing these facts we arrive at the conclusion that for each $y > 0$ there exist positive numbers $\alpha(y) < \beta(y)$ such that the function R_y is decreasing on the interval $(0, \alpha(y))$, increasing on the interval $(\alpha(y), \beta(y))$ to which $1/h(y)$ belongs and again decreasing on the interval $(\beta(y), \infty)$. Therefore to prove our main claim, i.e. that $T(y) \geq R_y(z)$ for any $y \in [0, 1.5], z \geq 0$ it is enough to prove that $T(y) \geq M(y)$, as in the points $z = 0$ and $z = 1/h(y)$ the claim is trivial.

Let us consider the following table:

k	y_k	T_1	T_2	Φ_1	Φ_2	h_1	h_2	z_k	Z_k	a_k	b_k	M_k
1	0.00	.500	.500	.500	.500	1.570	1.571	1.34	1.35	.256	.254	.393
2	0.25	.401	.402	.598	.599	1.075	1.081	1.78	1.81	.206	.202	.309
3	0.49	.311	.313	.687	.689	0.772	0.783	2.26	2.33	.162	.155	.242
4	0.69	.244	.246	.754	.756	0.602	0.613	2.70	2.80	.130	.123	.192
5	0.87	.192	.193	.807	.808	0.493	0.499	3.16	3.25	.104	.098	.149
6	1.04	.149	.150	.850	.851	0.411	0.417	3.62	3.72	.082	.077	.117
7	1.18	.118	.120	.880	.882	0.351	0.365	3.97	4.22	.069	.060	.104
8	1.25	.105	.106	.894	.895	0.330	0.337	4.23	4.40	.061	.055	.087
9	1.35	.088	.089	.911	.912	0.300	0.308	4.53	4..72	.052	.047	.075
10	1.43	.076	.077	.923	.924	0.280	0.288	4.76	4.99	.047	.041	.067
11	1.49	.068	.069	.931	.932	0.267	0.276	4.92	5.20	.043	.037	.064
12	1.52	.064										

Note: In the above table T_1 in the k-th row should be understood as $T_1(y_k)$, similarly one should understand five next columns.

We leave to the reader checking that for $k = 1, \ldots, 11$ the numbers in the table satisfy the following inequalities:

$$T_1(y_k) \leq T(y_k) \leq T_2(y_k), \quad \Phi_1(y_k) \leq \Phi(y_k) \leq \Phi_2(y_k),$$

$$h_1(y_k) \leq 2\pi T_1(y_k)^2 e^{y_k^2} \leq h(y_k), \quad h_2(y_k) \geq 2\pi T_2(y_k)^2 e^{y_k^2} \geq h(y_k),$$

$$z_k \leq Z_k, \quad \frac{1}{2}e^{-z_k/2} \leq a_k, \quad \frac{1}{2}e^{-Z_k/2} \geq b_k,$$

$$\Phi_1(y_k)h_1(y_k)z_k e^{-h_2(y_k)z_k^2/2} \geq a_k,$$

$$\Phi_2(y_k)h_2(y_k)Z_k e^{-h_1(y_k)Z_k^2/2} \le b_k$$

and

$$T_1(y_{k+1}) \ge M_k \ge e^{-z_k/2} - \Phi_1(y_k)e^{-h_2(y_k)Z_k^2/2}.$$

Note also that $T_1(y_{12}) \le T(y_{12})$.

Now we are in a position to prove our claim. For each $y \in [0, 1.5]$ we can find $k \in \{1, \dots, 11\}$ such that $y_k \le y \le y_{k+1}$. Note that

$$\begin{aligned} R'_{y_k}(z_k) &= -\frac{1}{2}e^{-z_k/2} + \Phi(y_k)h(y_k)z_k e^{-h(y_k)z_k^2/2} \\ &\ge -\frac{1}{2}e^{-z_k/2} + \Phi_1(y_k)h_1(y_k)z_k e^{-h_2(y_k)z_k^2/2} \ge -a_k + a_k = 0, \end{aligned}$$

while

$$\begin{aligned} R'_{y_k}(Z_k) &= -\frac{1}{2}e^{-Z_k/2} + \Phi(y_k)h(y_k)Z_k e^{-h(y_k)Z_k^2/2} \\ &\le -\frac{1}{2}e^{-Z_k/2} + \Phi_2(y_k)h_2(y_k)Z_k e^{-h_1(y_k)Z_k^2/2} \le -b_k + b_k = 0, \end{aligned}$$

which means that $z_k \le \beta_k = \beta(y_k) \le Z_k$. Therefore,

$$\begin{aligned} M(y) \le M(y_k) = R_{y_k}(\beta_k) &= e^{-\beta_k/2} - \Phi(y_k)e^{-h(y_k)\beta_k^2/2} \\ &\le e^{-z_k/2} - \Phi_1(y_k)e^{-h_2(y_k)Z_k^2/2} \le M_k \le T_1(y_{k+1}) \le T(y_{k+1}) \le T(y), \end{aligned}$$

which completes the proof in the case of $y < 1.5$. If $y \ge 1.5$, notice that we have already proved (8) for $0 \le z \le 1/h(y)$. Thus Lemma 2 implies (8) for $0 \le z \le y^2 + 1.5$.

LEMMA 12. *Let $w \geq a \geq x \geq 0$ and $y \in R$ satisfy the inequality*

$$\Phi(y)\Phi(w) + (1 - \Phi(y))\Phi(x) \geq \Phi(a). \tag{9}$$

Then if $y \leq 1.5$ or $a^2 - x^2 \leq y^2 + 1.5$ we have

$$w\sqrt{1+k^2}e^{-y^2/2} \geq \sqrt{2\pi}(a^2 - x^2)(1 - \Phi(y)) + kxe^{-y^2/2} \text{ for any } k \geq 0. \tag{10}$$

PROOF. Dividing both sides of (10) by $\sqrt{1+k^2}$ and taking supremum over k we have to prove that

$$w^2 \geq h(y)z^2 + x^2,$$

where $z = a^2 - x^2$. Suppose that this is not true, then by (9) we get that

$$\Phi(y)\Phi(\sqrt{h(y)z^2 + x^2}) + (1 - \Phi(y))\Phi(x) > \Phi(a)$$

so

$$\Phi(y)(\Phi(\sqrt{h(y)z^2 + x^2}) - \Phi(a)) > (1 - \Phi(y))(\Phi(a) - \Phi(x)). \tag{11}$$

Hence obviously $h(y)z^2 + x^2 > a^2$. Let us notice that

$$\begin{aligned} \sqrt{2\pi}(\Phi(a) - \Phi(x)) &= \int_{x^2}^{a^2} \frac{1}{2\sqrt{y}} e^{-y/2} dy \\ &\geq \frac{1}{a}(e^{-x^2/2} - e^{-a^2/2}) = \frac{1}{a}e^{-x^2/2}(1 - e^{-z/2}). \end{aligned} \tag{12}$$

In similar way we show that

$$\sqrt{2\pi}(\Phi(\sqrt{h(y)z^2 + x^2}) - \Phi(a)) \leq \frac{1}{a}e^{-x^2/2}(e^{-z/2} - e^{-h(y)z^2/2}). \tag{13}$$

By (11), (12) and (13) we obtain

$$e^{-z/2} > \Phi(y)e^{-h(y)z^2/2} + 1 - \Phi(y),$$

which contradicts Lemma 11.

LEMMA 13. *If $p > 0$ and q satisfy the condition*

$$\frac{1}{2}(1-\Phi(q)) = 1-\Phi(p)$$

then

$$4e^{q^2-p^2}p^2 - p^2 \leq \ln 4.$$

PROOF. Note that $q < p$. We will consider several cases

Case 1: $q^2 > p^2$. Then $-q > p$ and therefore $1-\Phi(q) = \Phi(-q) \geq \Phi(p) = \frac{1}{2}+\frac{1}{2}\Phi(q)$, i.e. $q \leq \Phi^{-1}(1/3) \leq -0.4$ and

$$\frac{1}{\sqrt{2\pi}}pe^{-p^2/2} \leq \Phi(p) - \frac{1}{2} = \frac{1}{2}\Phi(q).$$

So, by Lemma 1

$$4p^2e^{q^2-p^2} \leq 2\pi\Phi(q)^2e^{q^2} = h(-q) \leq h(0.4)$$

$$= 2\pi\Phi(-0.4)^2e^{0.16} \leq 0.876 < \ln 4.$$

Case 2: $q^2 \leq p^2$ and $q \leq 0$. Then $\Phi(q) \leq \frac{1}{2}$, so that $p \leq \Phi^{-1}(0.75) \leq 0.679$ and

$$4p^2e^{q^2-p^2} - p^2 \leq 3p^2 < \ln 4.$$

Case 3: $q > 0$. We will consider p as a function of q. Then we have

$$\frac{d}{dq}(p^2-q^2) = 2p\frac{dp}{dq} - 2q = pe^{(p^2-q^2)/2} - 2q.$$

However by Lemma 3, $q\sqrt{h(q)} < p\sqrt{h(p)}$ so

$$\frac{2q}{p} < \frac{2\sqrt{h(p)}}{\sqrt{h(q)}} = 2e^{(p^2-q^2)/2}\frac{T(p)}{T(q)} = e^{(p^2-q^2)/2}.$$

Thus $p^2 - q^2$ is an increasing function of q. Moreover by Lemma 1, $h(q) \geq h(p)$, hence $e^{p^2-q^2} \leq 4$ and $p^2 - q^2 \leq \ln 4$. Let us consider the following table

k	q_k	T_k	p_k	d_k
1	1.20	0.1152	1.58	1.057
2	0.52	0.3016	1.04	0.812
3	0.20	0.4208	0.81	0.617
4	0.00	0.5000		

One can easily check that for $k = 1, 2, 3$,

$$T_k \geq T(q_k) \geq 2T(p_k)$$

and

$$p_k^2 - q_k^2 \leq d_k \leq (2\pi T_{k+1}^2 e^{q_{k+1}^2})^{-1} - q_{k+1}^2 \leq h(q_{k+1})^{-1} - q_{k+1}^2.$$

Suppose that $q \in [q_k, q_{k-1})$ for some $k = 1, \ldots, 4$, where additionally we put $q_0 = \infty$. Then by Lemma 2 and monotonicity of $p^2 - q^2$ we get for $k = 2, 3, 4$

$$h(q)^{-1} - q^2 \geq h(q_k)^{-1} - q_k^2 \geq d_{k-1} \geq p_{k-1}^2 - q_{k-1}^2 \geq p^2 - q^2$$

and for $k = 1$,

$$h(q)^{-1} - q^2 \geq h(q_1)^{-1} - q_1^2 \geq (2\pi T_1^2 e^{q_1^2})^{-1} - q_1^2 \geq \ln 4 \geq p^2 - q^2.$$

Hence

$$p^2 h(q) \leq 1.$$

Moreover, by Lemma 4, $F(q) \geq F(p)$, so

$$h(p)^{-1} - h(q)^{-1} \leq 2\ln(T(q)/T(p)) = \ln 4.$$

Finally we get

$$4e^{q^2-p^2}p^2 - p^2 = p^2(\frac{h(q)}{h(p)} - 1) = p^2 h(q)(\frac{1}{h(p)} - \frac{1}{h(q)}) \leq 1 \cdot \ln 4.$$

Corollary 4. *If* $w^2 - p^2 \geq \ln 4$ *then*

$$w\gamma_2^+(A) \geq p\gamma_2^+(P) = \sqrt{\frac{2}{\pi}} p e^{-p^2/2}.$$

Proof. Suppose that $\gamma_2(A) = 2\Phi(p) - 1 = \Phi(q)$ then

$$\frac{1}{2}(1 - \Phi(q)) = 1 - \Phi(p)$$

and by isoperimetric inequality

$$\gamma_2^+(A) \geq \frac{1}{\sqrt{2\pi}} e^{-q^2/2}.$$

Hence if $w\gamma_2^+(A) < p\gamma_2^+(P)$ then $w < 2pe^{(q^2-p^2)/2}$ so by Lemma 13

$$w^2 - p^2 < 4p^2 e^{q^2-p^2} - p^2 \leq \ln 4$$

and we get a contradiction.

Proof of Theorem 2. By Corollary 4 we may and will assume that

$$w^2 - p^2 < \ln 4. \tag{14}$$

Let us define for $x \in [0, w)$

$$A(x) = \{(x_1, x_2) \in (-w, w) \times R : |x_1| < x \text{ or } x_2 \leq f(x_1)\},$$

$$\gamma(x) = \gamma_2(A(x))$$

and

$$d(x) = \frac{1}{2\pi} \int_x^w e^{-(t^2+f^2(t))/2} \sqrt{1 + (f'(t))^2} dt.$$

Let $a(x)$ and $g(x)$ be given by the formulae

$$\Psi(a(x)) = \gamma(x)$$

and

$$g(x) = 2\pi w d(x) + \sqrt{2\pi}x(1-\Phi(y))e^{-x^2/2} - \sqrt{2\pi}ae^{-a^2/2},$$

where $y = f(x)$. Then $A(0) = A$, $\gamma(0) = \gamma_2(A)$, $2d(0) = \gamma_2^+(A)$ and $a(0) = p$, so in order to prove the theorem we have to show that $g(0) \geq 0$. Since $a(w) = w$ and $d(w) = 0$ we have $g(w) = 0$, so it is enough to show that g is non-increasing on $[0, w)$.

Let us also notice that for $y = f(x)$ and $a = a(x)$ we have

$$\Phi(y)\Phi(w) + (1-\Phi(y))\Phi(x) \geq \frac{1}{2} + \frac{1}{2}\gamma(x) = \Phi(a). \tag{15}$$

Moreover if $k = -f'(x)$ then

$$d'(x) = -\frac{\sqrt{1+k^2}}{2\pi}e^{-(x^2+y^2)/2}$$

$$\gamma'(x) = 2\frac{1-\Phi(y)}{\sqrt{2\pi}}e^{-x^2/2}.$$

So since $a'(x)\Psi'(a) = \gamma'(x)$ we have

$$a'(x) = (1-\Phi(y))e^{(a^2-x^2)/2}.$$

So we get that

$$e^{x^2/2}g'(x) = \sqrt{2\pi}(a^2-x^2)(1-\Phi(y)) + kxe^{-y^2/2} - \sqrt{1+k^2}we^{-y^2/2}.$$

Therefore by Lemma 12 the proof will be completed if we establish the following claim

Claim Under the above notation it is not possible that $y = f(x) > 1.5$, $a(x)^2 > x^2 + y^2 + 1.5$ and $w^2 - p^2 < \ln 4$.

Proof of the claim. Suppose that it is possible, so for some $0 \le x < w$, we have $y = f(x) > 1.5, a = a(x) > \sqrt{x^2 + y^2 + 1.5}$ and $w^2 < p^2 + \ln 4$. Let the line l tangent to the set A in the point (x, y) intersect the y-axis at the point $(0, s)$. Then since the set A is convex it is contained in the halfplane below the line l. Therefore

$$\gamma_2(D) + \Phi(u) \ge \frac{1}{2} + \frac{1}{2}\gamma_2(A),$$

where D is a set of points which have negative first coordinate and lie above the line l and u is a distance from the origin to l. As $\gamma_2(D) \le \frac{1}{2}(1 - \Phi(s))$ and $\gamma_2(A) = 2\Phi(p) - 1$ we obtain

$$1 - \Phi(u) \le 1 - \Phi(p) + \frac{1}{2}(1 - \Phi(s)).$$

So by Lemma 8

$$\frac{1}{2}e^{(u^2-s^2)/2} + e^{(u^2-p^2)/2} \ge 1. \tag{16}$$

In particular, since $u^2 \le s^2$ and $u^2 \le x^2 + y^2$ we get that $w^2 \le p^2 + \ln 4 \le u^2 + 2\ln 4 \le x^2 + y^2 + 2\ln 4$. Let us notice that by (15)

$$(1 - \Phi(y))(\Phi(a) - \Phi(x)) \le \Phi(y)(\Phi(w) - \Phi(a)).$$

Since $a^2 > x^2 + y^2 + 1.5 > x^2 + 3.75$, by Lemma 7 we obtain

$$\Phi(a) - \Phi(x) \ge (1 - \Phi(x))(1 - e^{(x^2-a^2)/2}) \ge (1 - \Phi(x))(1 - e^{-1.875}).$$

Moreover (see the proof of Lemma 7)

$$\Phi(w) - \Phi(a) \leq \frac{1}{\sqrt{2\pi}a}(e^{-a^2/2} - e^{-w^2/2})$$

and by Lemma 2

$$1 - \Phi(y) \geq \frac{1}{\sqrt{2\pi(y^2+2)}}e^{-y^2/2}.$$

Hence

$$(1 - \Phi(x))e^{x^2/2} \leq (1 - e^{-1.875})^{-1}\frac{\sqrt{y^2+2}}{a}(e^{(x^2+y^2-a^2)/2} - e^{(x^2+y^2-w^2)/2}). \tag{17}$$

Suppose first that $x \leq 0.8$, then by Lemma 1

$$(1 - \Phi(x))e^{x^2/2} \geq (1 - \Phi(0.8))e^{0.32} \geq 0.29.$$

On the other hand, since $\sqrt{y^2+2}/a \leq \sqrt{(y^2+2)/(y^2+1.5)} \leq \sqrt{4.25/3.75}$ we get by (17) that

$$(1 - \Phi(x))e^{x^2/2} \leq (1 - e^{-1.875})^{-1}\sqrt{\frac{4.25}{3.75}}(e^{-0.75} - \frac{1}{4}) \leq 0.28.$$

This contradiction shows that $x > 0.8$ and then $a \geq \sqrt{x^2+y^2+1.5} \geq \sqrt{y^2+2}$. Thus by (17)

$$(1 - \Phi(x))e^{x^2/2} \leq (1 - e^{-1.875})^{-1}(e^{(x^2+y^2-a^2)/2} - e^{(x^2+y^2-w^2)/2}). \tag{18}$$

Let us consider the following table

k	d_k	x_k	T_k	c_k
1	1.50	3.23	0.0005	0.092
2	1.85	2.29	0.0109	0.149
3	2.12	1.80	0.0358	0.180
4	2.27	1.53	0.0629	0.202
5	2.39	1.31	0.0950	0.224
6	2.52	1.05	0.1468	0.254
7	2.71	0.51		

The reader may check that the numbers in the table satisfy the following inequalities for $k = 1, \ldots, 6$

$$T_k \leq 1 - \Phi(x_k), \quad c_k < T_k e^{x_k^2/2}$$

$$c_k > (1 - e^{-1.875})^{-1}(e^{-0.75} - e^{-d_{k+1}/2})$$

and

$$x_k \geq \sqrt{2\ln 4 - d_k} + \sqrt{-2\ln(2 - 4e^{-d_k/2})}.$$

The last inequality holds also for $k = 7$. Suppose that

$$w^2 - x^2 - y^2 \in [d_k, d_{k+1}] \text{ for some } k = 1, 2, \ldots, 7,$$

where we additionally define $d_8 = \infty$. Then

$$x^2 + y^2 - u^2 = x^2 + y^2 - w^2 + w^2 - u^2 \leq -d_k + 2\ln 4$$

and

$$u^2 - p^2 \leq x^2 + y^2 - p^2 \leq x^2 + y^2 - w^2 + \ln 4 \leq \ln 4 - d_k.$$

Thus by (16) we get

$$s^2 - u^2 \leq -2\ln(2 - 2e^{(u^2-p^2)/2}) \leq -2\ln(2 - 4e^{-d_k/2}).$$

Consider the triangle ABC with $A = (0,0), B = (x,y)$ and $C = (0,s)$, then by Pythagoras Theorem

$$x \leq |BC| \leq \sqrt{|AC|^2 - u^2} + \sqrt{|AB|^2 - u^2}$$

$$\leq \sqrt{-2\ln(2 - 4e^{-d_k/2})} + \sqrt{2\ln 4 - d_k} \leq x_k.$$

Hence if $k = 7$, $x < 0.8$ which contradicts our previous assumption. For $k < 7$ we have by Lemma 1

$$(1 - \Phi(x))e^{x^2/2} \geq (1 - \Phi(x_k))e^{x_k^2/2} > c_k$$

and

$$(1 - e^{-1.875})^{-1}(e^{(x^2+y^2-a^2)/2} - e^{(x^2+y^2-w^2)/2})$$

$$\leq (1 - e^{-1.875})^{-1}(e^{-0.75} - e^{-d_{k+1}/2}) < c_k.$$

The above inequalities contradict (18) and the proof is now completed.

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