

Tail and moment estimates for chaoses generated by symmetric random variables with logarithmically concave tails *

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Abstract

We present two-sided estimates of moments and tails of polynomial chaoses of order at most three generated by independent symmetric random variables with log-concave tails as well as for chaoses of arbitrary order generated by independent symmetric exponential variables. The estimates involve only deterministic quantities and are optimal up to constants depending only on the order of the chaos variable.

Keywords: Polynomial chaoses, tail and moment estimates, metric entropy

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1 Introduction

A (homogeneous) polynomial chaos of order d is a random variable defined as

$$\sum_{i_1, \dots, i_d=1}^n a_{i_1, \dots, i_d} X_{i_1} \cdots X_{i_d}, \quad (1)$$

where X_1, \dots, X_n is a sequence of independent real random variables and $(a_{i_1, \dots, i_d})_{1 \leq i_1, \dots, i_d \leq n}$ is a d -indexed symmetric array of real numbers, satisfying $a_{i_1, \dots, i_d} = 0$ whenever there exists $k \neq l$ such that $i_k = i_l$.

Random variables of this type appear in many branches of modern probability, e.g. as approximations of multiple stochastic integrals, elements of Fourier expansions in harmonic analysis on the discrete cube (when the underlying variables X_i 's are independent Rademachers), in subgraph counting problems for random graphs (in this case X_i 's are zero-one random variables) or in statistical physics.

Chaos of order one are just linear combinations of independent random variables and their behavior is well-understood. Chaos of higher orders behave in a more complex way as the summands in (1) are no longer independent. Nevertheless, due to their simple algebraic structure, many counterparts of classical results for sums of independent random variables are available. Among well known results there are Khinchine type inequalities and tail bounds involving the variance (see e.g. [18, 11, 4] or Chapter 3 of [7]). There are also two-sided inequalities in terms of expectations of suprema of empirical processes, which themselves are in general difficult to estimate (see [3, 5] for Gaussian chaoses and [17, 1] for the general case of chaoses generated by variables with log-concave tails).

In several cases, under additional assumptions on the distribution of X_i 's, even more precise results are known, which give two sided estimates on moments of polynomial chaoses in terms of deterministic quantities involving only the coefficients a_{i_1, \dots, i_d} (the estimates are accurate up to a constant depending only on d). Examples include Gaussian chaoses of arbitrary order [14], chaoses generated by nonnegative random variables with log-concave tails [15] and chaoses of order at most two, generated by symmetric random variables with log-concave tail ([10] for $d = 1$ and [13] for $d = 2$).

The aim of this paper is to provide some extensions of these results. In particular we provide two sided estimates for moments of chaoses of order

three generated by symmetric random variables with log-concave tails (Theorems 3.1 and 3.2) and for chaoses of arbitrary order, generated by symmetric exponential variables (Theorem 3.4).

Before we formulate precisely our main results let us recall the notion of decoupled chaos and decoupling inequalities. A decoupled chaos of order d is a random variable of the form

$$\sum_{i_1, \dots, i_d=1}^n a_{i_1, \dots, i_d} X_{i_1}^1 \cdots X_{i_d}^d, \quad (2)$$

where $(a_{i_1, \dots, i_d})_{1 \leq i_1, \dots, i_d \leq n}$ is a d -indexed array of real numbers and X_i^l , $i = 1, \dots, n$, $l = 1, \dots, d$, are independent random variables.

One can easily see that each decoupled chaos can be represented in the form (1) with a modified matrix and for suitably larger n . However it turns out that for the purpose of estimating tails or moments of chaoses it is enough to consider decoupled chaoses. More precisely, we have the following important result due to de la Peña and Montgomery-Smith [8].

Theorem 1.1. *Let $(a_{i_1, \dots, i_d})_{1 \leq i_1, \dots, i_d \leq n}$ be a symmetric d -indexed array such that $a_{i_1, \dots, i_d} = 0$ whenever there exists $k \neq l$ such that $i_k = i_l$. Let X_1, \dots, X_n be independent random variables and $(X_i^j)_{1 \leq i \leq n, j = 1, \dots, d}$, be independent copies of the sequence $(X_i)_{1 \leq i \leq n}$. Then for all $t \geq 0$,*

$$\begin{aligned} L_d^{-1} \mathbb{P} \left(\left| \sum_{i_1, \dots, i_d=1}^n a_{i_1, \dots, i_d} X_{i_1}^1 \cdots X_{i_d}^d \right| \geq L_d t \right) \\ \leq \mathbb{P} \left(\left| \sum_{i_1, \dots, i_d=1}^n a_{i_1, \dots, i_d} X_{i_1} \cdots X_{i_d} \right| \geq t \right) \\ \leq L_d \mathbb{P} \left(\left| \sum_{i_1, \dots, i_d=1}^n a_{i_1, \dots, i_d} X_{i_1}^1 \cdots X_{i_d}^d \right| \geq L_d^{-1} t \right), \end{aligned}$$

where $L_d \in (0, \infty)$ depends only on d . In particular, for all $p \geq 1$,

$$\begin{aligned} \tilde{L}_d^{-1} \left\| \sum_{i_1, \dots, i_d=1}^n a_{i_1, \dots, i_d} X_{i_1}^1 \cdots X_{i_d}^d \right\|_p &\leq \left\| \sum_{i_1, \dots, i_d=1}^n a_{i_1, \dots, i_d} X_{i_1} \cdots X_{i_d} \right\|_p \\ &\leq \tilde{L}_d \left\| \sum_{i_1, \dots, i_d=1}^n a_{i_1, \dots, i_d} X_{i_1}^1 \cdots X_{i_d}^d \right\|_p, \end{aligned}$$

where \tilde{L}_d depends only on d .

If we are not interested in the values of numerical constants, the above theorem reduces estimation of tails and moments of general chaoses of order d to decoupled chaoses. The importance of this result stems from the fact that the latter can be treated conditionally as chaoses of smaller order, which allows for induction with respect to d . Since the reduction is straightforward, in the sequel when formulating our results we will restrict our attention to the decoupled case.

Let us finish the introduction by remarking that two-sided bounds on moments of chaoses of the form (1) can be used to give two-sided estimates for more general random variables, i.e. tetrahedral polynomials in X_1, \dots, X_d , e.g. to polynomials in which every variable appears in a power at most 1. This is thanks to the following simple observation, which to our best knowledge has remained unnoticed.

Proposition 1.2. *For $j = 0, 1, \dots, d$ let $(a_{i_1, \dots, i_j}^j)_{1 \leq i_1, \dots, i_j \leq n}$ be a k -indexed symmetric array of real numbers (or more generally elements of some normed space), such that $a_{i_1, \dots, i_j}^j = 0$ if $i_k = i_l$ for some $1 \leq k < l \leq j$ (for $j = 0$ we have just a single number a_\emptyset^0). Let X_1, \dots, X_n be independent mean zero random variables. Then there exists a constant $L_d \in (0, \infty)$, depending only on d , such that for all $p \geq 1$,*

$$\sum_{j=0}^d \left\| \sum_{i_1, \dots, i_j=1}^n a_{i_1, \dots, i_j}^j X_{i_1} \cdots X_{i_j} \right\|_p \leq L_d \left\| \sum_{j=0}^d \sum_{i_1, \dots, i_j=1}^n a_{i_1, \dots, i_j}^j X_{i_1} \cdots X_{i_j} \right\|_p.$$

Note that a reverse inequality boils down just to the triangle inequality in L_p and so the above proposition immediately gives two-sided estimates of moments of tetrahedral polynomials from estimates for homogeneous chaoses. Since the details are straightforward we will not state explicitly the results which can be obtained from the inequalities we present. The easy (given general results on decoupling) but notationally involved proof of Proposition 1.2 is deferred to the appendix.

The organization of the article is as follows. After introducing the necessary notation (Section 2) we state our main results (Section 3) and devote the rest of the paper to their quite involved proof. In the course of the proof we provide entropy estimates for special kinds of metrics on subsets of certain product sets (Section 5.2) as well as bounds on empirical processes indexed by such sets (Section 6 and Section 7 where we also provide some partition theorems). We believe that these results may be of independent interest.

In Section 8 we conclude the proof of our result for chaoses of order three and in Section 9 we give a proof of estimates for chaoses of arbitrary order generated by exponential variables.

2 Definitions and notation

Let $(X_i^j)_{1 \leq i \leq n, 1 \leq j \leq d}$ be a matrix of independent symmetric random variables with logarithmically concave tails, i.e. such that the functions $N_i^j: [0, \infty) \rightarrow [0, \infty]$ defined by

$$N_i^j(t) = -\ln \mathbb{P}(|X_i^j| \geq t)$$

are convex. We assume that r.v.'s are normalized in such a way that

$$\inf\{t \geq 0: N_i^j(t) \geq 1\} = 1. \quad (3)$$

We set

$$\hat{N}_i^j(t) = \begin{cases} t^2 & \text{for } |t| \leq 1 \\ N_i^j(|t|) & \text{for } |t| > 1. \end{cases}$$

Remark When working with $d = 1$ we will suppress the upper index j and write simply X_i or N_i .

Recall that the p -th moment of a real random variable X is defined as $\|X\|_p^p = \mathbb{E}|X|^p$.

For a sequence $(x_i)_i$ of real numbers (sometimes multiindexed) we will denote $\|x\|_2 = \sqrt{\sum_i x_i^2}$ and $\|x\|_1 = \sum_i |x_i|$.

For $\mathbf{i} \in \{1, \dots, n\}^d$ and $I \subset \{1, \dots, d\}$ we write $\mathbf{i}_I = (i_k)_{k \in I}$. By P_d we will denote the family of all partitions of $\{1, \dots, d\}$ into nonempty, pairwise disjoint subsets. For $\mathcal{J} = \{I_1, \dots, I_k\} \in P_d$, $p \geq 2$ and a multiindexed matrix $(a_{\mathbf{i}})$ we define

$$\begin{aligned} & \|(a_{\mathbf{i}})\|_{\mathcal{J}, p}^{\mathcal{N}} & (4) \\ &= \sum_{s_1 \in I_1, \dots, s_k \in I_k} \sup \left\{ \sum_{\mathbf{i}} a_{\mathbf{i}} \prod_{l=1}^k x_{\mathbf{i}_{I_l}}^l : \sum_{i_{s_l}} \hat{N}_{i_{s_l}}^{s_l} (\|(x_{\mathbf{i}_{I_l}}^l)_{\mathbf{i}_{I_l \setminus \{s_l\}}}\|_2) \leq p, 1 \leq l \leq k \right\}. \end{aligned}$$

Remark When I_l is a singleton, i.e. $I_l = \{s_l\}$, then for any fixed value of i_{s_l} , $\|(x_{\mathbf{i}_{I_l}}^l)_{\mathbf{i}_{I_l \setminus \{s_l\}}}\|_2 = |x_{i_{s_l}}^l|$.

In particular for $d = 3$ we have

$$\begin{aligned} \|(a_{ijk})\|_{\{1,2,3\},p}^{\mathcal{N}} &= \sup \left\{ \sum_{ijk} a_{ijk} x_{ijk} : \sum_i \hat{N}_i^1 \left(\sqrt{\sum_{j,k} x_{ijk}^2} \right) \leq p \right\} \\ &+ \sup \left\{ \sum_{ijk} a_{ijk} x_{ijk} : \sum_j \hat{N}_j^2 \left(\sqrt{\sum_{i,k} x_{ijk}^2} \right) \leq p \right\} \\ &+ \sup \left\{ \sum_{ijk} a_{ijk} x_{ijk} : \sum_k \hat{N}_k^3 \left(\sqrt{\sum_{i,j} x_{ijk}^2} \right) \leq p \right\}, \end{aligned}$$

$$\begin{aligned} \|(a_{ijk})\|_{\{1,2\}\{3\},p}^{\mathcal{N}} &= \sup \left\{ \sum_{ijk} a_{ijk} x_{ij} y_k : \sum_i \hat{N}_i^1 \left(\sqrt{\sum_j x_{ij}^2} \right) \leq p, \sum_k \hat{N}_k^3(y_k) \leq p \right\} \\ &+ \sup \left\{ \sum_{ijk} a_{ijk} x_{ij} y_k : \sum_j \hat{N}_j^2 \left(\sqrt{\sum_i x_{ij}^2} \right) \leq p, \sum_k \hat{N}_k^3(y_k) \leq p \right\} \end{aligned}$$

and

$$\|(a_{ijk})\|_{\{1\}\{2\}\{3\},p}^{\mathcal{N}} = \sup \left\{ \sum_{ijk} a_{ijk} x_i y_j z_k : \sum_i \hat{N}_i^1(x_i) \leq p, \sum_j \hat{N}_j^2(y_j) \leq p, \sum_k \hat{N}_k^3(z_k) \leq p \right\}.$$

By B_1^n and B_2^n we will denote respectively the standard ℓ_1^n and ℓ_2^n balls, e.g. $B_1^n = \{x \in \mathbb{R}^n : \|x\|_1 \leq 1\}$ and $B_2^n = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$.

Throughout the article we will write L_d, L to denote constants depending only on d and universal constants respectively. In all cases the values of a constant may differ at each occurrence.

By $A \sim_d B$ we mean that there exists a constant $L_d \in (0, \infty)$, such that $L_d^{-1}B \leq A \leq L_d B$.

We will also denote $X^j = (X_i^j)_{1 \leq i \leq n}$ and write \mathbb{E}_j for the expectation with respect to X^j .

3 Main results

Theorem 3.1. *For any $d \geq 1$ and $p \geq 2$ we have*

$$\left\| \sum_{\mathbf{i}} a_{\mathbf{i}} X_{i_1}^1 \cdots X_{i_d}^d \right\|_p \geq \frac{1}{L_d} \sum_{\mathcal{J} \in P_d} \|(a_{\mathbf{i}})\|_{\mathcal{J}, p}^{\mathcal{N}}. \quad (5)$$

Theorem 3.2. *For $d \leq 3$ and $p \geq 2$,*

$$\left\| \sum_{\mathbf{i}} a_{\mathbf{i}} X_{i_1}^1 \cdots X_{i_d}^d \right\|_p \leq L_d \sum_{\mathcal{J} \in P_d} \|(a_{\mathbf{i}})\|_{\mathcal{J}, p}^{\mathcal{N}}. \quad (6)$$

Remark 1. Let $X_i^j = c g_i^j$, where g_i^j are i.i.d. $\mathcal{N}(0, 1)$ r.v.'s. and $1 < c < 10/9$ is a constant for which the normalization (3) holds. Then $t^2/L \leq \hat{N}_i^j(t) \leq Lt^2$ and for $\mathcal{J} = \{I_1, \dots, I_k\} \in P_d$, $p \geq 2$

$$\|(a_{\mathbf{i}})\|_{\mathcal{J}, p}^{\mathcal{N}} \sim_d p^{k/2} \|(a_{\mathbf{i}})\|_{\mathcal{J}},$$

where

$$\|(a_{\mathbf{i}})\|_{\mathcal{J}} = \sup \left\{ \sum_{\mathbf{i}} a_{\mathbf{i}} \prod_{l=1}^k x_{i_{I_l}}^l : \|x_{i_{I_l}}^l\|_2 \leq 1, 1 \leq l \leq k \right\}. \quad (7)$$

Theorems 3.1 and 3.2 (for arbitrary d) in this case were established in [14].

A standard application of the Paley-Zygmund inequality (see e.g. Corollary 3.3.2 of [7]) and the fact that p -th and $2p$ -th moments of chaoses generated by random variables with log-concave tails are comparable up to constants depending only on the order of the chaos yield the following corollary (for details see the proof of Corollary 1 in [14]).

Corollary 3.3. *For $d \leq 3$ and $t > 0$,*

$$\frac{1}{L_d} e^{-t/L_d} \leq \mathbb{P} \left(\left| \sum_{\mathbf{i}} a_{\mathbf{i}} X_{i_1}^1 \cdots X_{i_d}^d \right| \geq \sum_{\mathcal{J} \in P_d} \|(a_{\mathbf{i}})\|_{\mathcal{J}, t}^{\mathcal{N}} \right) \leq L_d e^{-tL_d}.$$

We are not able to show Theorem 3.2 for $d > 3$ in the general case. However we know that it holds for exponential random variables.

Theorem 3.4. *If $N_i^j(t) = t$ for all i, j and $t > 0$, then for any $d \geq 1$ and $p \geq 2$ the estimate (6) holds.*

4 Proof of Theorem 3.1

We will proceed by induction with respect to d . The case $d = 1$ was proved in [10]. Let us therefore assume that the theorem holds for all positive integers smaller than $d > 1$.

Note that since we allow the constants to depend on d , it is enough to show that the left-hand side of (5) is minorized by each of the summands on the right-hand side.

For any $\mathcal{J} = \{I_1, \dots, I_k\} \in P_d$, with $k \geq 2$, the induction assumption applied conditionally on $(X^j)_{j \in I_1}$ gives

$$\left\| \sum_{\mathbf{i}} a_{\mathbf{i}} X_{i_1}^1 \cdots X_{i_d}^d \right\|_p \geq \frac{1}{L_{d-\#I_1}} \left(\mathbb{E}_{I_1} \left(\left\| \left(\sum_{\mathbf{i}_{I_1}} a_{\mathbf{i}} \prod_{r \in I_1} X_{i_r}^r \right)_{\mathbf{i}_{I_1^c}} \right\|_{\mathcal{J} \setminus \{I_1\}, p}^{\mathcal{N}'} \right)^p \right)^{1/p}, \quad (8)$$

where $\mathcal{N}' = (N_i^j)_{1 \leq i \leq n, j \in I_1^c}$.

Let us fix arbitrary $s_1 \in I_1, \dots, s_k \in I_k$. We have

$$\begin{aligned} & \mathbb{E}_{I_1} \left(\left\| \left(\sum_{\mathbf{i}_{I_1}} a_{\mathbf{i}} \prod_{r \in I_1} X_{i_r}^r \right)_{\mathbf{i}_{I_1^c}} \right\|_{\mathcal{J} \setminus \{I_1\}, p}^{\mathcal{N}'} \right)^p \\ &= \mathbb{E}_{I_1} \left(\sup_{\mathbf{i}_{I_1^c}} \left\{ \left| \sum_{\mathbf{i}_{I_1}} \left(\sum_{r \in I_1} a_{\mathbf{i}} \prod_{r \in I_1} X_{i_r}^r \right) \prod_{l=2}^k x_{i_{I_l}}^l \right| : \right. \right. \\ & \quad \left. \left. \sum_{i_{s_l}} \hat{N}_{i_{s_l}}^l (\| (x_{\mathbf{i}_{I_l}}^l)_{\mathbf{i}_{I_l \setminus \{s_l\}} \|_2) \leq p, 2 \leq l \leq k \right\} \right)^p \\ & \geq \sup_{\mathbf{i}_{I_1^c}} \left\{ \mathbb{E}_{I_1} \left| \sum_{\mathbf{i}_{I_1}} \left(\sum_{r \in I_1} a_{\mathbf{i}} \prod_{r \in I_1} X_{i_r}^r \right) \prod_{l=2}^k x_{i_{I_l}}^l \right|^p : \right. \\ & \quad \left. \sum_{i_{s_l}} \hat{N}_{i_{s_l}}^l (\| (x_{\mathbf{i}_{I_l}}^l)_{\mathbf{i}_{I_l \setminus \{s_l\}} \|_2) \leq p, 2 \leq l \leq k \right\} \\ & \geq \frac{1}{L_{\#I_1}^p} \left(\sup_{\mathbf{i}} \left\{ \left| \sum_{\mathbf{i}} a_{\mathbf{i}} \prod_{l=1}^k x_{i_{I_l}}^l \right| : \sum_{i_{s_l}} \hat{N}_{i_{s_l}}^l (\| (x_{\mathbf{i}_{I_l}}^l)_{\mathbf{i}_{I_l \setminus \{s_l\}} \|_2) \leq p, 1 \leq l \leq k \right\} \right)^p, \end{aligned}$$

where the last inequality follows from another application of the induction assumption, this time to a chaos of order $\#I_1$. Since the indices s_1, \dots, s_d

run over sets of cardinality not exceeding d , the above estimate together with (8) imply that

$$\left\| \sum_{\mathbf{i}} a_{\mathbf{i}} X_{i_1}^1 \cdots X_{i_d}^d \right\|_p \geq \frac{1}{L_d} \|a_{\mathbf{i}}\|_{\mathcal{J}, p}^{\mathcal{N}}.$$

The case $k = 1$ requires a different approach. Again it is enough to show that for each $l \in \{1, \dots, d\}$,

$$\left\| \sum_{\mathbf{i}} a_{\mathbf{i}} X_{i_1}^1 \cdots X_{i_d}^d \right\|_p \geq \sup \left\{ \sum_{\mathbf{i}} a_{\mathbf{i}} x_{\mathbf{i}} : \sum_{i_l} \hat{N}_{i_l}^l \left(\left(\sum_{\mathbf{i}_{\{l\}^c} x_{\mathbf{i}}^2 \right)^{1/2} \right) \leq p \right\}.$$

Consider any $x_{\mathbf{i}}$ such that $\sum_{i_l} \hat{N}_{i_l}^l \left(\left(\sum_{\mathbf{i}_{\{l\}^c} x_{\mathbf{i}}^2 \right)^{1/2} \right) \leq p$. By the symmetry of $X_{i_l}^l$ we have

$$\begin{aligned} \left\| \sum_{\mathbf{i}} a_{\mathbf{i}} X_{i_1}^1 \cdots X_{i_d}^d \right\|_p^p &= \mathbb{E}_l \mathbb{E}_{\{l\}^c} \left| \sum_{i_l} X_{i_l}^l \left| \sum_{\mathbf{i}_{\{l\}^c} a_{\mathbf{i}} \prod_{k \neq l} X_{i_k}^k \right| \right|^p \\ &\geq \mathbb{E}_l \left| \sum_{i_l} X_{i_l}^l \mathbb{E}_{\{l\}^c} \left| \sum_{\mathbf{i}_{\{l\}^c} a_{\mathbf{i}} \prod_{k \neq l} X_{i_k}^k \right| \right|^p \\ &\geq \frac{1}{L_d^p} \mathbb{E}_l \left| \sum_{i_l} X_{i_l}^l \left(\sum_{\mathbf{i}_{\{l\}^c} a_{\mathbf{i}}^2 \right)^{1/2} \right|^p \\ &\geq \frac{1}{L_d^p} \left| \sup \left\{ \sum_{i_l} \left(\sum_{\mathbf{i}_{\{l\}^c} a_{\mathbf{i}}^2 \right)^{1/2} \alpha_{i_l} : \sum_{i_l} \hat{N}_{i_l}^l(\alpha_{i_l}) \leq p \right\} \right|^p \\ &\geq \frac{1}{L_d^p} \left| \sum_{i_l} \left(\sum_{\mathbf{i}_{\{l\}^c} a_{\mathbf{i}}^2 \right)^{1/2} \left(\sum_{\mathbf{i}_{\{l\}^c} x_{\mathbf{i}}^2 \right)^{1/2} \right|^p \\ &\geq \frac{1}{L_d^p} \left| \sum_{\mathbf{i}} a_{\mathbf{i}} x_{\mathbf{i}} \right|^p, \end{aligned}$$

where the first inequality follows from Jensen's inequality, the second one from hypercontractivity of chaoses generated by log-concave random variables combined with the contraction principle and the third one from the induction assumption.

5 Preliminary facts

In this section we present the basic notation and tools to be used in the proof of our main results.

5.1 Some additional notation

1. By $\gamma_{n,t}$ we will denote the distribution of tG_n , where $G_n = (g_1, \dots, g_n)$ is the standard Gaussian vector in \mathbb{R}^n . Let also $G_n^i = (g_1^i, \dots, g_n^i)$ be independent copies of G_n .
2. By $\nu_{n,t}$ we will denote the distribution of $t\mathcal{E}_n$, where $\mathcal{E}_n = (\xi_1, \dots, \xi_n)$ is a random vector in \mathbb{R}^n with independent coordinates distributed according to the symmetric exponential distribution with parameter 1. Thus $\nu_{n,t}$ has the density

$$d\nu_{n,t}(x) = (2t)^{-n} \exp\left(-\frac{1}{t} \sum_{i=1}^n |x_i|\right) dx.$$

We also put $\mathcal{E}_n^i = (\xi_1^i, \dots, \xi_n^i)$ for i.i.d. copies of \mathcal{E}_n .

Let us note that $\mathbb{E}\xi_i^2 = 2$.

3. For any norm α on $\mathbb{R}^{n_1 \cdots n_d} = \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_d}$ (which we will identify with the space of d -indexed matrices, i.e. $x_1 \otimes \cdots \otimes x_d = (\prod_{j=1}^d x_{i_j}^j)_{1 \leq i_1, \dots, i_d \leq n_j}$, where $x_j = (x_1^j, \dots, x_{n_j}^j)$), let ρ_α be the distance on $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}$ (which we will identify with $\mathbb{R}^{n_1 + \cdots + n_d}$), defined by

$$\rho_\alpha(\mathbf{x}, \mathbf{y}) = \alpha(x_1 \otimes \cdots \otimes x_d - y_1 \otimes \cdots \otimes y_d),$$

where $\mathbf{x} = (x_1, \dots, x_d)$, $\mathbf{y} = (y_1, \dots, y_d)$.

For $\mathbf{x} \in \mathbb{R}^{n_1 + \cdots + n_d}$ and $r \geq 0$ let $B_\alpha(\mathbf{x}, r)$ be the closed ball in the metric ρ_α with center \mathbf{x} and radius r .

4. Now, for $T \subset \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}$, $t > 0$, define

$$W_d^T(\alpha, t) = \sum_{k=1}^d t^k \sum_{I \subset \{1, \dots, d\}, \#I=k} W_I^T(\alpha),$$

where for $I \subset \{1, \dots, d\}$,

$$W_I^T(\alpha) = \sup_{\mathbf{x} \in T} \mathbb{E} \alpha \left(\left(\prod_{k \notin I} x_{i_k}^k \prod_{k \in I} g_{i_k}^k \right)_{i_1, \dots, i_d} \right).$$

5. Similarly, for $t > 0$, $T \subset \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_d}$ we put

$$V_d^T(\alpha, t) := \sum_{k=1}^d t^k \sum_{I \subset \{1, \dots, d\}: \#I=k} V_I^T(\alpha),$$

where

$$V_I^T(\alpha) := \sup_{\mathbf{x} \in T} \mathbb{E} \alpha \left(\left(\prod_{k \notin I} x_{i_k}^k \prod_{k \in I} \xi_{i_k}^k \right)_{i_1, \dots, i_d} \right).$$

6. For $s, t > 0$, $T \subset \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_d}$, we define

$$U_d^T(\alpha, s, t) := \sum_{k=1}^d \sum_{\substack{I, J \subset \{1, \dots, d\}, \\ \#(I \cup J) = k, I \cap J = \emptyset}} s^{\#I} t^{\#J} U_{I, J}^T(\alpha)$$

where

$$U_{I, J}^T(\alpha) := \sup_{\mathbf{x} \in T} \mathbb{E} \alpha \left(\left(\prod_{k \notin (I \cup J)} x_{i_k}^k \prod_{k \in I} g_{i_k}^k \prod_{k \in J} \xi_{i_k}^k \right)_{i_1, \dots, i_d} \right).$$

Remark Let us notice that $U_{\emptyset, I}^T(\alpha) = V_I^T(\alpha)$, whereas $U_{I, \emptyset}^T(\alpha) = W_I^T(\alpha)$.

The quantity W_I^T was defined in [14], where it played an important role in the analysis of moments of Gaussian chaoses. The quantities V_I^T and $U_{I, J}^T$ will play an analogous role for chaoses generated by general random variables with logarithmically concave tails (as will become clear in the next section, they will allow us to bound the covering numbers for more general sets than those which were important in the Gaussian case).

5.2 Entropy estimates

In this section we present some general entropy estimates which will be crucial for bounding suprema of stochastic processes in the proof of Theorem 3.2.

The first lemma we will need is a reformulation of Lemma 1 in [14]. The original statement from [14] is slightly weaker however the proof given therein justifies the version presented below.

Lemma 5.1. *For any norms α_1, α_2 on \mathbb{R}^n , $y \in B_2^n$ and $t > 0$,*

$$\gamma_{n, t} \left(x : \alpha_i(x - y) \leq 4t \mathbb{E} \alpha_i(G_n), i = 1, 2 \right) \geq \frac{1}{2} e^{-1/(2t^2)}.$$

Lemma 5.2. For any norms α_1, α_2 on \mathbb{R}^n , $y \in aB_1^n$ and $t > 0$,

$$\nu_{n,t}\left(x: \alpha_i(x - y) \leq 4t\mathbb{E}\alpha_i(\mathcal{E}_n), i = 1, 2\right) \geq \frac{1}{2}e^{-a/t}.$$

Proof. Let

$$K := \{x \in \mathbb{R}^n: \alpha_1(x) \leq 4t\mathbb{E}\alpha_1(\mathcal{E}_n), \alpha_2(x) \leq 4t\mathbb{E}\alpha_2(\mathcal{E}_n)\}.$$

By Chebyshev's inequality,

$$1 - \nu_{n,t}(K) \leq \mathbb{P}(\alpha_1(t\mathcal{E}_n) > 4\mathbb{E}\alpha_1(t\mathcal{E}_n)) + \mathbb{P}(\alpha_2(t\mathcal{E}_n) > 4\mathbb{E}\alpha_2(t\mathcal{E}_n)) < 1/2.$$

We get for any $y \in aB_1^n$,

$$\begin{aligned} \nu_{n,t}(y + K) &= (2t)^{-n} \int_K \exp\left(-\frac{1}{t} \sum_{i=1}^n |x_i + y_i|\right) dx \\ &\geq \exp\left(-\frac{1}{t} \sum_{i=1}^n |y_i|\right) \int_K d\nu_{n,t}(x) \\ &\geq \exp(-a/t) \nu_{n,t}(K) \geq \frac{1}{2} \exp(-a/t). \end{aligned}$$

Finally, notice that if $x \in y + K$, then $\alpha_i(x - y) \leq 4t\mathbb{E}\alpha_i(\mathcal{E}_n)$, $i = 1, 2$. \square

Before we formulate the next lemma, let us define $\mu_{n,s,t}$ (where $s, t > 0$) as the convolution of $\gamma_{n,s}$ and $\nu_{n,t}$.

Lemma 5.3. For any norms α_1, α_2 on \mathbb{R}^n , any $a > 0$, $y \in B_2^n + aB_1^n$ and $s, t > 0$, let

$$K = \{x: \alpha_1(x - y) \leq 4s\mathbb{E}\alpha_1(G_n) + 4t\mathbb{E}\alpha_1(\mathcal{E}_n), \\ \alpha_2(x) \leq 4s\mathbb{E}\alpha_2(G_n) + 4t\mathbb{E}\alpha_2(\mathcal{E}_n) + \alpha_2(y)\}.$$

Then

$$\mu_{n,s,t}(K) \geq \frac{1}{4}e^{-1/(2s^2)-a/t}.$$

Proof. We have $y = y_1 + y_2$ for some $y_1 \in B_2^n$, $y_2 \in aB_1^n$. Define

$$K_1 = \{x \in \mathbb{R}^n: \alpha_i(x - y_1) \leq 4s\mathbb{E}\alpha_i(G_n), i = 1, 2\}, \\ K_2 = \{x \in \mathbb{R}^n: \alpha_i(x - y_2) \leq 4t\mathbb{E}\alpha_i(\mathcal{E}_n), i = 1, 2\}.$$

For $x = x_1 + x_2$, where $x_j \in K_j$, $j = 1, 2$,

$$\alpha_1(x - y) \leq \alpha_1(x_1 - y_1) + \alpha_1(x_2 - y_2) \leq 4s\mathbb{E}\alpha_1(G_n) + 4t\mathbb{E}\alpha_1(\mathcal{E}_n)$$

and similarly

$$\alpha_2(x) \leq \alpha_2(x - y) + \alpha_2(y) \leq 4s\mathbb{E}\alpha_2(G_n) + 4t\mathbb{E}\alpha_2(\mathcal{E}_n) + \alpha_2(y),$$

therefore $K_1 + K_2 \subset K$. We thus have

$$\mu_{n,s,t}(K) \geq \mu_{n,s,t}(K_1 + K_2) \geq \gamma_{n,s}(K_1)\nu_{n,t}(K_2) \geq \frac{1}{4}e^{-1/(2s^2)-a/t},$$

where in the last inequality we used Lemmas 5.1 and 5.2. \square

Lemma 5.4. *For any $s, t > 0$, $a = (a_1, \dots, a_d) \in (0, \infty)^d$ and $\mathbf{x} \in (B_2^{n_1} + a_1 B_1^{n_1}) \times \dots \times (B_2^{n_d} + a_d B_1^{n_d})$ we have*

$$\mu_{n_1+\dots+n_d,s,t}(B_\alpha(\mathbf{x}, U_d^{\{\mathbf{x}\}}(\alpha, 4s, 4t))) \geq 4^{-d} \exp\left(-\frac{1}{2}ds^{-2} - \|a\|_1 t^{-1}\right). \quad (9)$$

Proof. We will proceed by induction on d . For $d = 1$, inequality (9) follows by Lemma 5.3. Now suppose that (9) holds for $d - 1$. We will show that it is also satisfied for d . Let us first notice that

$$\alpha\left(\bigotimes_{i=1}^d x^i - \bigotimes_{i=1}^d y^i\right) \leq \alpha^1(x^d - y^d) + \alpha_{y^d}\left(\bigotimes_{i=1}^{d-1} x^i - \bigotimes_{i=1}^{d-1} y^i\right), \quad (10)$$

where α^1 and α_{y^d} are norms on \mathbb{R}^{n_d} and $\mathbb{R}^{n_1 \dots n_{d-1}}$ respectively, defined by

$$\alpha^1(z) := \alpha\left(\bigotimes_{i=1}^{d-1} x^i \otimes z\right) \quad \text{and} \quad \alpha_{y^d}(z) := \alpha(z \otimes y^d).$$

Then

$$s\mathbb{E}\alpha^1(G_n) + t\mathbb{E}\alpha^1(\mathcal{E}_n) = sU_{\{d\},\emptyset}^{\{\mathbf{x}\}}(\alpha) + tU_{\emptyset,\{d\}}^{\{\mathbf{x}\}}(\alpha). \quad (11)$$

Moreover if we put $\pi(\mathbf{x}) = (x^1, \dots, x^{d-1})$ and define a norm $\alpha_{s,t}^2$ on \mathbb{R}^{n_d} by the formula

$$\alpha_{s,t}^2(y) := U_{d-1}^{\{\pi(\mathbf{x})\}}(\alpha_y, s, t)$$

then

$$\begin{aligned} & s\mathbb{E}\alpha_{s,t}^2(G_n) + t\mathbb{E}\alpha_{s,t}^2(\mathcal{E}_n) + \alpha_{s,t}^2(x^d) \\ &= \sum_{\substack{I, J \subset \{1, \dots, d\} \\ I \cup J \neq \emptyset, I \cap J = \emptyset}} s^{\#I} t^{\#J} U_{I,J}^{\{\mathbf{x}\}}(\alpha, t) - \left[sU_{\{d\},\emptyset}^{\{\mathbf{x}\}}(\alpha) + tU_{\emptyset,\{d\}}^{\{\mathbf{x}\}}(\alpha) \right]. \end{aligned} \quad (12)$$

Notice also that by the induction assumption we have for any $z \in \mathbb{R}^{n_d}$,

$$\begin{aligned} \mu_{n_1+\dots+n_{d-1},s,t} \left(\mathbf{y} \in \mathbb{R}^{n_1+\dots+n_{d-1}} : \alpha_z \left(\bigotimes_{i=1}^{d-1} x^i - \bigotimes_{i=1}^{d-1} y^i \right) \leq \alpha_{4s,4t}^2(z) \right) \\ \geq 4^{1-d} \exp(-(d-1)s^{-2}/2 - (a_1 + \dots + a_{d-1})t^{-1}). \end{aligned} \quad (13)$$

Finally let

$$\begin{aligned} A(\mathbf{x}) := \left\{ \mathbf{y} \in \mathbb{R}^{n_1+\dots+n_d} : \alpha^1(x^d - y^d) \leq 4s\mathbb{E}\alpha^1(G_{n_d}) + 4t\mathbb{E}\alpha^1(\mathcal{E}_{n_d}), \right. \\ \left. \alpha_{4s,4t}^2(y^d) \leq 4s\mathbb{E}\alpha_{4s,4t}^2(G_{n_d}) + 4t\mathbb{E}\alpha_{4s,4t}^2(\mathcal{E}_{n_d}) + \alpha_{4s,4t}^2(x^d), \right. \\ \left. \alpha_{y^d} \left(\bigotimes_{i=1}^{d-1} x^i - \bigotimes_{i=1}^{d-1} y^i \right) \leq \alpha_{4s,4t}^2(y^d) \right\}. \end{aligned}$$

By (10)-(12) we get $A(\mathbf{x}) \subset B_\alpha(\mathbf{x}, U_d^{\{\mathbf{x}\}}(\alpha, 4s, 4t))$ and therefore by (13), Lemma 5.3 and Fubini's theorem we get

$$\begin{aligned} \mu_{n_1+\dots+n_d,s,t} \left(B_\alpha \left(\mathbf{x}, U_d^{\{\mathbf{x}\}}(\alpha, 4s, 4t) \right) \right) \\ \geq \mu_{n_1+\dots+n_d,s,t}(A(\mathbf{x})) \\ \geq 4^{1-d} \exp(-(d-1)s^{-2}/2 - (a_1 + \dots + a_{d-1})t^{-1}) \cdot 4^{-1} \exp(-s^{-2}/2 - a_d t^{-1}) \\ = 4^{-d} \exp(-ds^{-2}/2 - \|a\|_1 t^{-1}). \end{aligned}$$

□

Corollary 5.5. *For any $T \subset (B_2^{n_1} + a_1 B_1^{n_1}) \times \dots \times (B_2^{n_d} + a_d B_1^{n_d})$ and $s, t \in (0, 1]$,*

$$N(T, \rho_\alpha, U_d^T(\alpha, s, t)) \leq \exp(Lds^{-2} + L\|a\|_1 t^{-1}).$$

Proof. Obviously $U_d^T(\alpha, s, t) \geq \sup_{\mathbf{x} \in T} U_d^{\{\mathbf{x}\}}(\alpha, s, t)$. Therefore by Lemma 5.4 we have for any $\mathbf{x} \in T$,

$$\mu_{n_1+\dots+n_d,s,t} \left(B_\alpha \left(\mathbf{x}, U_d^T(\alpha, 4s, 4t) \right) \right) \geq 4^{-d} \exp(-ds^{-2}/2 - \|a\|_1 t^{-1}). \quad (14)$$

Suppose that there exist $\mathbf{x}_1, \dots, \mathbf{x}_N \in T$ such that $\rho_\alpha(\mathbf{x}_i, \mathbf{x}_j) > U_d^T(\alpha, s, t) \geq 2U_d^T(\alpha, s/2, t/2)$ for $i \neq j$. Then the sets $B_\alpha(\mathbf{x}_i, U_d^T(\alpha, s/2, t/2))$ are disjoint, so by (14) we obtain $N \leq 4^d \exp(32ds^{-2} + 8\|a\|_1 t^{-1})$. Hence

$$N(T, \rho_\alpha, U_{d,I}^T(\alpha, s, t)) \leq 4^d \exp(32ds^{-2} + 8\|a\|_1 t^{-1}) \leq \exp(34ds^{-2} + 8\|a\|_1 t^{-1}).$$

□

We will also need the following standard lemma, whose proof we provide for the sake of completeness.

Lemma 5.6. *For any n and any norm α on \mathbb{R}^n , $\mathbb{E}\alpha(G_n) \leq 3\mathbb{E}\alpha(\mathcal{E}_n)$.*

Proof. Let g and ξ be respectively standard Gaussian and symmetric exponential random variables. For $t \geq 0$ we have $\mathbb{P}(|g| \geq t) \leq e^{-t^2/2}$ and $\mathbb{P}(|\xi| \geq t) = e^{-t}$. Thus for $t \geq 2$ we have $\mathbb{P}(|g| \geq t) \leq \mathbb{P}(|\xi| \geq t)$.

Consider now $G_n = (g_1, \dots, g_n)$, $\mathcal{E}_n = (\xi_1, \dots, \xi_n)$. Define moreover independent random variables X_1, \dots, X_n distributed as $|g|\mathbf{1}_{\{|g|>2\}}$. Since for all $t \geq 0$, $\mathbb{P}(X_i \geq t) \leq \mathbb{P}(|\xi_i| \geq t)$ we can assume that X_i 's, g_i 's and ξ_i 's are defined on the same probability space together with a sequence $\varepsilon_1, \dots, \varepsilon_n$ of independent Rademacher variables, in such a way that for all i , $X_i \leq |\xi_i|$ pointwise, g_i 's, ξ_i 's, ε_i 's are independent and X_i 's are independent of ε_i 's. We can write

$$\begin{aligned} \mathbb{E}\alpha(G_n) &= \mathbb{E}\alpha(\varepsilon_1|g_1|, \dots, \varepsilon_n|g_n|) \\ &\leq \mathbb{E}\alpha(\varepsilon_1|g_1|\mathbf{1}_{\{|g_1|\leq 2\}}, \dots, \varepsilon_n|g_n|\mathbf{1}_{\{|g_n|\leq 2\}}) \\ &\quad + \mathbb{E}\alpha(\varepsilon_1|g_1|\mathbf{1}_{\{|g_1|>2\}}, \dots, \varepsilon_n|g_n|\mathbf{1}_{\{|g_n|>2\}}) \\ &\leq 2\mathbb{E}\alpha(\varepsilon_1, \dots, \varepsilon_n) + \mathbb{E}\alpha(\varepsilon_1 X_1, \dots, \varepsilon_n X_n) \\ &\leq 2\mathbb{E}_\varepsilon \alpha(\varepsilon_1 \mathbb{E}_\xi |\xi_1|, \dots, \varepsilon_n \mathbb{E}_\xi |\xi_n|) + \mathbb{E}\alpha(\varepsilon_1 |\xi_1|, \dots, \varepsilon_n |\xi_n|) \\ &\leq 3\mathbb{E}\alpha(\xi_1, \dots, \xi_n), \end{aligned}$$

where in the second and third inequality we used (conditionally) the contraction principle. \square

Corollary 5.5 together with Lemma 5.6 yield

Corollary 5.7. *For any $T \subset (B_2^n + aB_1^n)^d$ and any $t \in (0, 1]$,*

$$N(T, \rho_\alpha, V_d^T(\alpha, t)) \leq \exp(L_d t^{-2} + L_d a t^{-1}).$$

We would like to remark that by applying Corollary 5.5 with $t_i a_i$ instead of a_i and letting t_i tend to 0 or infinity we can obtain similar results for Cartesian products of the form $\times_{i=1}^d K_i$ where K_i is either B_2^n or $a_i B_1^n$. Such results can be also obtained directly by following the proof of Corollary 5.5 and using Lemmas 5.1 and 5.2 instead of Lemma 5.3. We will need such entropy estimates only for $d = 1$ and $K = aB_1^n$. This case, described in the next corollary, follows just from Lemma 5.2.

Corollary 5.8. *For any $a > 0$, $T \subset aB_1^n$ and $t \in (0, 1]$,*

$$N(T, \rho_\alpha, t\mathbb{E}\alpha(\mathcal{E})) \leq 2 \exp(8at^{-1}).$$

5.3 Concentration of measure for linear combinations of independent random variables with log-concave tails

Similarly as in [14], the proof of our main results will rely on induction with respect to d , the order of the chaos variable. The base of the induction, i.e. the case $d = 1$ was obtained in [10] by Gluskin and Kwapien and later extended in [12] to linear combinations of independent symmetric random variables with log-concave tails with vector valued coefficients. Below we present the more general vector-valued version, together with some of its rather standard consequences, which provide the toolbox to be used in the proof. All the lemmas below contain the special case of Gaussian variables and reduce in this case to standard facts about the concentration and integrability for suprema of Gaussian processes.

In the rest of this section we will use the assumptions and notation introduced in Section 2 specialized to the case of $d = 1$. In particular we will suppress upper indices (see the remark after the definition of the functions \hat{N}_i^j).

Lemma 5.9 (Theorem 1 in [12]). *For any bounded set $T \subset \mathbb{R}^n$ and all $p \geq 2$ we have*

$$\begin{aligned} \frac{1}{L} \left\| \sup_{t \in T} \left\| \sum_{i=1}^n t_i X_i \right\| \right\|_p &\leq \left\| \sup_{t \in T} \left\| \sum_{i=1}^n t_i X_i \right\| \right\|_1 \\ &\quad + \sup \left\{ \sum_{i=1}^n t_i x_i : t \in T, x \in \mathbb{R}^n, \sum_{i=1}^n \hat{N}_i(x_i) \leq p \right\} \\ &\leq L \left\| \sup_{t \in T} \left\| \sum_{i=1}^n t_i X_i \right\| \right\|_p. \end{aligned}$$

Thus, for any $u > 0$,

$$\begin{aligned} \mathbb{P} \left(\sup_{t \in T} \left\| \sum_{i=1}^n t_i X_i \right\| \geq \right. \\ \left. L \left[\left\| \sup_{t \in T} \left\| \sum_{i=1}^n t_i X_i \right\| \right\|_1 + \sup \left\{ \sum_{i=1}^n t_i x_i : t \in T, x \in \mathbb{R}^n, \sum_{i=1}^n \hat{N}_i(x_i) \leq u \right\} \right] \right) \leq e^{-u}. \end{aligned}$$

Remark Using the notation of Section 2, we can write

$$\sup \left\{ \sum_{i=1}^n t_i x_i : t \in T, x \in \mathbb{R}^n, \sum_{i=1}^n \hat{N}_i(x_i) \leq p \right\} = \sup_{t \in T} \|t\|_{\{1\}, p}^N,$$

which shows that the above lemma is indeed a strengthening of the case $d = 1$ of Theorems 3.1 and 3.2.

Lemma 5.10. *Consider arbitrary sets $T_1, \dots, T_m \subset \mathbb{R}^n$ and let $T = \bigcup_{j=1}^m T_j$. Then*

$$\begin{aligned} \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i X_i &\leq L \left(\max_{j \leq m} \mathbb{E} \sup_{t \in T_j} \sum_{i=1}^n t_i X_i \right. \\ &\quad \left. + \sup \left\{ \sum_{i=1}^n (t_i - s_i) x_i : t, s \in T, x \in \mathbb{R}^n, \sum_{i=1}^n \hat{N}_i(x_i) \leq \log m \right\} \right) \end{aligned}$$

Proof. For $m = 1$ the theorem is obvious, so we will assume that $m \geq 2$. Let us fix arbitrary $s \in T$. Since $\mathbb{E} X_i = 0$, we have

$$\mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i X_i = \mathbb{E} \max_{j \leq m} \sup_{t \in T_j} \sum_{i=1}^n (t_i - s_i) X_i \leq \mathbb{E} \max_{j \leq m} \sup_{t \in T_j} \left| \sum_{i=1}^n (t_i - s_i) X_i \right|.$$

Let $A = \sup \left\{ \sum_{i=1}^n (t_i - s_i) x_i : t \in T, x \in \mathbb{R}^n, \sum_{i=1}^n \hat{N}_i(x_i) \leq \log m \right\}$ and note that by the convexity of N_i and the definition of \hat{N}_i , for any $u \geq 1$,

$$\hat{N}_i(x/u) \leq \hat{N}_i(x)/u, \tag{15}$$

which implies that for $u \geq 1$,

$$\sup \left\{ \sum_{i=1}^n (t_i - s_i) x_i : t \in T, x \in \mathbb{R}^n, \sum_{i=1}^n \hat{N}_i(x_i) \leq 2u \log m \right\} \leq 2uA.$$

Thus by Lemma 5.9 and the union bound, for any $u \geq 1$,

$$\begin{aligned} \mathbb{P} \left(\max_{j \leq m} \sup_{t \in T_j} \left| \sum_{i=1}^n (t_i - s_i) X_i \right| \geq L \max_j \mathbb{E} \sup_{t \in T_j} \left| \sum_{i=1}^n (t_i - s_i) X_i \right| + LuA \right) \\ \leq m e^{-2u \log m} \leq \frac{1}{2u}, \end{aligned}$$

which by integration by parts gives

$$\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n (t_i - s_i) X_i \right| \leq L \left(\max_{j \leq m} \mathbb{E} \sup_{t \in T_j} \left| \sum_{i=1}^n (t_i - s_i) X_i \right| + A \right).$$

To finish the proof of the lemma it is therefore sufficient to show that for all $j \leq m$,

$$\mathbb{E} \sup_{t \in T_j} \left| \sum_{i=1}^n (t_i - s_i) X_i \right| \leq L \left(\mathbb{E} \sup_{t \in T_j} \sum_{i=1}^n t_i X_i + A \right). \quad (16)$$

Let us choose any $z \in T_j$. We have

$$\begin{aligned} \mathbb{E} \sup_{t \in T_j} \left| \sum_{i=1}^n (t_i - s_i) X_i \right| &\leq \mathbb{E} \sup_{t \in T_j} \left| \sum_{i=1}^n (t_i - z_i) X_i \right| + \mathbb{E} \left| \sum_{i=1}^n (z_i - s_i) X_i \right| \\ &\leq \mathbb{E} \sup_{t \in T_j} \left| \sum_{i=1}^n (t_i - z_i) X_i \right| + L \left(\sum_{i=1}^n (z_i - s_i)^2 \right)^{1/2} \\ &\leq \mathbb{E} \sup_{t \in T_j} \left| \sum_{i=1}^n (t_i - z_i) X_i \right| + LA, \end{aligned} \quad (17)$$

where in the first inequality we used the fact that variances of X_i 's are bounded by a universal constants, whereas in the second one, the estimate $(\sum_{i=1}^n (z_i - s_i)^2)^{1/2} = \sup \{ \sum_{i=1}^n (z_i - s_i) u_i : \sum_{i=1}^n u_i^2 \leq 1 \} \leq (\log 2)^{-1/2} A$ for $m \geq 2$, which is an easy consequence of the fact that $\hat{N}_i(u) = u^2$ for $|u| \leq 1$.

Let us now notice that

$$\begin{aligned} &\mathbb{E} \sup_{t \in T_j} \left| \sum_{i=1}^n (t_i - z_i) X_i \right| \\ &= \mathbb{E} \max \left(\sup_{t \in T_j} \left(\sum_{i=1}^n (t_i - z_i) X_i \right)_+, \sup_{t \in T_j} \left(\sum_{i=1}^n (t_i - z_i) X_i \right)_- \right) \\ &\leq \mathbb{E} \sup_{t \in T_j} \left(\sum_{i=1}^n (t_i - z_i) X_i \right)_+ + \mathbb{E} \sup_{t \in T_j} \left(\sum_{i=1}^n (t_i - z_i) X_i \right)_- \\ &= 2 \mathbb{E} \sup_{t \in T_j} \left(\sum_{i=1}^n (t_i - z_i) X_i \right)_+ = 2 \mathbb{E} \sup_{t \in T_j} \sum_{i=1}^n (t_i - z_i) X_i, \end{aligned}$$

where in the second equality we used the symmetry of X_i 's and in the last one the fact that $z \in T_j$.

The above inequality together with (17) proves (16) and ends the proof of the lemma. \square

Let us finish this section with a version of Lemma 5.10 in the special case of Gaussian variables. It improves on the inequality of Lemma 5.10, as it asserts that the constant in front of $\max_j \mathbb{E} \sup_{x \in T_j} \sum_{i=1}^n x_i g_i$ may be taken to be equal to one. This result is again pretty standard and its proof can be found e.g. in [14] (see Lemma 3 therein). It is analogous to the argument presented above, but instead of Lemma 5.9 it uses the Gaussian concentration inequality.

Lemma 5.11. *Let g_1, \dots, g_n be independent standard Gaussian variables and let $T = \bigcup_{j=1}^m T_j \subset \mathbb{R}^n$. Then*

$$\mathbb{E} \sup_{x \in T} \sum_{i=1}^n t_i g_i \leq \max_{j \leq m} \mathbb{E} \sup_{t \in T_j} \sum_{i=1}^n t_i g_i + L \sqrt{\log m} \sup_{s, t \in T} \left(\sum_{i=1}^n (s_i - t_i)^2 \right)^{1/2}.$$

6 Suprema of some Gaussian processes

The main result of this section is Proposition 6.1 below, which is a strengthening of Theorem 3 of [12] in the special case $d = 3$. Before stating the proposition we need some additional definitions.

For a triple indexed matrix $A = (a_{ijk})$ and a set $T \subset \mathbb{R}^n \times \mathbb{R}^n$, let us define

$$\Delta_A(T) = \sup \left\{ \left(\sum_k \left(\sum_{ij} a_{ijk} (x_i y_j - \tilde{x}_i \tilde{y}_j) \right)^2 \right)^{1/2} : (x, y), (\tilde{x}, \tilde{y}) \in T \right\}$$

and

$$s_2^T(A) = \sup_{(x, y) \in T} \left[\left(\sum_{jk} \left(\sum_i a_{ijk} x_i \right)^2 \right)^{1/2} + \left(\sum_{ik} \left(\sum_j a_{ijk} y_j \right)^2 \right)^{1/2} \right].$$

Proposition 6.1. *For any $p \geq 2$ and any set $T \subset (B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)$,*

$$\mathbb{E} \sup_{(x, y) \in T} \sum_{ijk} a_{ijk} x_i y_j g_k \leq L \left[\sqrt{p} \Delta_A(T) + s_2^T(A) + \frac{1}{\sqrt{p}} \left(\sum_{ijk} a_{ijk}^2 \right)^{1/2} \right].$$

Before we pass to the proof of Proposition 6.1 we will prove its counterpart for double-indexed matrices. This simpler result will be used in the proof of Proposition 6.1. Recall the notation introduced in (7), according to which $\|\cdot\|_{\{1,2\}}$ is the Hilbert-Schmidt norm of a matrix (i.e. for $B = (b_{ij})$, $\|B\|_{\{1,2\}} = \sqrt{\sum_{ij} b_{ij}^2}$).

Lemma 6.2. *For any matrix $B = (b_{ij})_{i,j \leq n}$, $a \geq 1$ and $T \subset aB_1^n$,*

$$\begin{aligned} \mathbb{E} \sup_{x \in T} \sum_{ij=1}^n b_{ij} x_i g_j &\leq La^{1/2} \|B\|_{\{1,2\}}^{1/2} (\|B\|_{\{1,2\}} \wedge \Delta_B(T))^{1/2} + La^{1/2} \Delta_B(T), \\ &\leq L \left(\|B\|_{\{1,2\}} + a \Delta_B(T) \right). \end{aligned}$$

where $\Delta_B(T) = \sup_{x, x' \in T} \left(\sum_{j=1}^n \left(\sum_{i=1}^n b_{ij} (x_i - x'_i) \right)^2 \right)^{1/2}$.

Proof. Let us consider the process $Z_x = \sum_{i,j=1}^n b_{ij} x_i g_j$ and the associated metric

$$d_Z(x, x') = \|Z_x - Z_{x'}\|_2 = \left(\sum_{j=1}^n \left(\sum_{i=1}^n b_{ij} (x_i - x'_i) \right)^2 \right)^{1/2}.$$

We have $\Delta_B(T) = \text{diam}_{d_Z} T$. Since $\mathbb{E} \left(\sum_{i=1}^n \left(\sum_{j=1}^n b_{ij} \xi_j \right)^2 \right)^{1/2} \leq \sqrt{2} \|B\|_{\{1,2\}}$, by Corollary 5.8, we have for $t \in (0, 1]$,

$$N(T, d_Z, t \|B\|_{\{1,2\}}) \leq \exp(Lat^{-1}),$$

so for $\varepsilon \leq \|B\|_{\{1,2\}}$,

$$N(T, d_Z, \varepsilon) \leq \exp(L \|B\|_{\{1,2\}} a \varepsilon^{-1}).$$

By Dudley's bound (see [9] or e.g. Corollary 5.1.6 in [7]) we have

$$\begin{aligned} \mathbb{E} \sup_{x \in T} Z_x &\leq L \int_0^{\Delta_B(T)} \sqrt{\log N(T, d_Z, \varepsilon)} d\varepsilon \\ &\leq L \int_0^{\|B\|_{\{1,2\}} \wedge \Delta_B(T)} a^{1/2} \|B\|_{\{1,2\}}^{1/2} \varepsilon^{-1/2} d\varepsilon \\ &\quad + L \int_{\|B\|_{\{1,2\}} \wedge \Delta_B(T)}^{\Delta_B(T)} a^{1/2} d\varepsilon \\ &= La^{1/2} \|B\|_{\{1,2\}}^{1/2} (\|B\|_{\{1,2\}} \wedge \Delta_B(T))^{1/2} + La^{1/2} \Delta_B(T). \end{aligned}$$

The second estimate of the lemma follows from the inequality $2\sqrt{xy} \leq a^{-1/2}x + a^{1/2}y$. \square

Lemma 6.3. *For any matrix $B = (b_{ij})_{i,j \leq n}$, any $T \subset B_2^n + \sqrt{p}B_1^n$ and $p \geq 1$,*

$$\mathbb{E} \sup_{x \in T} \sum_{i,j=1}^n b_{ij} x_i g_j \leq L \left(\|B\|_{\{1,2\}} + \sqrt{p} \Delta_B(T) \right),$$

where Δ_B is as in Lemma 6.2.

Proof. Since $\mathbb{E}(\sum_{i=1}^n (\sum_{j=1}^n b_{ij} \xi_j)^2)^{1/2} \leq \sqrt{2} \|B\|_{\{1,2\}}$, by Corollary 5.8 (with $a = \sqrt{p}$ and $t = 1/(\sqrt{2p})$) there exist sets $K_i \subset \sqrt{p}B_1^n$, $i = 1, \dots, N \leq \exp(Lp)$, such that

$$\sqrt{p}B_1^n = \bigcup_{i=1}^N K_i$$

and

$$\Delta_B(K_i) \leq p^{-1/2} \|B\|_{\{1,2\}}. \quad (18)$$

By Lemma 5.11 we have

$$\begin{aligned} \mathbb{E} \sup_{x \in T} \sum_{ij} b_{ij} x_i g_j &= \mathbb{E} \max_{i \leq N} \sup_{x \in T \cap (B_2^n + K_i)} \sum_{ij} b_{ij} x_i g_j \\ &\leq \max_{i \leq N} \mathbb{E} \sup_{x \in T \cap (B_2^n + K_i)} \sum_{ij} b_{ij} x_i g_j + L \sqrt{\log N} \Delta_B(T) \\ &\leq \max_{i \leq N} \left(\mathbb{E} \sup_{x \in B_2^n} \sum_{ij} b_{ij} x_i g_j + \mathbb{E} \sup_{x \in K_i} \sum_{ij} b_{ij} x_i g_j \right) + L \sqrt{p} \Delta_B(T) \\ &\leq \|B\|_{\{1,2\}} + L(\|B\|_{\{1,2\}} + \sqrt{p} \Delta_B(K_i)) + L \sqrt{p} \Delta_B(T), \end{aligned} \quad (19)$$

where in the last inequality we used Lemma 6.2 and the fact that

$$\mathbb{E} \sup_{x \in B_2^n} \sum_{ij} b_{ij} x_i g_j = \mathbb{E} \sqrt{\sum_i (\sum_j b_{ij} g_j)^2} \leq \|B\|_{\{1,2\}}.$$

Inequalities (18) and (19) imply the lemma. \square

For a triple indexed matrix $A = (a_{ijk})_{i,j,k}$, let α_A be a norm on \mathbb{R}^{n^2} , given by

$$\alpha_A(z) = \left(\sum_k \left(\sum_{i,j} a_{ijk} z_{ij} \right)^2 \right)^{1/2}.$$

To simplify the notation we will write ρ_A for ρ_{α_A} . Note that

$$\rho_A((x_1, y_1), (x_2, y_2)) = (\mathbb{E}(X_{(x_1, y_1)} - X_{(x_2, y_2)})^2)^{1/2},$$

where

$$X_{(x,y)} = \sum_{ijk} a_{ijk} x_i y_j g_k.$$

We will also need a norm on $\mathbb{R}^n \times \mathbb{R}^n$ defined by

$$\tilde{\alpha}_A((x, y)) = \left(\sum_{jk} \left(\sum_i a_{ijk} x_i \right)^2 \right)^{1/2} + \left(\sum_{ik} \left(\sum_j a_{ijk} y_j \right)^2 \right)^{1/2}.$$

The corresponding distance on $\mathbb{R}^n \times \mathbb{R}^n$ will be denoted by $\tilde{\rho}_A$.

We will use the following consequences of Corollary 5.7.

Corollary 6.4. *For any $p \geq 1$, any set $T \subset (B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)$ and any $t \in (0, 1]$,*

$$N(T, \rho_A, t^2 \|A\|_{\{1,2,3\}} + ts_2^T(A)) \leq \exp(Lt^{-2} + L\sqrt{p}t^{-1}).$$

Proof. It is enough to notice that

$$\begin{aligned} V_{\{1,2\}}^T(\alpha_A) &= \mathbb{E} \left(\sum_k \left(\sum_{ij} a_{ijk} \xi_i^1 \xi_j^2 \right)^2 \right)^{1/2} \\ &\leq \left(\mathbb{E} \sum_k \left(\sum_{ij} a_{ijk} \xi_i^1 \xi_j^2 \right)^2 \right)^{1/2} = 2 \|A\|_{\{1,2,3\}}, \end{aligned}$$

whereas

$$\begin{aligned} &V_{\{1\}}^T(\alpha_A) + V_{\{2\}}^T(\alpha_A) \\ &= \sup_{(x,y) \in T} \mathbb{E} \left(\sum_k \left(\sum_{ij} a_{ijk} \xi_i y_j \right)^2 \right)^{1/2} + \sup_{(x,y) \in T} \mathbb{E} \left(\sum_k \left(\sum_{ij} a_{ijk} x_i \xi_j \right)^2 \right)^{1/2} \\ &\leq \sqrt{2} \sup_{(x,y) \in T} \left(\sum_{jk} \left(\sum_i a_{ijk} x_i \right)^2 \right)^{1/2} + \sqrt{2} \sup_{(x,y) \in T} \left(\sum_{ik} \left(\sum_j a_{ijk} y_j \right)^2 \right)^{1/2} \\ &\leq 2\sqrt{2} s_2^T(A). \end{aligned}$$

The statement of the corollary follows now from Corollary 5.7 applied with $d = 2$. \square

Corollary 6.5. *For any $p \geq 1$, any set $T \subset (B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)$ and any $t \in (0, 1]$,*

$$N(T, \tilde{\rho}_A, t\|A\|_{\{1,2,3\}}) \leq \exp(Lt^{-2} + L\sqrt{p}t^{-1}).$$

Proof. Let $(\mathcal{E}^1, \mathcal{E}^2)$ be a standard exponential random vector with values in $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. We have

$$\mathbb{E}\tilde{\alpha}(\mathcal{E}^1, \mathcal{E}^2) \leq 2\sqrt{2}\|A\|_{\{1,2,3\}},$$

hence the corollary follows from Corollary 5.7 with $d = 1$ and the fact that

$$(B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n) \subset \sqrt{2}B_2^{2n} + 2\sqrt{p}B_1^{2n}. \quad (20)$$

\square

To simplify the formulation of the next lemmas let us denote

$$F_A^G(T) = \mathbb{E} \sup_{(x,y) \in T} \sum_k \sum_{ij} a_{ijk} x_i y_j g_k.$$

Lemma 6.6. *For $p \geq 1$ let $(x, y) \in (B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)$ and let $T \subset (B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)$. Then, for any $l \geq 0$, there exists a decomposition*

$$T = \bigcup_{i=l}^N T_i,$$

with $N \leq \exp(L2^{2l}p)$, such that for all $l \leq N$,

$$F_A^G((x, y) + T_l) \leq F_A^G(T_l) + L\tilde{\alpha}((x, y)). \quad (21)$$

and

$$\Delta_A(T_l) \leq 2^{-l}p^{-1/2}s_2^T(A) + 2^{-2l}p^{-1}\|A\|_{\{1,2,3\}}. \quad (22)$$

Proof. We apply Corollary 6.4 with $t = 2^{-l}p^{-1/2}$, which gives us a partition of T into $N \leq \exp(L2^{2l}p)$ sets, satisfying the required diameter bound (22). Let $B^1 = (b_{ik}^1)$, $B^2 = (b_{jk}^2)$ where

$$b_{ik}^1 = \sum_j a_{ijk}y_j, \quad b_{jk}^2 = \sum_i a_{ijk}x_i$$

We have

$$\begin{aligned} & \mathbb{E} \left(\sum_k \left(\sum_i b_{ik}^1 \xi_i \right)^2 \right)^{1/2} + \mathbb{E} \left(\sum_k \left(\sum_j b_{jk}^2 \xi_j \right)^2 \right)^{1/2} \\ & \leq \sqrt{2} \left(\sum_{ik} \left(\sum_j a_{ijk}y_j \right)^2 \right)^{1/2} + \sqrt{2} \left(\sum_{jk} \left(\sum_i a_{ijk}x_i \right)^2 \right)^{1/2} \\ & = \sqrt{2} \tilde{\alpha}((x, y)), \end{aligned}$$

therefore (taking into account (20)) by Corollary 5.7 (with $d = 1$, $a = \sqrt{p}$ and $t = 1/(L\sqrt{p})$), there exists a partition of T into at most e^{Lp} sets S_l such that for all l ,

$$\begin{aligned} & \sup_{(x', y'), (x'', y'') \in S_l} \left[\left(\sum_k \left(\sum_i b_{ik}^1 (x'_i - x''_i) \right)^2 \right)^{1/2} + \left(\sum_k \left(\sum_j b_{jk}^2 (y'_j - y''_j) \right)^2 \right)^{1/2} \right] \\ & \leq \frac{1}{\sqrt{p}} \tilde{\alpha}((x, y)). \end{aligned}$$

We can intersect this partition with the previous one to obtain a partition of T into at most $e^{L2^{2l}p}$ sets T_l , such that (22) holds and the above inequality is satisfied with T_l instead of S_l .

Let π_1, π_2 be the projections from $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ onto the first and the second n coordinates respectively and note that

$$\begin{aligned} \Delta_{B^1}(\pi_1(T_l)) + \Delta_{B^2}(\pi_2(T_l)) & \leq 2 \sup_{(x', y'), (x'', y'') \in T_l} \left[\left(\sum_k \left(\sum_i b_{ik}^1 (x'_i - x''_i) \right)^2 \right)^{1/2} \right. \\ & \quad \left. + \left(\sum_k \left(\sum_j b_{jk}^2 (y'_j - y''_j) \right)^2 \right)^{1/2} \right] \\ & \leq \frac{2}{\sqrt{p}} \tilde{\alpha}((x, y)). \end{aligned} \tag{23}$$

By the equality $\mathbb{E} \sum_{ijk} a_{ijk} x_i y_j g_k = 0$ we get for any l ,

$$\begin{aligned}
F_A^G((x, y) + T_l) &\leq F_A^G(T_l) + \mathbb{E} \sup_{(\tilde{x}, \tilde{y}) \in T_l} \sum_{ijk} a_{ijk} x_i \tilde{y}_j g_k + \mathbb{E} \sup_{(\tilde{x}, \tilde{y}) \in T_l} \sum_{ijk} a_{ijk} \tilde{x}_i y_j g_k \\
&\leq F_A^G(T_l) + L \left(\left(\sum_{jk} \left(\sum_i a_{ijk} x_i \right)^2 \right)^{1/2} + \left(\sum_{ik} \left(\sum_j a_{ijk} y_j \right)^2 \right)^{1/2} \right) \\
&\quad + \sqrt{p} \Delta_{B^1}(\pi_1(T_l)) + \sqrt{p} \Delta_{B^2}(\pi_2(T_l)) \\
&\leq F_A^G(T_l) + L \tilde{\alpha}((x, y)),
\end{aligned}$$

where in the second inequality we used the assumption $T \subset (B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)$ and Lemma 6.3 (applied to matrices B^1, B^2) and in the last inequality the estimate (23). \square

Lemma 6.7. *Let S be a finite subset of $(B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)$ of cardinality at least 2, such that $S - S \subset (B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)$. Then, for any $l \geq 0$, there exist finite sets $S_i \subset (B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)$, and points $(x_i, y_i) \in S_i$ $i = 1, \dots, N$, such that*

- (i) $2 \leq N \leq \exp(L2^{2l}p)$,
- (ii) $S = \bigcup_{i=1}^N ((x_i, y_i) + S_i)$, $S_i - S_i \subset S - S$, $\#S_i \leq \#S - 1$,
- (iii) $\Delta_A(S_i) \leq 2^{-2l}p^{-1} \|A\|_{\{1,2,3\}}$,
- (iv) $s_2^{S_i}(A) \leq 2^{-l}p^{-1/2} \|A\|_{\{1,2,3\}}$,
- (v) $F_A^G((x_i, y_i) + S_i) \leq F_A^G(S_i) + Ls_2^S(A)$.

Proof. Corollary 6.5, applied with $t = 2^{-l-1}p^{-1/2}$, gives us a decomposition

$$S = \bigcup_{i=1}^{N_1} ((x_i, y_i) + T_i),$$

where $N_1 \leq \exp(L2^{2l}p)$, $(x_i, y_i) \in S$ and $s_2^{T_i}(A) \leq 2^{-l-1}p^{-1/2} \|A\|_{\{1,2,3\}}$. Since $\#S \geq 2$ we can assume that $N_1 \geq 2$. We can also assume that the sets $(x_i, y_i) + T_i$ are pairwise disjoint and nonempty, which implies that $\#T_i \leq \#S - 1$.

Since $T_i \subset S - (x_i, y_i) \subset (B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)$, by Lemma 6.6, it can be further decomposed into the union

$$T_i = \bigcup_{j=1}^{N_2} T_{ij},$$

with $N_2 \leq \exp(L2^{2l}p)$, where for all j ,

$$\Delta_A(T_{ij}) \leq 2^{-l-1}p^{-1/2}s_2^{T_i}(A) + 2^{-2l-2}p^{-1}\|A\|_{\{1,2,3\}} \leq 2^{-2l}p^{-1}\|A\|_{\{1,2,3\}}$$

and such that

$$F_A^G((x_i, y_i) + T_{ij}) \leq F_A^G(T_{ij}) + Ls_2^S(A).$$

Notice that $N = N_1N_2 \leq \exp(L2^{2l}p)$, moreover $T_{ij} - T_{ij} \subset S - S$ and $s_2^{T_{ij}}(A) \leq s_2^{T_i}(A) \leq 2^{-l}p^{-1}\|A\|_{\{1,2,3\}}$. Since $\#T_{ij} \leq \#T_i \leq \#S - 1$, to get the covering S_i it is enough to renumerate the sets T_{ij} . \square

We are now ready to prove Proposition 6.1.

Proof of Proposition 6.1. Define the numbers $\Delta_l, \tilde{\Delta}_l, l \geq 0$ as

$$\Delta_0 = \Delta_A(T), \quad \tilde{\Delta}_0 = s_2^T(A)$$

and

$$\Delta_l = 2^{2-2l}p^{-1}\|A\|_{\{1,2,3\}}, \quad \tilde{\Delta}_l = 2^{1-l}p^{-1/2}\|A\|_{\{1,2,3\}}.$$

Assume first that $T \subset \frac{1}{2}[(B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)]$ and define for $r, l \in \mathbb{N}$,

$$c_T(r, l) = \sup\{F_A^G(S) : S \subset (B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n), \\ S - S \subset T - T, \#S \leq r, \Delta_A(S) \leq \Delta_l, s_2^S(A) \leq \tilde{\Delta}_l\}.$$

We have $c_T(1, l) = 0$. Moreover

$$c_T(r, 0) \geq \sup\{F_A^G(S) : S \subset T, \#S \leq r\}. \quad (24)$$

Notice now, that for any S satisfying the constraints from the definition of $c_T(r, l)$, by Lemma 6.7, we can find a decomposition $S = \bigcup_{i=1}^N ((x_i, y_i) + S_i)$, satisfying (i)–(v). Thus

$$\begin{aligned} F_A^G(S) &\leq \max_i F_A^G((x_i, y_i) + S_i) + L\sqrt{\log N}\Delta_A(S) \\ &\leq \max_i F_A^G(S_i) + Ls_2^S(A) + L2^l\sqrt{p}\Delta_l \\ &\leq c_T(r-1, l+1) + L\tilde{\Delta}_l + L2^l\sqrt{p}\Delta_l. \end{aligned}$$

Taking the supremum yields

$$c_T(r, l) \leq c_T(r-1, l+1) + L\tilde{\Delta}_l + L2^l\sqrt{p}\Delta_l,$$

which gives

$$\begin{aligned} c_T(r, 0) &\leq c_T(1, r-1) + L \sum_{l=0}^{\infty} (\tilde{\Delta}_l + 2^l\sqrt{p}\Delta_l) \\ &\leq L \left(\sqrt{p}\Delta_A(T) + s_2^T(A) + 2p^{-1/2}\|A\|_{\{1,2,3\}} \right). \end{aligned}$$

To finish the proof it is now enough to notice that for $T \subset (B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)$,

$$F_A^G(T) = 4 \sup_{r \geq 1} c_{\frac{1}{2}T}(r, 0).$$

□

Remark Note that the only place in the above argument where the quantities $\Delta_A(T)$ and $s_2^T(A)$ appear is the first step of the induction, when we pass from $l = 0$ to $l = 1$. All the other steps contribute just proper multiples of $\|A\|_{\{1,2,3\}}$ which are upper bounds on the parameters $\Delta_A(S)$ and $s_2^S(A)$ of the set S considered there.

7 The partition theorem

In this section we present partition results which will allow us to pass from the bounds on expectations of suprema of Gaussian processes developed so far to empirical processes involving general random variables with bounded fourth moments (in particular all random variables with log-concave tails).

Lemma 7.1. *Let α and $\tilde{\alpha}$ be two norms on \mathbb{R}^{n^2} and \mathbb{R}^{2n} respectively. For any $p \geq 1$ and $T \subset (B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)$ we can find a decomposition $T = \bigcup_{l=1}^N (T_l + (x_l, y_l))$ with $N \leq \exp(Lp)$, $(x_l, y_l) \in T$ such that for any $(x, y), (\tilde{x}, \tilde{y}) \in T_l$*

$$\alpha(x \otimes y - \tilde{x} \otimes \tilde{y}) \leq \frac{1}{p} \mathbb{E} \alpha(\mathcal{E}^1 \otimes \mathcal{E}^2)$$

and

$$\tilde{\alpha}(x, y) \leq \frac{1}{\sqrt{p}} \mathbb{E} \tilde{\alpha}(\mathcal{E}^1, \mathcal{E}^2).$$

Proof. Let

$$M := \mathbb{E}\alpha(\mathcal{E}^1 \otimes \mathcal{E}^2) \text{ and } \widetilde{M} := \mathbb{E}\tilde{\alpha}(\mathcal{E}^1, \mathcal{E}^2).$$

Define norm β on \mathbb{R}^{2n} by

$$\beta((x, y)) = \mathbb{E}\alpha(x \otimes \mathcal{E}^2) + \mathbb{E}\alpha(\mathcal{E}^1 \otimes y).$$

By Corollary 5.7 with $d = 1$, $a = \sqrt{p}$ and $t = p^{-1/2}$ we can find a decomposition $T = \bigcup_{l=1}^{N_0} S_l$ in such a way that $N_0 \leq \exp(Lp)$ and

$$\beta(x - \tilde{x}, y - \tilde{y}) \leq \frac{1}{\sqrt{p}}M, \quad \tilde{\alpha}(x - \tilde{x}, y - \tilde{y}) \leq \frac{1}{\sqrt{p}}\widetilde{M}$$

for any $(x, y), (\tilde{x}, \tilde{y}) \in S_l$. Let us choose any $(x_l, y_l) \in S_l$, put $\tilde{S}_l = S_l - (x_l, y_l)$ and notice that

$$V_2^{\tilde{S}_l}(\alpha, \frac{1}{2\sqrt{p}}) = \frac{1}{4p}M + \frac{1}{2\sqrt{p}} \sup_{(x,y) \in \tilde{S}_l} \beta((x, y)) \leq \frac{1}{p}M.$$

We have $\tilde{S}_l \subset 2(B_2^n + \sqrt{p}B_1^n)$, hence again by Corollary 5.7 with $t = p^{-1/2}/2$ we can decompose $\tilde{S}_l = \bigcup_{k=1}^{N_l} T_{l,k}$ with $N_l \leq \exp(Lp)$ and $\alpha(x \otimes y - \tilde{x} \otimes \tilde{y}) \leq \frac{1}{p}M$ for all $(x, y), (\tilde{x}, \tilde{y}) \in T_{l,k}$. \square

Theorem 7.2. *For any $p \geq 1$ and $T \subset (B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)$ we can find a decomposition $T = \bigcup_{l=1}^N (T_l + (x_l, y_l))$ with $N \leq \exp(Lp)$, $(x_l, y_l) \in T$ such that for any z_k ,*

$$\mathbb{E} \sup_{(x,y) \in T_l} \sum_{ijk} a_{ijk} x_i y_j z_k g_k \leq \frac{L}{\sqrt{p}} \left(\sum_{ijk} a_{ijk}^2 z_k^4 \right)^{1/4} \left(\sum_{ijk} a_{ijk}^2 \right)^{1/4}.$$

Proof. Let

$$\alpha_z(x) := \left(\sum_k z_k^2 \left(\sum_{ij} a_{ijk} x_{ij} \right)^2 \right)^{1/2}$$

and

$$\tilde{\alpha}_z(x, y) := \left(\sum_{j,k} z_k^2 \left(\sum_i a_{ijk} x_i \right)^2 \right)^{1/2} + \left(\sum_{i,k} z_k^2 \left(\sum_j a_{ijk} y_j \right)^2 \right)^{1/2}.$$

Notice that by the Schwarz inequality

$$\alpha_z(x) \leq \left(\sum_{ijk} a_{ijk}^2 z_k^4 \right)^{1/4} \beta(x), \quad \tilde{\alpha}_z(x, y) \leq \left(\sum_{ijk} a_{ijk}^2 z_k^4 \right)^{1/4} \tilde{\beta}(x, y), \quad (25)$$

where

$$\beta(x) := \left(\sum_k \frac{(\sum_{ij} a_{ijk} x_{ij})^4}{\sum_{ij} a_{ijk}^2} \right)^{1/4}$$

and

$$\tilde{\beta}(x, y) := \left(\sum_{j,k} \frac{(\sum_i a_{ijk} x_i)^4}{\sum_i a_{ijk}^2} \right)^{1/4} + \left(\sum_{i,k} \frac{(\sum_j a_{ijk} y_j)^4}{\sum_j a_{ijk}^2} \right)^{1/4}.$$

Notice that (since the 4-th and 2-nd moments of chaoses generated by exponential variables are comparable) we have

$$\mathbb{E}\beta(\mathcal{E}^1 \otimes \mathcal{E}^2) \leq (\mathbb{E}\beta^4(\mathcal{E}^1 \otimes \mathcal{E}^2))^{1/4} \leq L \left(\sum_{ijk} a_{ijk}^2 \right)^{1/4}$$

and

$$\mathbb{E}\tilde{\beta}(\mathcal{E}^1, \mathcal{E}^2) \leq (\mathbb{E}\tilde{\beta}^4(\mathcal{E}^1, \mathcal{E}^2))^{1/4} \leq L \left(\sum_{ijk} a_{ijk}^2 \right)^{1/4}.$$

Hence by Lemma 7.1 we may decompose $T = \bigcup_{l=1}^N (\tilde{T}_l + (x_l, y_l))$ with $N \leq \exp(Lp)$, $(x_l, y_l) \in T$ in such a way that for any $(x, y), (\tilde{x}, \tilde{y}) \in \tilde{T}_l$,

$$\beta(x \otimes y - \tilde{x} \otimes \tilde{y}) \leq \frac{1}{Lp} \left(\sum_{ijk} a_{ijk}^2 \right)^{1/4} \text{ and } \tilde{\beta}(x, y) \leq \frac{1}{L\sqrt{p}} \left(\sum_{ijk} a_{ijk}^2 \right)^{1/4}.$$

The assertion follows by Proposition 6.1 and (25). \square

Corollary 7.3. *Let Z_1, \dots, Z_n be independent mean zero random variables. For any $p \geq 1$ and $T \subset (B_2^n + \sqrt{p}B_1^n) \times (B_2^n + \sqrt{p}B_1^n)$ there exists a decomposition $T = \bigcup_{l \leq N} ((x_l, y_l) + T_l)$, where $N \leq \exp(Lp)$, $(x_l, y_l) \in T$ and for every l ,*

$$\begin{aligned} \mathbb{E} \sup_{(x,y) \in T_l} \sum_{ijk} a_{ijk} x_i y_j Z_k &\leq \frac{L}{\sqrt{p}} \left(\sum_{ijk} a_{ijk}^2 \right)^{1/4} \mathbb{E} \left(\sum_{ijk} a_{ijk}^2 Z_k^4 \right)^{1/4} \\ &\leq \frac{L}{\sqrt{p}} \|A\|_{\{1,2,3\}} \max_k \|Z_k\|_4. \end{aligned}$$

Proof. It is enough to take the decomposition given by Theorem 7.2 and notice that by classical symmetrization inequalities and comparison of Gaussian

and Rademacher averages, we have

$$\begin{aligned} \mathbb{E} \sup_{(x,y) \in T_l} \sum_{ijk} a_{ijk} x_i y_j Z_k &\leq 2 \mathbb{E} \sup_{(x,y) \in T_l} \sum_{ijk} a_{ijk} x_i y_j Z_k \varepsilon_k \\ &\leq \sqrt{2\pi} \mathbb{E} \sup_{(x,y) \in T_l} \sum_{ijk} a_{ijk} x_i y_j Z_k g_k, \end{aligned}$$

where ε_k (resp. g_k) are sequences of i.i.d Rademacher (resp. standard Gaussian) random variables, independent of the sequence Z_k . \square

8 Proof of Theorem 3.2

The case $d = 1$ of the theorem has been proved in [10], whereas the case $d = 2$ in [13], thus it remains to prove the case $d = 3$.

To simplify the notation we will write X_i, Y_j, Z_k instead of $X_{i_1}^1, X_{i_2}^2, X_{i_3}^3$ respectively. Applying the theorem in the (already known) case of chaoses of order two, conditionally on Z_k 's yields

$$\begin{aligned} \mathbb{E} \left| \sum_{ijk} a_{ijk} X_i Y_j Z_k \right|^p &\leq L^p \left(\mathbb{E} \left(\left\| \left(\sum_k a_{ijk} Z_k \right)_{ij} \right\|_{\mathcal{N}'} \right)^p \right. \\ &\quad \left. + \mathbb{E} \left(\left\| \left(\sum_k a_{ijk} Z_k \right)_{ij} \right\|_{\{1\}\{2\},p} \right)^p \right), \end{aligned}$$

where $\mathcal{N}' = (N_i^j)_{i \leq n, j \leq 2}$.

Using the definition (4) of the norms $\|\cdot\|_{\{1,2\},p}^{\mathcal{N}'}$ and $\|\cdot\|_{\{1\},\{2\},p}^{\mathcal{N}'}$ one can express

$$\left\| \left(\sum_k a_{ijk} Z_k \right)_{ij} \right\|_{\{1,2\},p}^{\mathcal{N}'} \quad \text{and} \quad \left\| \left(\sum_k a_{ijk} Z_k \right)_{ij} \right\|_{\{1\}\{2\},p}^{\mathcal{N}'}$$

in terms of suprema of empirical processes. Lemma 5.9 then yields

$$\begin{aligned} \left\| \sum_{ijk} a_{ijk} X_i Y_j Z_k \right\|_p &\leq L \left(\mathbb{E} \left\| \left(\sum_k a_{ijk} Z_k \right)_{ij} \right\|_{\{1,2\},p}^{\mathcal{N}'} + \mathbb{E} \left\| \left(\sum_k a_{ijk} Z_k \right)_{ij} \right\|_{\{1\}\{2\},p}^{\mathcal{N}'} \right. \\ &\quad \left. + \|A\|_{\{1,2\}\{3\},p}^{\mathcal{N}'} + \|A\|_{\{1\}\{2\}\{3\},p}^{\mathcal{N}'} \right). \end{aligned} \quad (26)$$

We are therefore left with the problem of estimation of the expectations on the right hand side of the above inequality. This will be achieved in Lemmas 8.2 and 8.6 below.

Let us first state a simple lemma which will be used repeatedly in the sequel. It is an almost immediate consequence of the inequality (15), therefore we will skip its proof.

Lemma 8.1. *If \mathcal{J} is a partition of $\{1, 2, 3\}$ and $\#\mathcal{J} = r$, then for any $t \geq 1$, $\|A\|_{\mathcal{J},tp}^{\mathcal{N}} \leq t^r \|A\|_{\mathcal{J},p}^{\mathcal{N}}$.*

Lemma 8.2. *Let $\mathcal{N}' = (N_i^j)_{i \leq n, j \leq 2}$. Then for any $p \geq 2$,*

$$\mathbb{E} \left\| \left(\sum_k a_{ijk} Z_k \right)_{ij} \right\|_{\{1,2\},p}^{\mathcal{N}'} \leq L(\|A\|_{\{1,2,3\},p}^{\mathcal{N}} + \|A\|_{\{1,2\}\{3\},p}^{\mathcal{N}}).$$

Before we prove the above lemma let us state the following well-known auxiliary result, whose proof we provide for completeness.

Lemma 8.3. *For any $p \geq 2$ and any $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$,*

$$\sup \left\{ \sum_{i=1}^n a_i t_i : \sum_{i=1}^n t_i^2 \leq p, |t_i| \leq 1 \right\} \geq \frac{1}{3} \left(\sum_{i \leq p} a_i + \sqrt{p} \left(\sum_{p < i \leq n} a_i^2 \right)^{1/2} \right). \quad (27)$$

Proof. Denote the left hand side of (27) by M . By choosing $t_i = 1$ for $i \leq p$ and $t_i = 0$ for $i > p$ we see that $M \geq \sum_{i \leq p} a_i$.

Now if $\sqrt{p} a_{\lfloor p+1 \rfloor} \leq (\sum_{i > p} a_i^2)^{1/2}$, then $M \geq (p \sum_{i > p} a_i^2)^{1/2}$ (as 'witnessed' by the sequence $t_i = 0$ for $i \leq p$ and $t_i = p^{1/2} a_i (\sum_{i > p} a_i^2)^{-1/2}$ for $i > p$). Otherwise, by monotonicity of a_i , we get $M \geq \sum_{i \leq p} a_i \geq \lfloor p \rfloor a_{\lfloor p+1 \rfloor} \geq 2^{-1} \sqrt{p} (\sum_{i > p} a_i^2)^{1/2}$, which ends the proof. \square

Proof of Lemma 8.2. By symmetry it is enough to prove that

$$\begin{aligned} \mathbb{E} \sup \left\{ \sum_{ijk} a_{ijk} Z_k x_{ij} : \sum_i \hat{N}_i^1 \left(\left(\sum_j x_{ij}^2 \right)^{1/2} \right) \leq p \right\} \\ \leq L(\|A\|_{\{1,2,3\},p}^{\mathcal{N}} + \|A\|_{\{1,2\}\{3\},p}^{\mathcal{N}}). \end{aligned} \quad (28)$$

Moreover, we may and will assume that $\sum_{jk} a_{ijk}^2$ is decreasing in i .

Let us first notice that

$$\begin{aligned}
\|A\|_{\{1,2,3\},p}^{\mathcal{N}} &\geq \sup \left\{ \sum_{ijk} a_{ijk} x_{ijk} : \sum_i \sum_{jk} x_{ijk}^2 \leq p, \sum_{jk} x_{ijk}^2 \leq 1, 1 \leq i \leq n \right\} \\
&= \sup \left\{ \sum_i t_i \left(\sum_{jk} a_{ijk}^2 \right)^{1/2} : \sum_i t_i^2 \leq p, |t_i| \leq 1, 1 \leq i \leq n \right\} \\
&\geq \frac{1}{L} \left(\sum_{i \leq p} \left(\sum_{jk} a_{ijk}^2 \right)^{1/2} + \sqrt{p} \left(\sum_{i > p} \sum_{jk} a_{ijk}^2 \right)^{1/2} \right), \tag{29}
\end{aligned}$$

where the first inequality follows from the definition of the norm $\|\cdot\|_{\{1,2,3\},p}^{\mathcal{N}}$ and the inequality $N_i^1(t) = t^2$ for $|t| \leq 1$, whereas the last one from Lemma 8.3.

Let $A_p = \{t \in \mathbb{R}^n : \sum_i \hat{N}_i^1(t_i) \leq p\}$ and note that

$$\begin{aligned}
&\mathbb{E} \sup \left\{ \sum_{ijk} a_{ijk} Z_k x_{ij} : \sum_i \hat{N}_i^1 \left(\left(\sum_j x_{ij}^2 \right)^{1/2} \right) \leq p \right\} \\
&= \mathbb{E} \sup \left\{ \sum_i t_i \sqrt{\sum_j \left(\sum_k a_{ijk} Z_k \right)^2} : t \in A_p \right\}.
\end{aligned}$$

Define

$$\begin{aligned}
A_p^1 &= \{t \in A_p : |t_i| \leq 1\}, \\
A_p^2 &= \{t \in A_p : \forall_{l \in \mathbb{N}, l \geq 1} \forall_i i \in (2^l p, 2^{l+1} p] \Rightarrow (t_i = 0 \text{ or } |t_i| \geq l^3)\}, \\
A_p^3 &= \{t \in A_p : t_i = 0 \text{ for } i \leq 2p, \\
&\quad \forall_{l \in \mathbb{N}, l \geq 1} \forall_i i \in (2^l p, 2^{l+1} p] \Rightarrow (1 \leq |t_i| \leq l^3 \text{ or } t_i = 0)\}
\end{aligned}$$

and for $m = 1, 2, 3$,

$$S_m := \mathbb{E} \sup \left\{ \sum_i t_i \sqrt{\sum_j \left(\sum_k a_{ijk} Z_k \right)^2} : t \in A_p^m \right\}.$$

Since $A_p \subset A_p^1 + A_p^2 + A_p^3$ (where $+$ stands for the Minkowski sum of sets), we have

$$\mathbb{E} \sup \left\{ \sum_{ijk} a_{ijk} Z_k x_{ij} : \sum_i \hat{N}_i^1 \left(\left(\sum_j x_{ij}^2 \right)^{1/2} \right) \leq p \right\} \leq S_1 + S_2 + S_3. \tag{30}$$

Step 1 For $|t| \leq 1$, $\hat{N}_i^1(t) = t^2$, so

$$\begin{aligned} S_1 &= \mathbb{E} \sup \left\{ \sum_i t_i \sqrt{\sum_j \left(\sum_k a_{ijk} Z_k \right)^2} : \sum_i t_i^2 \leq p, \forall_i |t_i| \leq 1 \right\} \\ &\leq \mathbb{E} \sum_{i \leq p} \sqrt{\sum_j \left(\sum_k a_{ijk} Z_k \right)^2} + \mathbb{E} \sqrt{p} \left(\sum_{i > p} \sum_j \left(\sum_k a_{ijk} Z_k \right)^2 \right)^{1/2} \\ &\leq L \left(\sum_{i \leq p} \left(\sum_{jk} a_{ijk}^2 \right)^{1/2} + \sqrt{p} \left(\sum_{i > p} \sum_{jk} a_{ijk}^2 \right)^{1/2} \right), \end{aligned}$$

where in the second inequality we used the fact that $\mathbb{E} Z_n^2 \leq L$. By (29) this implies that

$$S_1 \leq L \|A\|_{\{1,2,3\},p}^{\mathcal{N}}. \quad (31)$$

Step 2 We will now estimate S_2 . To this end let us note that since for $t \geq 1$, $\hat{N}_i^1(t) \geq |t|$, for every $t \in A_p^2$, the set $I(t) = \text{supp } t = \{i \leq n : t_i \neq 0\}$, satisfies

$$\#I(t) \leq 3p \quad \text{and} \quad \forall_{l \in \mathbb{N}, l \geq 1} \#(I(t) \cap (2^l p, 2^{l+1} p]) \leq p/l^3.$$

Let us denote the family of subsets of $\{1, \dots, n\}$ satisfying the above conditions by \mathcal{I} . We have

$$\#\mathcal{I} \leq 2^{2p} \prod_{l \geq 1} \left(\sum_{s \leq p/l^3} \binom{2^l p}{s} \right) \leq 2^{2p} \prod_{l \geq 1} \left(\frac{e 2^l p}{p/l^3} \right)^{p/l^3} \leq L^p.$$

For each $I \in \mathcal{I}$ let $B_I = \text{conv}\{t \in \mathbb{R}^n : \text{supp } t \subset I, \sum_i \hat{N}_i^1(t_i) \leq p\}$. Then

$$S_2 \leq \mathbb{E} \max_{I \in \mathcal{I}} \sup_{t \in B_I} \sum_i t_i \sqrt{\sum_j \left(\sum_k a_{ijk} Z_k \right)^2}.$$

For each $I \in \mathcal{I}$, the set B_I admits a $1/2$ -net \mathcal{M}_I (with respect to the semi-norm induced by B_I) of cardinality at most $5^{\#I} \leq 5^{3p}$. By standard approximation arguments we have

$$\sup_{t \in B_I} \sum_i t_i \sqrt{\sum_j \left(\sum_k a_{ijk} Z_k \right)^2} \leq 2 \sup_{t \in \mathcal{M}_I} \sum_i t_i \sqrt{\sum_j \left(\sum_k a_{ijk} Z_k \right)^2}.$$

Therefore

$$S_2 \leq 2\mathbb{E} \sup_{t \in \bigcup_{I \in \mathcal{I}} \mathcal{M}_I} \sum_i t_i \sqrt{\sum_j \left(\sum_k a_{ijk} Z_k \right)^2},$$

which by Lemma 5.10 is up to a universal constant majorized by

$$\begin{aligned} & \sup_{t \in \bigcup_{I \in \mathcal{I}} \mathcal{M}_I} \sum_i t_i \mathbb{E} \sqrt{\sum_j \left(\sum_k a_{ijk} Z_k \right)^2} \\ & + \sup_{t \in \bigcup_{I \in \mathcal{I}} \mathcal{M}_I} \sup_{r: \sum_k \hat{N}_k^3(r_k) \leq Lp} \sum_i t_i \sqrt{\sum_j \left(\sum_k a_{ijk} r_k \right)^2} \\ & \leq L \sup_{t \in A_p} \sum_i t_i \sqrt{\sum_{jk} a_{ijk}^2} + \|A\|_{\{1,2\},\{3\},Lp}^{\mathcal{N}} \leq L(\|A\|_{\{1,2,3\},p}^{\mathcal{N}} + \|A\|_{\{1,2\},\{3\},Lp}^{\mathcal{N}}). \end{aligned}$$

Since for $t \geq 1$, $\|A\|_{\{1,2\},\{3\},tp}^{\mathcal{N}} \leq t^2 \|A\|_{\{1,2\},\{3\},p}^{\mathcal{N}}$, the above inequality implies that

$$S_2 \leq L(\|A\|_{\{1,2,3\},p}^{\mathcal{N}} + \|A\|_{\{1,2\},\{3\},p}^{\mathcal{N}}). \quad (32)$$

Step 3 For $|t| \geq 1$, $\hat{N}_i^1(t) \geq t$, so

$$\begin{aligned} S_3 & \leq \sum_{l \geq 1} \mathbb{E} \sup \left\{ \sum_{2^l p < i \leq 2^{l+1} p} t_i \sqrt{\sum_j \left(\sum_k a_{ijk} Z_k \right)^2} : \sum_i |t_i| \leq p, |t_i| \leq l^3 \right\} \\ & \leq L \sum_{l \geq 1} \min(l^3, p) \mathbb{E} \max_{I \subset (2^l p, 2^{l+1} p], \#I \leq \lceil p/l^3 \rceil} \sum_{i \in I} \sqrt{\sum_j \left(\sum_k a_{ijk} Z_k \right)^2} \\ & \leq L \sum_{l \geq 1} \min(l^3, p) \lceil p/l^3 \rceil^{3/4} \mathbb{E} \max_{I \subset (2^l p, 2^{l+1} p], \#I \leq \lceil p/l^3 \rceil} \left(\sum_{i \in I} \left(\sum_j \left(\sum_k a_{ijk} Z_k \right)^2 \right)^2 \right)^{1/4} \\ & \leq L \sum_{l \geq 1} (pl)^{3/4} \mathbb{E} \left(\sum_{2^l p < i \leq 2^{l+1} p} \left(\sum_j \left(\sum_k a_{ijk} Z_k \right)^2 \right)^2 \right)^{1/4} \\ & \leq L \sum_{l \geq 1} (pl)^{3/4} \left(\sum_{2^l p < i \leq 2^{l+1} p} \mathbb{E} \left(\sum_j \left(\sum_k a_{ijk} Z_k \right)^2 \right)^2 \right)^{1/4} \\ & \leq L \sum_{l \geq 1} (pl)^{3/4} \left(\sum_{2^l p < i \leq 2^{l+1} p} \left(\sum_{jk} a_{ijk}^2 \right)^2 \right)^{1/4}, \end{aligned}$$

where in the last inequality we used the comparison of the 4-th and the second moment of norms of linear combinations of independent random variables with log-concave tails.

Now, denote $B = \sqrt{\sum_{i>p} \sum_{ij} a_{ijk}^2}$ and notice that by the assumption on monotonicity of $\sum_{jk} a_{ijk}^2$, we have for $i > p$

$$\sum_{jk} a_{ijk}^2 \leq \frac{B^2}{i-p}.$$

Therefore, we have

$$S_3 \leq L \sum_{l \geq 1} (pl)^{3/4} \left(\sum_{i > 2^l p} \frac{B^4}{(i-p)^2} \right)^{1/4} \leq LB \sum_{l \geq 1} (pl)^{3/4} \frac{1}{(2^l p)^{1/4}} \leq L\sqrt{p}B,$$

which by (29) implies that

$$S_3 \leq L \|A\|_{\{1,2,3\},p}^{\mathcal{N}}. \quad (33)$$

Inequalities (30-33) imply (28) and conclude the proof of the lemma. \square

We will also need the following lemma, proven in [13] (Corollary 3, therein). We would like to remark in passing that the approach in [13] was different that in the present article and that the tools developed in the previous sections could be used to give another proof of this lemma (in the spirit of the argument we provide below for Lemma 8.6). It seems a little bit more natural since Lemmas 8.2 and Lemma 8.6 play in the proof of Theorem 3.2 for $d = 3$ a role analogous to the role played by Lemma 8.4 in the proof of its counterpart for $d = 2$.

Lemma 8.4 (Corollary 3 in [13]). *Consider any matrix $A = (a_{ij})_{ij \leq n}$ and let $\mathcal{N}_1 = (N_i^1)_{i \leq n}$, $\mathcal{N}' = (N_i^j)_{i \leq n, j \leq 2}$. Then, for any $p \geq 2$,*

$$\mathbb{E} \left\| \left(\sum_j a_{ij} Y_j \right) \right\|_{\{1\},p}^{\mathcal{N}_1} \leq L (\|A\|_{\{1,2\},p}^{\mathcal{N}'} + \|A\|_{\{1\}\{2\},p}^{\mathcal{N}'}).$$

Lemma 8.5. *Let $\mathcal{N}' = (N_i^j)_{i \leq n, j \leq 2}$. Then*

$$\mathbb{E} \left\| \left(\sum_{k \leq p} a_{ijk} Z_k \right)_{i,j} \right\|_{\{1\}\{2\},p}^{\mathcal{N}'} \leq L \|A\|_{\{1\}\{2\}\{3\},p}^{\mathcal{N}'}$$

Proof. Consider the norm on $\mathbb{R}^{[p]}$ given by

$$\|(z_1, \dots, z_k)\| = \left\| \left(\sum_{k \leq p} a_{ijk} z_k \right)_{i,j} \right\|_{\{1\}\{2\},p}^{\mathcal{N}'}$$

and let K be the unit ball of the dual norm $\|\cdot\|_*$. Let M be a $1/2$ net in K (with respect to $\|\cdot\|_*$) of cardinality not larger than $3^{[p]}$ (M exists by standard volumetric arguments). Then for all $z \in \mathbb{R}^{[p]}$,

$$\|z\| \leq 2 \sup_{u \in M} \sum_{k \leq p} u_k z_k.$$

Thus

$$\mathbb{E} \left\| \left(\sum_{k \leq p} a_{ijk} Z_k \right)_{i,j} \right\|_{\{1\}\{2\},p}^{\mathcal{N}'} \leq 2 \mathbb{E} \sup_{u \in M} \sum_{k \leq p} u_k Z_k,$$

which by Lemma 5.10 does not exceed

$$\begin{aligned} & L \sup \left\{ \sum_{k \leq p} u_k z_k : u \in M - M, \sum_{k \leq p} \hat{N}_k^3(z_k) \leq p \right\} \\ & \leq L \sup \left\{ \|z\| : \sum_{k \leq p} \hat{N}_k^3(z_k) \leq p \right\} = L \|A\|_{\{1\}\{2\}\{3\},p}^{\mathcal{N}'} \end{aligned}$$

□

Lemma 8.6. *Let $\mathcal{N}' = (N_i^j)_{i \leq n, j \leq 2}$. Then for any $p \geq 2$,*

$$\begin{aligned} & \mathbb{E} \left\| \left(\sum_k a_{ijk} Z_k \right)_{ij} \right\|_{\{1\}\{2\},p}^{\mathcal{N}'} \\ & \leq L \left(\|A\|_{\{1,2,3\},p}^{\mathcal{N}'} + \|A\|_{\{1\}\{2,3\},p}^{\mathcal{N}'} + \|A\|_{\{2\}\{1,3\},p}^{\mathcal{N}'} + \|A\|_{\{1\}\{2\}\{3\},p}^{\mathcal{N}'} \right). \end{aligned} \quad (34)$$

Proof. Let us first notice that it's enough to prove the formally weaker estimate

$$\begin{aligned} & \mathbb{E} \left\| \left(\sum_k a_{ijk} Z_k \right)_{ij} \right\|_{\{1\},\{2\},p}^{\mathcal{N}'} \\ & \leq L \left(\sqrt{p} \|A\|_{\{1,2,3\}} + \|A\|_{\{1\}\{2,3\},p}^{\mathcal{N}'} + \|A\|_{\{2\}\{1,3\},p}^{\mathcal{N}'} + \|A\|_{\{1\}\{2\}\{3\},p}^{\mathcal{N}'} \right). \end{aligned} \quad (35)$$

Indeed, suppose that the above inequality holds for all triple-indexed matrices, and assume additionally (without loss of generality) that $\sum_{ij} a_{ijk}^2$ decreases in k . We have

$$\begin{aligned} & \mathbb{E} \left\| \left(\sum_k a_{ijk} Z_k \right)_{ij} \right\|_{\{1\},\{2\},p}^{\mathcal{N}'} \\ & \leq \mathbb{E} \left\| \left(\sum_{k \leq p} a_{ijk} Z_k \right)_{ij} \right\|_{\{1\},\{2\},p}^{\mathcal{N}'} + \mathbb{E} \left\| \left(\sum_{k > p} a_{ijk} Z_k \right)_{ij} \right\|_{\{1\},\{2\},p}^{\mathcal{N}'}. \end{aligned}$$

By Lemma 8.5 we have

$$\mathbb{E} \left\| \left(\sum_{k \leq p} a_{ijk} Z_k \right)_{ij} \right\|_{\{1\}\{2\},p}^{\mathcal{N}'} \leq L \|A\|_{\{1\}\{2\}\{3\},p}^{\mathcal{N}}.$$

Moreover, by our assumption (35),

$$\mathbb{E} \left\| \left(\sum_{k > p} a_{ijk} Z_k \right)_{ij} \right\|_{\{1\}\{2\},p}^{\mathcal{N}'} \leq L \sqrt{p} \left(\sum_{k > p} \sum_{ij} a_{ijk}^2 \right)^{1/2} + \sum_{\substack{\mathcal{J} \in \mathcal{P}_3, \mathcal{J} \neq \{\{1,2,3\}\} \\ \mathcal{J} \neq \{\{1,2\},\{3\}\}}} \|A\|_{\mathcal{J},p}^{\mathcal{N}}.$$

Monotonicity of $\sum_{ij} a_{ijk}^2$ implies that

$$\begin{aligned} & \sqrt{p} \left(\sum_{k > p} \sum_{ij} a_{ijk}^2 \right)^{1/2} \\ & \leq \sup \left\{ \sum_k t_k \sqrt{\sum_{ij} a_{ijk}^2} : \sum_k t_k^2 \leq p, |t_k| \leq 1 \right\} \leq \|A\|_{\{1,2,3\},p}^{\mathcal{N}}, \end{aligned}$$

which together with the previous three inequalities proves (34).

We will now prove (35). To this end let us denote

$$A_p^j = \left\{ t \in \mathbb{R}^n : \sum_{i=1}^n \hat{N}_i^j(t_i) \leq p \right\}, \quad j = 1, 2, 3.$$

Since $\hat{N}_i^j(t) \geq |t|$ for $t > 1$, it is easy to see that $A_p^j \subset \sqrt{p} B_2^n + p B_1^n$. Hence, by Corollary 7.3 and the fact that $\mathbb{E} Z_k^4 \leq L$, there exists a partition

$$A_p^1 \times A_p^2 = \bigcup_{l \leq N} ((x^l, y^l) + T_l),$$

with $N \leq \exp(Lp)$, $(x^l, y^l) \in A_p^1 \times A_p^2$, such that

$$\max_{l \leq N} \mathbb{E} \sup_{(x,y) \in T_l} \sum_{ijk} a_{ijk} x_i y_j Z_k \leq L\sqrt{p} \|A\|_{\{1,2,3\}}. \quad (36)$$

Now, by Lemma 5.10,

$$\begin{aligned} & \mathbb{E} \sup_{(x,y) \in A_p^1 \times A_p^2} \sum_{ijk} a_{ijk} x_i y_j Z_k \\ & \leq \max_{l \leq N} \mathbb{E} \sup_{(x,y) \in (x^l, y^l) + T_l} \sum_{ijk} a_{ijk} x_i y_j Z_k + 2 \sup_{(x,y) \in A_p^1 \times A_p^2, z \in A_{Lp}^3} \sum_{ijk} a_{ijk} x_i y_j z_k \\ & \leq \max_{l \leq N} \mathbb{E} \sup_{(x,y) \in (x^l, y^l) + T_l} \sum_{ijk} a_{ijk} x_i y_j Z_k + L \|A\|_{\{1\}\{2\}\{3\}, p}^{\mathcal{N}}, \end{aligned}$$

where in the second inequality we used the fact that $A_{Lp}^3 \subset LA_p^3$.

Thus it remains to estimate $\max_{l \leq N} \mathbb{E} \sup_{(x,y) \in (x^l, y^l) + T_l} \sum_{ijk} a_{ijk} x_i y_j Z_k$. Denote by $\pi_1(T)$, $\pi_2(T)$ respectively projections of T_l onto the first n and the last n coordinates and let $\mathcal{N}_j = (N_i^j)_{i \leq n}$, $j = 1, 2$. We have

$$\begin{aligned} & \mathbb{E} \sup_{(x,y) \in (x^l, y^l) + T_l} \sum_{ijk} a_{ijk} x_i y_j Z_k \\ & \leq \mathbb{E} \sup_{(x,y) \in T_l} \sum_{ijk} a_{ijk} x_i y_j Z_k + \mathbb{E} \sup_{x \in \pi_1(T)} \sum_{ijk} a_{ijk} x_i y_j^l Z_k + \mathbb{E} \sup_{y \in \pi_2(T)} \sum_{ijk} a_{ijk} x_i^l y_j Z_k \\ & \leq L\sqrt{p} \|A\|_{\{1,2,3\}} + 2 \left\| \left(\sum_{ik} a_{ijk} x_i^l Z_k \right)_j \right\|_{\{2\}, p}^{\mathcal{N}_2} + 2 \left\| \left(\sum_{jk} a_{ijk} y_j^l Z_k \right)_i \right\|_{\{1\}, p}^{\mathcal{N}_1} \\ & \leq L\sqrt{p} \|A\|_{\{1,2,3\}} + L \|A\|_{\{1\}\{2,3\}, p}^{\mathcal{N}} + L \|A\|_{\{2\}\{1,3\}, p}^{\mathcal{N}} + L \|A\|_{\{1\}\{2\}\{3\}, p}^{\mathcal{N}}, \end{aligned}$$

where the second inequality follows from (36) and the fact that $\pi_j(T_l) \subset LA_p^j$, and the third inequality from Lemma 8.4 (applied to Z_k instead of Y_j , which corresponds to an appropriate permutation of the array N_i^j). This proves (35) and ends the proof of the lemma. \square

Conclusion of the proof of Theorem 3.2. By lemmas 8.2 and 8.6, the right hand side of (26) does not exceed

$$\sum_{\mathcal{J} \in \mathcal{P}_3} \|(a_{ijk})\|_{\mathcal{J}, p}^{\mathcal{N}},$$

which ends the proof. \square

9 Proof of Theorem 3.4

In this section we restrict our attention to the special case of symmetric exponential variables and consider polynomial chaoses of arbitrary order. For exponential variables, the function $N_i^j(t) = t$, which allows us to replace quantities $\|(a_{\mathbf{i}})\|_{\mathcal{J},p}^{\mathcal{N}}$ by simpler quantities.

Proposition 9.1. *If for all $i \leq n, j \leq d$, $N_i^j(t) = t$, then for every $\mathcal{J} = \{J_1, \dots, J_k\} \in P_d$ and every $p \geq 2$,*

$$\begin{aligned} & L_d^{-1} \sum_{I \in Q(\mathcal{J})} p^{\#I^c + (k - \#I^c)/2} \max_{\mathbf{i}_{I^c}} \|(a_{\mathbf{i}})_{\mathbf{i}_I}\|_{S(\mathcal{J}, I)} \\ & \leq \|(a_{\mathbf{i}})\|_{\mathcal{J},p}^{\mathcal{N}} \leq L_d \sum_{I \in Q(\mathcal{J})} p^{\#I^c + (k - \#I^c)/2} \max_{\mathbf{i}_{I^c}} \|(a_{\mathbf{i}})_{\mathbf{i}_I}\|_{S(\mathcal{J}, I)} \end{aligned}$$

where $Q(\mathcal{J}) = \{I \subset \{1, \dots, d\} : \forall_{i \leq k} \#(I^c \cap J_i) \leq 1\}$ and $S(\mathcal{J}, I)$ is the partition of I obtained from \mathcal{J} by removing from the sets J_i all the elements of I^c .

Proof. It is enough to prove that

$$\begin{aligned} & L^{-1} \left(p \max_{i_1} \|(a_{\mathbf{i}})_{i_2, \dots, i_d}\|_{\{2, \dots, d\}} + \sqrt{p} \|(a_{\mathbf{i}})\|_{\{1, \dots, d\}} \right) \\ & \leq \sup \left\{ \sum_{\mathbf{i}} a_{\mathbf{i}} x_{\mathbf{i}} : \sum_{i_1} \min \left(\left(\sum_{i_2, \dots, i_d} x_{\mathbf{i}}^2 \right)^{1/2}, \sum_{i_2, \dots, i_d} x_{\mathbf{i}}^2 \right) \leq p \right\} \\ & \leq L \left(p \max_{i_1} \|(a_{\mathbf{i}})_{i_2, \dots, i_d}\|_{\{2, \dots, d\}} + \sqrt{p} \|(a_{\mathbf{i}})\|_{\{1, \dots, d\}} \right). \end{aligned}$$

The proposition follows easily by an iterative application of this inequality.

To prove the above inequality it suffices to notice that

$$\begin{aligned} & \left\{ x_{\mathbf{i}} : \sum_{i_1} \min \left(\left(\sum_{i_2, \dots, i_d} x_{\mathbf{i}}^2 \right)^{1/2}, \sum_{i_2, \dots, i_d} x_{\mathbf{i}}^2 \right) \leq p \right\} \\ & = \left\{ z_{i_1} y_{\mathbf{i}} : \sum_i \min(|z_i|, z_i^2) \leq p, \forall_{i_1} \sum_{i_2, \dots, i_d} y_{\mathbf{i}}^2 \leq 1 \right\} \end{aligned}$$

and

$$(\sqrt{p}B_2^n) \cup (pB_1^n) \subset \left\{ z \in \mathbb{R}^n : \sum_i \min(|z_i|, z_i^2) \leq p \right\} \subset \sqrt{p}B_2^n + pB_1^n.$$

We leave the details to the reader. \square

For a nonempty set I , let us denote by P_I the set of all partitions of I into pairwise disjoint, nonempty sets. In particular $P_{\{1,\dots,d\}} = P_d$, $P_\emptyset = \{\emptyset\}$.

The above proposition yields the following

Corollary 9.2. *If for all $i \leq n, j \leq d$, $N_i^j(t) = t$, then for every $p \geq 2$,*

$$\begin{aligned} & L_d^{-1} \sum_{I \subset \{1,\dots,d\}} \sum_{\mathcal{J} \in P_I} p^{\#I^c + \#\mathcal{J}/2} \max_{\mathbf{i}_{I^c}} \|(a_{\mathbf{i}})_{\mathbf{i}_I}\|_{\mathcal{J}} \\ & \leq \sum_{\mathcal{J} \in P_d} \|(a_{\mathbf{i}})\|_{\mathcal{J},p}^{\mathcal{N}} \leq L_d \sum_{I \subset \{1,\dots,d\}} \sum_{\mathcal{J} \in P_I} p^{\#I^c + \#\mathcal{J}/2} \max_{\mathbf{i}_{I^c}} \|(a_{\mathbf{i}})_{\mathbf{i}_I}\|_{\mathcal{J}}. \end{aligned}$$

From the above corollary and Theorem 3.1 it follows that to prove Theorem 3.4 it is enough to demonstrate the following

Proposition 9.3. *If $(X_i^j)_{i \leq n, j \leq d}$ are independent symmetric exponential random variables, then for every $p \geq 2$,*

$$\left\| \sum_{\mathbf{i}} a_{\mathbf{i}} X_{i_1}^1 \cdots X_{i_d}^d \right\|_p \leq L_d \sum_{I \subset \{1,\dots,d\}} \sum_{\mathcal{J} \in P_I} p^{\#I^c + \#\mathcal{J}/2} \max_{\mathbf{i}_{I^c}} \|(a_{\mathbf{i}})_{\mathbf{i}_I}\|_{\mathcal{J}}. \quad (37)$$

The proof of Proposition 9.3 will be based on induction with respect to d . It will require several additional lemmas. Throughout the rest of this section we will assume that $(X_i^j)_{i \leq n, j \leq d}$ are independent symmetric exponential random variables.

Lemma 9.4. *For any $d = 2, 3, \dots$,*

$$\begin{aligned} & \mathbb{E} \left\| \left(\sum_{i_d} a_{\mathbf{i}} X_{i_d}^d \right)_{\mathbf{i}_{\{d\}^c}} \right\|_{\{1\} \dots \{d-1\}} \\ & \leq L_d \sum_{\mathcal{J} \in P_d} p^{(1+\#\mathcal{J}-d)/2} \|(a_{\mathbf{i}})\|_{\mathcal{J}} + L_d \sum_{\mathcal{J} \in P_{d-1}} p^{1+(1+\#\mathcal{J}-d)/2} \max_{i_d} \|(a_{\mathbf{i}})_{\mathbf{i}_{\{d\}^c}}\|_{\mathcal{J}}. \end{aligned}$$

We will need the following technical fact.

Lemma 9.5. *Let $Y_i^{(1)}$ be independent standard symmetric exponential variables and $Y_i^{(2)} = g_i^2$, $Y_i^{(3)} = g_i \tilde{g}_i$, where g_i, \tilde{g}_i are i.i.d $\mathcal{N}(0, 1)$ variables and ε_i - i.i.d Rademacher variables independent of $Y_i^{(j)}$. Then for any normed space E and any vectors $v_1, \dots, v_n \in E$ the quantities*

$$\mathbb{E} \left\| \sum_i v_i \varepsilon_i Y_i^{(j)} \right\|, \quad j = 1, 2, 3,$$

are comparable up to universal multiplicative factors.

Proof. Since we can symmetrize all variables, and by the contraction principle and Jensen's inequality

$$\mathbb{E} \left\| \sum_i v_i \varepsilon_i |Y_i^{(j)}| \right\| \geq c \mathbb{E} \left\| \sum_i v_i \varepsilon_i \right\|,$$

it is enough to show that one can define copies of the variables $Y_i^{(j)}$ (which we will identify with the variables) on a common probability space in such a way that for any $j, k = 1, 2, 3$,

$$|Y_i^{(j)}| \leq L(1 + |Y_i^{(k)}|).$$

This is possible by using the inverse of the distribution function, since

$$\mathbb{P}(|Y_i^{(1)}| \geq t) = e^{-t},$$

$$L^{-1}e^{-Lt} \leq \mathbb{P}(Y_i^{(2)} \geq t) \leq e^{-t/2}$$

and

$$L^{-1}e^{-Lt} \leq \mathbb{P}(Y_i^{(3)} \geq t) \leq 2e^{-t/2},$$

where in the last inequality we used the estimates $\mathbb{P}(|g_i|, |\tilde{g}_i| \geq \sqrt{t}) \leq \mathbb{P}(|Y_i^{(3)}| \geq t) \leq \mathbb{P}(|g_i| \geq \sqrt{t}) + \mathbb{P}(|\tilde{g}_i| \geq \sqrt{t})$. \square

The proof of Lemma 9.4 will be based on a conditional application of the following result from [14] (see [2] for a similar approach in the context of moment inequalities for U -statistics).

Lemma 9.6 ([14], Theorem 2). *For any $p \geq 2$,*

$$\mathbb{E} \left\| \left(\sum_{i_d} a_i g_{i_d} \right)_{i_{d-1}} \right\|_{\{1\} \dots \{d-1\}} \leq L_d \sum_{\mathcal{J} \in \mathcal{P}_d} p^{(1+\#\mathcal{J}-d)/2} \mathbb{E} \|(a_i)\|_{\mathcal{J}}.$$

Proof of Lemma 9.4. Lemmas 9.5 and 9.6 give

$$\begin{aligned} \mathbb{E} \left\| \left(\sum_{i_d} a_i X_{i_d}^d \right) \right\|_{\{1\} \dots \{d-1\}} &\leq L \mathbb{E} \left\| \left(\sum_{i_d} a_i g_{i_d} \tilde{g}_{i_d} \right) \right\|_{\{1\} \dots \{d-1\}} \\ &\leq L_d \sum_{\mathcal{J} \in \mathcal{P}_d} p^{(1+\#\mathcal{J}-d)/2} \mathbb{E} \|(a_i g_{i_d})\|_{\mathcal{J}}. \end{aligned} \quad (38)$$

Take $\mathcal{J} \in P_d$ of the form $\mathcal{J} = \{I_1 \cup \{d\}, \dots, I_k\}$ where $\{I_1, \dots, I_k\} \setminus \{\emptyset\} \in P_{d-1}$. We have

$$\begin{aligned} \mathbb{E} \|(a_i g_{i_d})\|_{\mathcal{J}}^2 &= \mathbb{E} \sup_{\|x_{\mathbf{i}_{I_j}}^j\|_2 \leq 1, j=2, \dots, k} \sum_{i_d} g_{i_d}^2 \sum_{\mathbf{i}_{I_1}} \left(\sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j \right)^2 \\ &\leq \|(a_{\mathbf{i}})\|_{\mathcal{J}}^2 + \mathbb{E} \sup_{\|x_{\mathbf{i}_{I_j}}^j\|_2 \leq 1, j=2, \dots, k} \sum_{i_d} (g_{i_d}^2 - 1) \sum_{\mathbf{i}_{I_1}} \left(\sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j \right)^2. \end{aligned}$$

Since $\mathbb{E} g_{i_d}^2 = 1$, standard symmetrization arguments applied to the second term on the right hand side give

$$\begin{aligned} \mathbb{E} \|(a_i g_{i_d})\|_{\mathcal{J}}^2 &\leq \|(a_{\mathbf{i}})\|_{\mathcal{J}}^2 + 2\mathbb{E} \sup_{\|x_{\mathbf{i}_{I_j}}^j\|_2 \leq 1, j=2, \dots, k} \sum_{i_d} \varepsilon_{i_d} g_{i_d}^2 \sum_{\mathbf{i}_{I_1}} \left(\sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j \right)^2 \\ &\leq \|(a_{\mathbf{i}})\|_{\mathcal{J}}^2 + L\mathbb{E} \sup_{\|x_{\mathbf{i}_{I_j}}^j\|_2 \leq 1, j=2, \dots, k} \sum_{i_d} g_{i_d} \tilde{g}_{i_d} \sum_{\mathbf{i}_{I_1}} \left(\sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j \right)^2, \end{aligned} \tag{39}$$

where in the second inequality we used again Lemma 9.5. Let now

$$M = \max_{i_d} \sup_{\|x_{\mathbf{i}_{I_j}}^j\|_2 \leq 1, j=2, \dots, k} \sqrt{\sum_{\mathbf{i}_{I_1}} \left(\sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j \right)^2} = \max_{i_d} \|(a_{\mathbf{i}})_{\mathbf{i}_{\{d\}^c}}\|_{I_1, \dots, I_k},$$

where in the case $I_1 = \emptyset$ we slightly abuse the notation, by identifying the set $\{I_1, \dots, I_k\}$ with the partition $\{I_2, \dots, I_k\}$. For fixed g_{i_d} consider functions $\varphi_k: \mathbb{R} \rightarrow \mathbb{R}$, given by the formula

$$\varphi_{i_d}(t) = \begin{cases} \frac{t^2}{2Mg_{i_d}} & \text{for } |t| \leq |g_{i_d}|M, \\ g_{i_d}M/2 & \text{for } |t| > |g_{i_d}|M. \end{cases}$$

We have $|\varphi'_{i_d}(t)| = |t|/(|g_{i_d}|M) \leq 1$ for $|t| \leq |g_{i_d}|M$, moreover φ_{i_d} is constant for $t \geq |g_{i_d}|M$, so φ_{i_d} is 1-Lipschitz. Thus, by the contraction principle (see

Corollary 3.17 in [16]),

$$\begin{aligned} & \mathbb{E}_{\tilde{g}} \sup_{\|x_{\mathbf{i}_{I_j}}^j\|_2 \leq 1, j=2, \dots, k} \left| \sum_{i_d} \tilde{g}_{i_d} \varphi_{i_d} \left(g_{i_d} \left(\sum_{\mathbf{i}_{I_1}} \left(\sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j \right)^2 \right)^{1/2} \right) \right| \quad (40) \\ & \leq 4 \mathbb{E}_{\tilde{g}} \sup_{\|x_{\mathbf{i}_{I_j}}^j\|_2 \leq 1, j=2, \dots, k} \left| \sum_{i_d} \tilde{g}_{i_d} g_{i_d} \left(\sum_{\mathbf{i}_{I_1}} \left(\sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j \right)^2 \right)^{1/2} \right|, \end{aligned}$$

which implies that

$$\begin{aligned} & \mathbb{E}_{\tilde{g}} \sup_{\|x_{\mathbf{i}_{I_j}}^j\|_2 \leq 1, j=2, \dots, k} \left| \sum_{i_d} \tilde{g}_{i_d} g_{i_d} \sum_{\mathbf{i}_{I_1}} \left(\sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j \right)^2 \right| \quad (41) \\ & \leq 8M \mathbb{E}_{\tilde{g}} \sup_{\|x_{\mathbf{i}_{I_j}}^j\|_2 \leq 1, j=2, \dots, k} \left| \sum_{i_d} \tilde{g}_{i_d} g_{i_d} \left(\sum_{\mathbf{i}_{I_1}} \left(\sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j \right)^2 \right)^{1/2} \right|. \end{aligned}$$

Denote $T = \prod_{j=2}^k B_{H_j}$, where B_{H_j} is the unit ball of the Hilbert space $\otimes_{t \in I_j} \mathbb{R}^n$. For $t \in T$, $t = (x_{\mathbf{i}_{I_j}}^j)_{j=2}^k$ let

$$X_t = \sum_{i_d} \tilde{g}_{i_d} g_{i_d} \left(\sum_{\mathbf{i}_{I_1}} \left(\sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j \right)^2 \right)^{1/2}.$$

Then, conditionally on g_{i_d} , $(X_t)_{t \in T}$ is a Gaussian process. It induces a metric on T given by

$$d_X(t, s) = \|X_t - X_s\|_2.$$

More explicitly if $t = (x_{\mathbf{i}_{I_j}}^j)_{j=2}^k$, $s = (y_{\mathbf{i}_{I_j}}^j)_{j=2}^k$, then

$$\begin{aligned} & d_X(t, s)^2 \\ & = \sum_{i_d} g_{i_d}^2 \left[\left(\sum_{\mathbf{i}_{I_1}} \left(\sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j \right)^2 \right)^{1/2} - \left(\sum_{\mathbf{i}_{I_1}} \left(\sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k y_{\mathbf{i}_{I_j}}^j \right)^2 \right)^{1/2} \right]^2 \\ & \leq \sum_{i_d} g_{i_d}^2 \sum_{\mathbf{i}_{I_1}} \left(\sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \left(\prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j - \prod_{j=2}^k y_{\mathbf{i}_{I_j}}^j \right) \right)^2, \end{aligned}$$

where to obtain the last inequality for each fixed i_d we used the triangle inequality in the space $\ell_2(\{1, \dots, n\}^{I_1})$ for vectors $a_{\mathbf{i}_{I_1}} = \sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j$ and $b_{\mathbf{i}_{I_1}} = \sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k y_{\mathbf{i}_{I_j}}^j$.

Now, the right-hand side above is equal to $d_{\tilde{X}}(t, s) = \|\tilde{X}_t - \tilde{X}_s\|_2$, where $(\tilde{X}_t)_{t \in T}$ is a (conditionally) Gaussian process defined as

$$\tilde{X}_t = \sum_{\mathbf{i}_{I_1 \cup \{d\}}} g_{i_d} \tilde{g}_{\mathbf{i}_{I_1 \cup \{d\}}} \sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j$$

where $t = (x_{\mathbf{i}_{I_j}}^j)_{j=2}^k$ and $(\tilde{g}_{\mathbf{i}_{I_1 \cup \{d\}}})_{\mathbf{i}_{I_1 \cup \{d\}}}$ is an array of i.i.d. standard Gaussian variables independent of g_{i_d} .

Thus by the Slepian lemma we have

$$\mathbb{E}_{\tilde{g}} \sup_{t \in T} X_t \leq \mathbb{E}_{\tilde{g}} \sup_{t \in T} \tilde{X}_t.$$

Moreover, since $0 \in T$, $X_0 = 0$ and the distribution of X is the same as of $-X$, we have

$$\mathbb{E} \sup_{t \in T} |X_t| = \mathbb{E} \max(\sup_{t \in T} X_t, \sup_{t \in T} (-X_t)) \leq \mathbb{E} \sup_{t \in T} X_t + \mathbb{E} \sup_{t \in T} (-X_t) = 2 \sup_{t \in T} X_t.$$

Thus, we have

$$\begin{aligned} & \mathbb{E}_{\tilde{g}} \sup_{\|x_{\mathbf{i}_{I_j}}^j\|_{2 \leq j \leq k} \leq 1} \left| \sum_{i_d} \tilde{g}_{i_d} g_{i_d} \left(\sum_{\mathbf{i}_{I_1}} \left(\sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j \right)^2 \right)^{1/2} \right| \\ & \leq 2 \mathbb{E}_{\tilde{g}} \sup_{\|x_{\mathbf{i}_{I_j}}^j\|_{2 \leq j \leq k} \leq 1} \sum_{\mathbf{i}_{I_1 \cup \{d\}}} \tilde{g}_{\mathbf{i}_{I_1 \cup \{d\}}} \sum_{\mathbf{i}_{(I_1 \cup \{d\})^c}} g_{i_d} a_{\mathbf{i}} \prod_{j=2}^k x_{\mathbf{i}_{I_j}}^j \\ & \leq L_d \sum_{\mathcal{K} \in P_d} p^{(1+\#\mathcal{K}-k)/2} \|(a_{\mathbf{i}} g_{i_d})\|_{\mathcal{K}}, \end{aligned}$$

where the last inequality follows from another application of Lemma 9.6, conditionally on g_{i_d} . Going now back to (39) and (41), we obtain that for all

$\varepsilon \in (0, 1)$ and all $\mathcal{J} = \{I_1 \cup \{d\}, I_2, \dots, I_k\} \in P_d$,

$$\begin{aligned}
& p^{(1+k-d)/2} \mathbb{E} \|(a_{\mathbf{i}} g_{i_d})\|_{\mathcal{J}} \\
& \leq p^{(1+k-d)/2} \|(a_{\mathbf{i}})\|_{\mathcal{J}} \\
& \quad + L_d \sqrt{p^{(2+k-d)/2} \max_{i_d} \|(a_{\mathbf{i}})_{i_{\{d\}^c}}\|_{\mathcal{J}'}} \sqrt{\sum_{\mathcal{K} \in P_d} p^{(1+\#\mathcal{K}-d)/2} \mathbb{E} \|(a_{\mathbf{i}} g_{i_d})\|_{\mathcal{K}}} \\
& \leq p^{(1+k-d)/2} \|(a_{\mathbf{i}})\|_{\mathcal{J}} \\
& \quad + L_d p^{1+(1+\#\mathcal{J}'-d)/2} \varepsilon^{-1} \max_{i_d} \|(a_{\mathbf{i}})_{i_{\{d\}^c}}\|_{\mathcal{J}'} + \varepsilon L_d \sum_{\mathcal{K} \in P_d} p^{(1+\#\mathcal{K}-d)/2} \mathbb{E} \|(a_{\mathbf{i}} g_{i_d})\|_{\mathcal{K}},
\end{aligned}$$

where $\mathcal{J}' = \{I_1, \dots, I_k\} \setminus \{d\}$. Summing the above inequalities over all $\mathcal{J} \in P_d$ and choosing ε to be a sufficiently small number depending on d , we get

$$\begin{aligned}
\sum_{\mathcal{J} \in P_d} p^{(1+\#\mathcal{J}-d)/2} \mathbb{E} \|(a_{\mathbf{i}} g_{i_d})\|_{\mathcal{J}} & \leq L_d \sum_{\mathcal{J} \in P_d} p^{(1+\#\mathcal{J}-d)/2} \|(a_{\mathbf{i}})\|_{\mathcal{J}} \\
& \quad + L_d \sum_{\mathcal{J} \in P_{d-1}} p^{1+(1+\#\mathcal{J}-d)/2} \max_{i_d} \|(a_{\mathbf{i}})_{i_{\{d\}^c}}\|_{\mathcal{J}}.
\end{aligned}$$

Together with (38) this ends the proof of the lemma. \square

To prove Proposition 9.3 we will also use a technical fact proved in [2] in greater generality (see Lemma 5 therein).

Lemma 9.7. *For $\alpha > 0$ and arbitrary nonnegative numbers r_{i_1, \dots, i_d} and $p > 1$ we have*

$$p^{\alpha p} \sum_{\mathbf{i}} r_{\mathbf{i}}^p \leq L_d^p p^{\alpha d} \left[p^{\alpha p} \max_{\mathbf{i}} r_{\mathbf{i}}^p + \sum_{I \subsetneq \{1, \dots, d\}} p^{\#I p} \max_{\mathbf{i}_I} \left(\sum_{\mathbf{i}_{I^c}} r_{\mathbf{i}} \right)^p \right].$$

Proof of Proposition 9.3. The argument is similar to the proof of Theorem 6 in [2] therefore we will only sketch the main steps.

Since for $p = 2$ the proposition is trivial (recall that $(\|(a_{\mathbf{i}})\|_{\{1, \dots, d\}} = (\sum_{\mathbf{i}} a_{\mathbf{i}}^2)^{1/2})$, we will assume that $p > 2$.

Let us first note that to prove the proposition it is enough to show that

$$\mathbb{E} \left| \sum_{\mathbf{i}} a_{\mathbf{i}} X_{i_1}^1 \cdots X_{i_d}^d \right|^p \leq L_d^p \sum_{I \subset \{1, \dots, d\}} \sum_{\mathcal{J} \in P_I} p^{p(\#I^c + \#\mathcal{J}/2)} \sum_{\mathbf{i}_{I^c}} \|(a_{\mathbf{i}})_{\mathbf{i}_{I^c}}\|_{\mathcal{J}}^p. \quad (42)$$

Indeed, for fixed I let us apply Lemma 9.7 (with $p/2$ instead of p , $\#I^c$ instead of d and $r_{\mathbf{i}_{I^c}} = \|(a_{\mathbf{i}})_{\mathbf{i}_I}\|_{\mathcal{J}}^2$). We get

$$\begin{aligned} & \sum_{\mathbf{i}_{I^c}} \|(a_{\mathbf{i}})_{\mathbf{i}_I}\|_{\mathcal{J}}^p \\ & \leq L_{\#I^c}^p (p/2)^{\alpha\#I^c} \left(\max_{\mathbf{i}_{I^c}} \|(a_{\mathbf{i}})_{\mathbf{i}_I}\|_{\mathcal{J}}^p + \sum_{J \subsetneq I^c} (p/2)^{\#Jp/2 - \alpha p/2} \max_{\mathbf{i}_J} \left(\sum_{\mathbf{i}_{I^c \setminus J}} \|(a_{\mathbf{i}})_{\mathbf{i}_I}\|_{\mathcal{J}}^2 \right)^{p/2} \right). \end{aligned}$$

Note that for $J \subsetneq I^c$ we have $\sum_{\mathbf{i}_{I^c \setminus J}} \|(a_{\mathbf{i}})_{\mathbf{i}_I}\|_{\mathcal{J}}^2 \leq \sum_{\mathbf{i}_{J^c}} a_{\mathbf{i}}^2 = \|(a_{\mathbf{i}})_{\mathbf{i}_{J^c}}\|_{\{J^c\}}^2$ and that $p^{\#J+1/2} \max_{\mathbf{i}_J} \|(a_{\mathbf{i}})_{\mathbf{i}_{J^c}}\|_{\{J^c\}}$ appears among the summands on the right hand side of (37). Thus the above inequality with α sufficiently large (depending only on d) implies that the right hand side of (42) is majorized by the p -th power of the right hand side of the inequality asserted in the proposition (we use the fact that if α depends only on d then $p^{\alpha\#I^c} \leq L_d^p$).

It remains to prove (42). We will proceed by induction on d . For $d = 1$, the proposition (which is stronger than (42) for $d = 1$) is a special case of Theorem 3.2 (it also follows from the Gluskin-Kwapień estimate).

Let us thus assume that (42) holds for chaoses of order at most $d - 1$. We will show that then it holds for chaoses of order d . Applying the induction assumption conditionally on $(X_i^d)_i$ together with the Fubini theorem and Lemma 5.10 we obtain

$$\begin{aligned} & \mathbb{E} \left| \sum_{\mathbf{i}} a_{\mathbf{i}} X_{i_1}^1 \cdots X_{i_d}^d \right|^p \\ & \leq L_{d-1}^p \sum_{I \subset \{1, \dots, d-1\}} \sum_{\mathcal{J} \in P_I} p^{p(d-1-\#I+\#\mathcal{J}/2)} \sum_{\mathbf{i}_{\{1, \dots, d-1\} \setminus I}} \mathbb{E} \left\| \left(\sum_{i_d} a_{\mathbf{i}} X_{i_d}^d \right)_{\mathbf{i}_I} \right\|_{\mathcal{J}}^p \\ & \leq L_d^p \sum_{I \subset \{1, \dots, d-1\}} \sum_{\mathcal{J} \in P_I} p^{p(d-1-\#I+\#\mathcal{J}/2)} \sum_{\mathbf{i}_{\{1, \dots, d-1\} \setminus I}} \left(\mathbb{E} \left\| \left(\sum_{i_d} a_{\mathbf{i}} X_{i_d}^d \right)_{\mathbf{i}_I} \right\|_{\mathcal{J}} \right)^p \\ & \quad + L_d^p \sum_{I \subset \{1, \dots, d\}} \sum_{\mathcal{J} \in P_I} p^{p(\#I^c+\#\mathcal{J}/2)} \sum_{\mathbf{i}_{I^c}} \|(a_{\mathbf{i}})_{\mathbf{i}_I}\|_{\mathcal{J}}^p. \end{aligned}$$

By Lemma 9.4 the first sum on the right hand side above is majorized by the second one, which proves (42). \square

10 Appendix

Proof of Proposition 1.2. Note that

$$\sum_{j=0}^d \sum_{i_1, \dots, i_j=1}^n a_{i_1, \dots, i_j}^j X_{i_1} \cdots X_{i_j} = \sum_{\substack{1 \leq i_1, \dots, i_d \leq n \\ \text{pairwise distinct}}} H_{i_1, \dots, i_d}(X_{i_1}, \dots, X_{i_d}),$$

where

$$H_{i_1, \dots, i_d}(x_1, \dots, x_d) = \frac{1}{d!} \sum_{j=0}^d \frac{(n-d)!}{(n-j)!} \sum_{\pi \in S_d} a_{i_{\pi(1)}, \dots, i_{\pi(j)}}^j x_{\pi(1)} \cdots x_{\pi(j)}$$

and S_d denotes the set of all permutations of the set $\{1, \dots, d\}$ (for $j = 0$ and every $\pi \in S_d$ we set $a_{\pi(1), \dots, \pi(j)}^0 = a_{\emptyset}^0$).

Note that for every $\pi \in S_d$, $h_{i_{\pi(1)} \dots i_{\pi(d)}}(x_{\pi(1)}, \dots, x_{\pi(d)}) = h_{i_1, \dots, i_d}(x_1, \dots, x_d)$. Therefore by general decoupling inequalities for U -statistics (see [6] or Theorem 3.1.1. in [7]), we have

$$L_d \left\| \sum_{j=0}^d \sum_{i_1, \dots, i_j=1}^n a_{i_1, \dots, i_j}^j X_{i_1} \cdots X_{i_j} \right\|_p \geq \left\| \sum_{\substack{1 \leq i_1, \dots, i_d \leq n \\ \text{pairwise distinct}}} H_{i_1, \dots, i_d}(X_{i_1}^1, \dots, X_{i_d}^d) \right\|_p.$$

The right hand side of the above inequality is equal to

$$\left\| \sum_{j=0}^d \frac{1}{\binom{d}{j}} \sum_{1 \leq r_1 < \dots < r_j \leq d} \sum_{i_1, \dots, i_j=1}^n a_{i_1, \dots, i_j}^j X_{i_1}^{r_1} \cdots X_{i_j}^{r_j} \right\|_p$$

(we used the symmetry of the coefficients a_{i_1, \dots, i_j}^j). Since (again by decoupling) for any $1 \leq r_1 < \dots < r_j \leq d$,

$$L_d \left\| \sum_{i_1, \dots, i_j=1}^n a_{i_1, \dots, i_j}^j X_{i_1}^{r_1} \cdots X_{i_j}^{r_j} \right\|_p \geq \left\| \sum_{i_1, \dots, i_j=1}^n a_{i_1, \dots, i_j}^j X_{i_1} \cdots X_{i_j} \right\|_p,$$

to finish the proof it is enough to show that

$$\begin{aligned} L_d \left\| \sum_{j=0}^d \sum_{1 \leq r_1 < \dots < r_j \leq d} \sum_{i_1, \dots, i_j=1}^n b_{i_1, \dots, i_j}^j X_{i_1}^{r_1} \cdots X_{i_j}^{r_j} \right\|_p & \quad (43) \\ & \geq \sum_{j=0}^d \sum_{1 \leq r_1 < \dots < r_j \leq d} \left\| \sum_{i_1, \dots, i_j=1}^n b_{i_1, \dots, i_j}^j X_{i_1}^{r_1} \cdots X_{i_j}^{r_j} \right\|_p \end{aligned}$$

for any coefficients b_{i_1, \dots, i_j}^j .

We will proceed by induction on d . For $d = 0$, (43) reads as $L_0|b_\emptyset^0| \geq |b_\emptyset^0|$, which is obviously true. Let us thus assume that (43) holds for all numbers smaller than d . Consider any $k \in \{1, \dots, d\}$. By the Fubini theorem, Jensen's inequality (applied to the integration with respect to $(X_i^k)_i$) and the assumption that X_i^k has mean zero, we get

$$\begin{aligned} & \left\| \sum_{j=0}^d \sum_{1 \leq r_1 < \dots < r_j \leq d} \sum_{i_1, \dots, i_j=1}^n b_{i_1, \dots, i_j}^j X_{i_1}^{r_1} \dots X_{i_j}^{r_j} \right\|_p \\ & \geq \left\| \sum_{j=0}^{d-1} \sum_{\substack{1 \leq r_1 < \dots < r_j \leq d \\ r_l \neq k, l=1, \dots, j}} \sum_{i_1, \dots, i_j=1}^n b_{i_1, \dots, i_j}^j X_{i_1}^{r_1} \dots X_{i_j}^{r_j} \right\|_p, \end{aligned}$$

which by the induction assumption is greater than or equal to

$$L_d^{-1} \sum_{j=0}^{d-1} \sum_{\substack{1 \leq r_1 < \dots < r_j \leq d \\ r_l \neq k, l=1, \dots, j}} \left\| \sum_{i_1, \dots, i_j=1}^n b_{i_1, \dots, i_j}^j X_{i_1}^{r_1} \dots X_{i_j}^{r_j} \right\|_p.$$

Thus, since k in the above inequality is arbitrary, we get

$$\begin{aligned} & L_d \left\| \sum_{j=0}^d \sum_{1 \leq r_1 < \dots < r_j \leq d} \sum_{i_1, \dots, i_j=1}^n b_{i_1, \dots, i_j}^j X_{i_1}^{r_1} \dots X_{i_j}^{r_j} \right\|_p \\ & \geq \sum_{j=0}^{d-1} \sum_{1 \leq r_1 < \dots < r_j \leq d} \left\| \sum_{i_1, \dots, i_j=1}^n b_{i_1, \dots, i_j}^j X_{i_1}^{r_1} \dots X_{i_j}^{r_j} \right\|_p. \end{aligned}$$

To finish the proof of (43) it is now enough to notice that for any norm $\|\cdot\|$, vectors x, y and number $K > 0$, $\|x\| \leq K\|x + y\|$ implies that $\|x\| + \|y\| \leq (2K + 1)\|x + y\|$. This ends the proof of the proposition. \square

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