On the best constant in the Khintchine-Kahane inequality *

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Abstract

We prove that if r_i is the Rademacher system of functions then

$$(\int \|\sum_{i=1}^{n} x_i r_i(t)\|^2 dt)^{1/2} \leq \sqrt{2} \int \|\sum_{i=1}^{n} x_i r_i(t)\| dt$$

for any sequence of vectors x_i in any linear normed space F.

Introduction: The classical result of Khintchine [3] states that for each $p, q > 0$ there exists a constant $c_{p,q}$ such that for any real numbers $x_1, \dots, x_n$

$$(\int |\sum_{i=1}^{n} x_i r_i(t)|^p dt)^{1/p} \leq c_{p,q} (\int |\sum_{i=1}^{n} x_i r_i(t)|^q dt)^{1/q} \tag{1}$$

The smallest constant $c_{p,q}$ we will denote by $C_{p,q}^R$ and will call the best one. Obviously for $p \leq q$ $C_{p,q}^R = 1$, but it took some efforts to calculate the other best constants. The especially interesting case $p = 2, q = 1$ was first solved by S.J. Szarek [4], who proved $C_{2,1}^R = \sqrt{2}$. A simpler proof was given by U. Haagerup [1] who also found $C_{p,2}^R$ and $C_{2,p}^R$ for each $p > 0$. A simple and elementary proof that $C_{2,1}^R = \sqrt{2}$ was also presented by B. Tomaszewski [6].

J.-P. Kahane [2] generalized the result of Khintchine to Banach space valued sequences $x_1, \dots, x_n$ replacing in (1) the absolute value by the norm. Let $C_{p,q}$ denote the smallest constant in the vector-valued inequality. It is of interest if the constants are the same in the vector and the real case. As far as we know the best result for p=2 and q=1 known up to now was obtained by B. Tomaszewski [5], who proved that $C_{2,1} \leq \sqrt{3}$. In this paper we show that $C_{2,1} = \sqrt{2}$, and we think that our proof is simpler than the ones known for real numbers.

Notation: For $\sigma = (\sigma_1, \dots, \sigma_n) \in \{0,1\}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in R^n$ $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n), \eta = (\eta_1, \dots, \eta_n) \in \{-1,1\}^n$ and $x_1, \dots, x_n \in F$ let us define

$|\sigma| = \sum_{i=1}^{n} \sigma_i$

$\alpha^\sigma = \prod_{i=1}^{n} \alpha_i^{\sigma_i}$ (where $x^0 = 1$ for any $x \in R$)

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$-\varepsilon = (-\varepsilon_1, \dots, -\varepsilon_n)$
$X_\varepsilon = \| \sum_{i=1}^n \varepsilon_i x_i \|$
$d(\varepsilon, \eta) = \text{card}\{i : \varepsilon_i \neq \eta_i\}$
By E(.) we will denote the mean value of (.).

Results: We will prove the following theorem:

Theorem 1 *Let $S = \sum_{i=1}^n \varepsilon_i x_i$, where ε_i are independent Bernoulli random variables and x_i are vectors of a normed linear space F. Then*

$$(E\|S\|^2)^{1/2} \leq \sqrt{2} E\|S\|$$

The constant $\sqrt{2}$ is the best possible.

Proof:

Differentiating in t both sides of the equality

$$t^2 \prod_{i=1}^n (1 + t^{-1}\alpha_i) = \sum_{\sigma \in \{0,1\}^n} t^{2-|\sigma|} \alpha^\sigma$$

and evaluating them at t=1 we get

$$2 \prod_{i=1}^n (1 + \alpha_i) - \sum_{j=1}^n \alpha_j \prod_{i=1, i \neq j}^n (1 + \alpha_i) = \sum_{\sigma \in \{0,1\}^n} (2 - |\sigma|) \alpha^\sigma$$

Hence we obtain

$$\sum_{\varepsilon, \eta \in \{-1,1\}^n} (2 \prod_{i=1}^n (1+\varepsilon_i \eta_i) - \sum_{j=1}^n \varepsilon_j \eta_j \prod_{i=1, i \neq j}^n (1+\varepsilon_i \eta_i)) X_\varepsilon X_\eta = \sum_{\varepsilon, \eta \in \{-1,1\}^n} \sum_{\sigma \in \{0,1\}^n} (2-|\sigma|) \varepsilon^\sigma \eta^\sigma X_\varepsilon X_\eta$$

$$= \sum_{\sigma \in \{0,1\}^n} (2 - |\sigma|) (\sum_{\varepsilon \in \{-1,1\}^n} \varepsilon^\sigma X_\varepsilon)^2 \leq 2(\sum_{\varepsilon \in \{-1,1\}^n} X_\varepsilon)^2$$

The last inequality holds because $X_\varepsilon = X_{-\varepsilon}$ for each ε, so that obviously

$$\sum_{\varepsilon \in \{-1,1\}^n} \varepsilon^\sigma X_\varepsilon = 0$$

for each σ with $|\sigma| = 1$

Since the left-hand side of the above inequality is equal to

$$2^{n+1} \sum_{\varepsilon \in \{-1,1\}^n} X_\varepsilon^2 - n 2^{n-1} \sum_{\varepsilon \in \{-1,1\}^n} X_\varepsilon^2 + 2^{n-1} \sum_{\varepsilon, \eta \in \{-1,1\}^n, d(\varepsilon,\eta)=1} X_\varepsilon X_\eta$$

we arrive at

$$2^n \sum_{\varepsilon \in \{-1,1\}^n} X_\varepsilon^2 + 2^{n-1} \sum_{\varepsilon \in \{-1,1\}^n} X_\varepsilon (\sum_{\eta \in \{-1,1\}^n, d(\varepsilon,\eta)=1} X_\eta - (n-2) X_\varepsilon) \leq 2(\sum_{\varepsilon \in \{-1,1\}^n} X_\varepsilon)^2 \quad (2)$$

By the triangle inequality for each fixed ε we get

$$(n-2)X_\varepsilon \leq \sum_{\eta\in\{-1,1\}^n, d(\varepsilon,\eta)=1} X_\eta$$

So inequality (2) yields

$$2^n \sum_{\varepsilon\in\{-1,1\}^n} X_\varepsilon^2 \leq 2(\sum_{\varepsilon\in\{-1,1\}^n} X_\varepsilon)^2$$

Dividing by 2^{2n} we get

$$E\|S\|^2 \leq 2(E\|S\|)^2$$

To see that the constant $\sqrt{2}$ is the best possible it suffices to take $n = 2, x_1 = x_2 \neq 0$.

Remark 1: If we replace in the above proof X_ε by X_ε^p and use the inequality

$$X_\varepsilon \leq \frac{1}{n-2} \sum_{\eta\in\{-1,1\}^n, d(\varepsilon,\eta)=1} X_\eta \leq \frac{n}{n-2}(\frac{1}{n} \sum_{\eta\in\{-1,1\}^n, d(\varepsilon,\eta)=1} X_\eta^p)^{1/p}$$

we will obtain that for $p \in [1, 2)$

$$(E\|S\|^{2p})^{1/2p} \leq (1-p/2)^{-1/2p}(E\|S\|^p)^{1/p}$$

but we do not think that the above constants are optimal for $p > 1$.

Remark 2: Since for each bounded real random variable X the function $f(r) = r \ln E|X|^{1/r}$ is convex Theorem 1 yields that for each $q \in (0, 1], p \in (0, 2], q \leq p$ the following inequality holds:

$$(E\|S\|^p)^{1/p} \leq 2^{1/q-1/p}(E\|S\|^q)^{1/q}$$

and the constants $2^{1/q-1/p}$ are optimal.

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