On the almost sure boundedness of norms of some empirical operators *†

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Abstract

Let $X_1, X_2, \ldots$ be i.i.d. random variables and h be a symmetric measurable real function. We show that the norms of operators on l_2^n given by the matrix $(\frac{1}{n}h(X_i, X_j)\delta_{i\neq j})_{1\leq i,j\leq n}$ are a.s. bounded if and only if h is square integrable.

Introduction. Let $(E, \mathcal{F}, \mu)$ be a probability space, $\mu_2 = \mu \otimes \mu$, $h : E \times E \to \mathbb{R}$ be a symmetric measurable function and let H be an operator on $L_2(\mu) = L_2(E, \mathcal{F}, \mu)$ defined by the formula

$$Hg(x) = \int_E h(x,y)g(y)d\mu(y), \quad g \in L_2(\mu), x \in E.$$

Let $X_1, Y_1, X_2, Y_2, \ldots$ be i.i.d. (μ) random variables with values in E. We may then define random operators H_n on l_2^n, defining their matrices in canonical basis by

$$H_n = (\frac{1}{n}\delta_{i\neq j}h(X_i, X_j))_{1\leq i,j\leq n}.$$

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Operators H_n can be see as natural empirical counterparts to H, as it was explained in Kolchinski and Giné (1995).

In their paper V. Kolchinsky and E. Giné proved that if $h \in L_2(\mu_2)$ (so in other words if H is a Hilbert-Schmidt operator) then the spectrum of H may be a.s. approximated in very strong sence by the spectrum of H_n (e.g. $\|H_n\|_{HS} \to \|H\|_{HS}$ a.s.). In particular $\sup \|H_n\| < \infty$ a.s.

They posed also the question if the converse is true - namely if $\sup \|H_n\| < \infty$ a.s. imply that $h \in L^2(\mu_2)$. In this paper we give a positive answer to this question.

Lemma 1 *Let $Z_1, Z_2, \dots$ be a sequence of i.i.d. nonnegative random variables such that $Z_i \leq 4n^2$ a.s. and $EZ_i \geq n$. Then*

$$P(\sum_{i=1}^{n} Z_i \geq \frac{1}{2}n^2) \geq \frac{1}{20}$$

Proof. We have

$$E(\sum_{i=1}^{n} Z_i)^2 \leq (\sum_{i=1}^{n} EZ_i)^2 + \sum_{i=1}^{n} EZ_i^2 \leq (\sum_{i=1}^{n} EZ_i)^2 + 4n^2(\sum_{i=1}^{n} EZ_i) \leq 5(\sum_{i=1}^{n} EZ_i)^2.$$

Hence by Paley-Zygmund inequality (cf. Kahane (1985), p.8)

$$P(\sum_{i=1}^{n} Z_i \geq \frac{1}{2}n^2) \geq P(\sum_{i=1}^{n} Z_i \geq \frac{1}{2}(E\sum_{i=1}^{n} Z_i)) \geq$$

$$\frac{1}{4}\frac{(E\sum_{i=1}^{n} Z_i)^2}{E(\sum_{i=1}^{n} Z_i)^2} \geq \frac{1}{20}.$$

Lemma 2 *Suppose that a measurable set $A \subset E \times E$ satisfy the following conditions*

$$\forall_{x \in E}\ \mu(A_x) \leq n^{-1} \tag{1}$$

and

$$\forall_{y \in E}\ \mu(A^y) \leq n^{-1}. \tag{2}$$

Then

$$P(\exists_{1 \leq i,j \leq n}\ (X_i, Y_j) \in A) \geq \frac{1}{16}\min(1, n^2\mu_2(A)).$$

Proof. Obviously we may assume that $\mu_2(A) > 0$. Let us first notice that

$$P(\exists_{1\le i,j\le n}\ (X_i, Y_j) \in A) = P(\sum_{1\le i,j\le n} I_A(X_i, Y_j) > 0).$$

We have

$$E \sum_{1\le i,j\le n} I_A(X_i, Y_j) = n^2 E I_A(X_1, Y_1) = n^2 \mu_2(A)$$

and

$$E(\sum_{1\le i,j\le n} I_A(X_i, Y_j))^2 = n^2 E I_A(X_1, Y_1) + n^2(n-1)^2 E I_A(X_1, Y_1) I_A(X_2, Y_2)$$

$$+ n^2(n-1) E I_A(X_1, Y_1) I_A(X_1, Y_2) + n^2(n-1) E I_A(X_1, Y_1) I_A(X_2, Y_1).$$

But

$$E I_A(X_1, Y_1) I_A(X_2, Y_2) = \mu_2^2(A),$$

$$E I_A(X_1, Y_1) I_A(X_1, Y_2) = E\mu^2(A_{X_1}) \le n^{-1} E\mu(A_{X_1}) = n^{-1}\mu_2(A)$$

and

$$E I_A(X_1, Y_1) I_A(X_2, Y_1) = E\mu^2(A^{Y_1}) \le n^{-1} E\mu(A^{Y_1}) = n^{-1}\mu_2(A).$$

Hence

$$E(\sum_{1\le i,j\le n} I_A(X_i, Y_j))^2 \le 3n^2\mu_2(A) + n^4\mu_2^2(A).$$

So by Paley-Zygmund inequality

$$P(\sum_{1\le i,j\le n} I_A(X_i, Y_j) \ge \frac{1}{2} E \sum_{1\le i,j\le n} I_A(X_i, Y_j)) \ge \frac{1}{4} \frac{(E\sum_{1\le i,j\le n} I_A(X_i, Y_j))^2}{E(\sum_{1\le i,j\le n} I_A(X_i, Y_j))^2}$$

$$\ge \frac{1}{16} \min(1, n^2\mu_2(A)).$$

Theorem 1 *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with values in $(E, \mathcal{F})$ and h be a symmetric measurable function on $E \times E$. If a sequence*

$$\|\frac{1}{n}(h(X_i, X_j)\delta_{i\ne j})_{1\le i,j\le n}\|_{l_n^2 \to l_n^2}$$

is a.s. bounded then

$$Eh^2(X_1, X_2) < \infty.$$

Proof. Let $Y_1, Y_2, \ldots$ be an independent copy of X_i. Let $I_l = \{2^l + 1, 2^l + 2, \ldots, 2^{l+1}\}$. Let us notice that

$$\|2^{-l}(h(X_i,X_j)\delta_{i\neq j})_{1\leq i,j\leq 2^l}\|_{l^2_{2^l}\to l^2_{2^l}} \geq \|2^{-l}(h(X_i,X_j)\delta_{i\neq j})_{i,j\in I_{l-1}}\|_{l^2_{2^{l-1}}\to l^2_{2^{l-1}}}$$

and random variables

$$\|2^{-l}(h(X_i,X_j)\delta_{i\neq j})_{i,j\in I_{l-1}}\|_{l^2_{2^{l-1}}\to l^2_{2^{l-1}}},\ l = 1, 2, \ldots$$

are independent. Hence by Borel-Cantelli Lemma we get that for some $C < \infty$

$$\sum_{l=1}^{\infty} P(\|2^{-l}(h(X_i,X_j)\delta_{i\neq j})_{i,j\in I_{l-1}}\|_{l^2_{2^{l-1}}\to l^2_{2^{l-1}}} \geq C) < 2^{-4}.$$

But

$$\|(h(X_i,X_j)\delta_{i\neq j})_{i,j\in I_l}\|_{l^2_{2^l}\to l^2_{2^l}} \geq \max_{i\in I_l}\Big(\sum_{j\in I_l, j\neq i} h^2(X_i,X_j)\Big)^{1/2} \geq$$

$$\max_{2^l < i \leq 2^l + 2^{l-1}}\Big(\sum_{j=2^l+2^{l-1}+1}^{2^{l+1}} h^2(X_i,X_j)\Big)^{1/2}.$$

Therefore

$$P(\|2^{-l-2}(h(X_i,X_j)\delta_{i\neq j})_{i,j\in I_{l+1}}\|_{l^2_{2^{l+1}}\to l^2_{2^{l+1}}} \geq C)$$

$$\geq P\Big(\max_{1\leq i\leq 2^l}\sum_{j=1}^{2^l} h^2(X_i,Y_j) \geq C^2 2^{2l+4}\Big).$$

We may obviously assume that $C^2 = 2^{-5}$ and hence get

$$\sum_{l=1}^{\infty} P\Big(\max_{1\leq i\leq 2^l}\sum_{j=1}^{2^l} h^2(X_i,Y_j) \geq 2^{2l-1}\Big) < 2^{-4}.$$

Let us define the following sets

$$A_l = \{(x,y) : 2^{2l+2} > h^2(x,y) \geq 2^{2l}\},$$

$$B_l = \{(x,y) \in A_l : \mu((A_l)_x) \geq 2^{-l}\},$$

$$C_l = \{(x,y) \in A_l : \mu((A_l)^y) \geq 2^{-l}\}$$

and

$$D_l = A_l \setminus (B_l \cup C_l).$$

Then

$$\forall_{x,y\in E}\ \mu((D_l)_x), \mu((D_l)^y) \le 2^{-l}. \tag{3}$$

Let

$$H_1(x,y) = \sum_{l=1}^{\infty} h^2(x,y) I_{\{(x,y)\in B_l\}},$$

$$H_2(x,y) = \sum_{l=1}^{\infty} h^2(x,y) I_{\{(x,y)\in C_l\}}$$

and

$$H_3(x,y) = \sum_{l=1}^{\infty} h^2(x,y) I_{\{(x,y)\in D_l\}}.$$

Then

$$Eh^2(X,Y) \le 4 + EH_1(X,Y) + EH_2(X,Y) + EH_3(X,Y)$$

and by the symmetry of h^2 we have $EH_1(X,Y) = EH_2(X,Y)$.

Let us notice that by the definition of A_l

$$EH_3(X,Y) \le \sum_{l=1}^{\infty} 2^{2l+2}\mu(D_l).$$

By Lemma 2 and (3)

$$2^{-4}\min(1, 2^{2l}\mu(D_l)) \le P(\exists_{1\le i,j\le 2^l}\ (X_i,Y_j)\in D_l) \le$$

$$P(\max_{1\le i,j\le 2^l} h^2(X_i,Y_j) \ge 2^{2l}) \le P(\max_{1\le i\le 2^l}\sum_{j=1}^{2^l} h^2(X_i,Y_j) \ge 2^{2l}).$$

Hence

$$EH_3(X,Y) \le 2^6 \sum_{l=1}^{\infty} P(\max_{1\le i\le 2^l}\sum_{j=1}^{2^l} h^2(X_i,Y_j) \ge 2^{2l-1}) < \infty.$$

To estimate $EH_1(X,Y)$ let us define

$$h_1(x) = E_Y H_1(x,Y)$$

and

$$V_l = \{x : 2^l \le h_1(x) < 2^{l+1}\}.$$

Let us notice that by our definition of $H_1(x,y)$ if for some x, y we have $2^{2k} \le H_1(x,y) < 2^{2k+2}$ then $P_Y(H_1(x,Y) \ge 2^{2k}) \ge 2^{-k}$, so $h_1(x) \ge 2^k$. Therefore

$$H_1(x,y) \le 2^{2l+2} \text{ for } \ x \in V_l.$$

Now by Lemma 1 we obtain that

$$P_Y(\sum_{j=1}^{2^l} H_1(x,Y_j) \ge 2^{2l-1}) \ge \frac{1}{20} \text{ for } \ x \in V_l.$$

Since $H_1 \le h^2$ and

$$P(\exists_{1 \le i \le 2^l} \ X_i \in V_l) \ge \frac{1}{2} \min(1, 2^l \mu(V_l))$$

we get that

$$P(\max_{1 \le i \le 2^l} \sum_{j=1}^{2^l} h^2(X_i, Y_j) \ge 2^{2l-1}) \ge \frac{1}{40} \min(1, 2^l \mu(V_l)).$$

But this means that

$$EH_1(X,Y) = Eh_1(X) \le 2 + \sum_{l=1}^{\infty} 2^{l+1} \mu(V_l) \le$$

$$2 + 80 P(\max_{1 \le i \le 2^l} \sum_{j=1}^{2^l} h^2(X_i, Y_j) \ge 2^{2l-1}) < \infty.$$

From Theorem 1 and the results of Kolchinsky and Giné (1995) immediately follows the following Corollary, which may be seen as some type of law of large numbers for U-statistics.

Corollary 1 *If $h \in L^2(\mu_2)$ then $\lim_{n\to\infty} \|H_n\| = \|H\|$ a.s.. Conversely if $\limsup_{n\to\infty} \|H_n\| < \infty$ a.s. then $h \in L^2(\mu_2)$.*

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References

Kahane, J.-P. (1985), *Some random series of functions*, Cambridge University Press, 2nd ed.

Kolchinsky, V., Giné E. (1995), *Random matrix approximation of spectra of compact integral operators*, preprint.