Numerical investigation of movable singularities
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Radosław A. Kycia
rkycia@mimuw.edu.pl

The Faculty of Mathematics, Informatics and Mechanics
The Univeristy of Warsaw
Poland

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Outline

Singularities of ODEs (Ordinary Differential Equations)

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Nonlinear ODEs posses two types of singularities:

- fixed - singularities of the coefficients of ODE
- movable - the singularities of solutions; position depends on initial data; not present in linear ODEs;

Example [Goriely]

The equation

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The Painlevé property/Painlevé test

- Cauchy approach - local existence
- Painlevé approach - global existence, finite form and single valuedness
- Solutions can be globally defined only when we know how to define its Riemann surface, i.e., the only movable singularities are poles.
- Deduce global structure of solution (types of singularities) from the local behaviour around some points in the complex plane. Only sufficient conditions $\rightarrow$ by the contraposition - it gives a result when it fails.
- In application, global existence is sometimes important.

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Problem Statement

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- ODE as a system of first order DE

$$\frac{d\vec{y}(x)}{dx} = \vec{f}(\vec{y}; x), \quad \vec{y}(x) : x \in \mathbb{C} \to \mathbb{C}^n. \tag{1}$$

- Initial value $\vec{y}(x_0) = \vec{y}_0$.
- Path, e.g., $(t \in \mathbb{R}^+)$
  - Semiline $x(t) = x_0 + (t + shift) \cdot e^{i\phi}$
  - Spiral $x(t) = (x_0 + (at + b)e^{i \cdot dir \cdot t})e^{i\phi}$
- Domain - path connected region (ideally connected by paths along which integration is performed).
- Condition for singularity proximity - the crude estimation $||\vec{y}|| < \mathrm{Large\ \ const}$. Not the state of art, but it can be improved.

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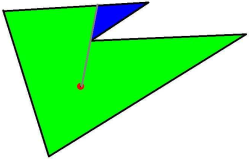

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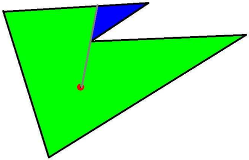
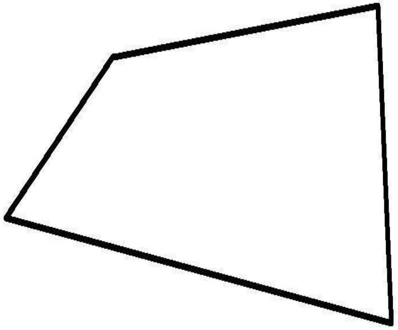
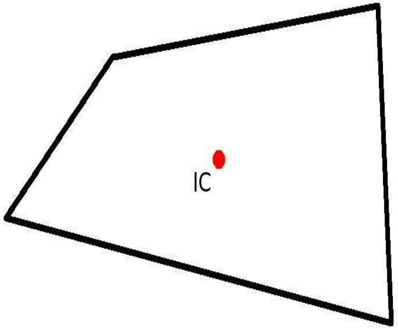


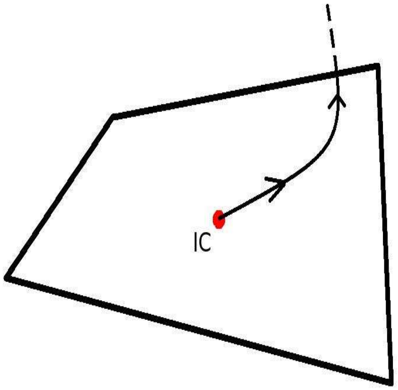
Domain



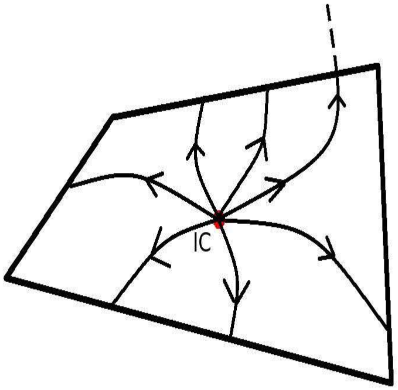
Initial Conditions



Integration along path



Full integration



Implementation

$$\frac{d\vec{y}(x)}{dx} = \vec{f}(\vec{y};x), \quad \vec{y}(x) : x \in \mathbb{C} \to \mathbb{C}^n. \tag{2}$$

If path if prescribed by C^1 curve then the integration can be viewed as an integration on $\mathbb{R}^+$ of the **pullback** of the equation on the path

$$\begin{cases} \frac{d\vec{y}}{dt} = p'(t)\vec{f}(x(t), \vec{y}(t)) \\ x'(t) = p'(t), \end{cases} \tag{3}$$

where $' = \frac{d}{dt}$.

This system of ODEs can be integrated using standard Mathematica algorithms.

In the monitor function the conjunction of x being in the domain **and** proximity of a singularity have to be checked.

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C++

- Implemented using Object Oriented approach...
- ...however, can be used in a functional way.
- Mapper class - maps a path (using ODE, IC, Domain constraints, numerical solver) onto the solution along the path. Some resemblance to the higher-order functions.
- Easy to parallelize as the integrations along curves are independent - parallel producer-consumer problem.

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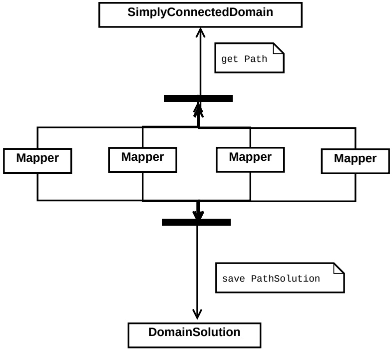
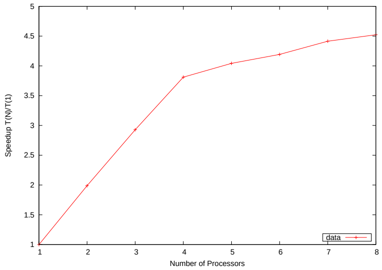
C++ (parallel)





The Amdahl's law is preserved at the beginning up to 4 threads - parallel executed code is about 90%. Then processes start to block each other. New way of parallelization needed.

Examples

Equations [Kycia, Filipuk]

The Emden-Fowler equation

$$\frac{d^2u(x)}{dx^2} + \frac{\alpha}{x}\frac{du(x)}{dx} + x^n u(x)^p = 0 \qquad (4)$$

Generalized Isothermal Sphere equation

$$\frac{d^2u(x)}{dx^2} + \frac{\alpha}{x}\frac{du(x)}{dx} - x^n e^{-u(x)} = 0, \qquad (5)$$

$u(0) = 0$

Location of singularities

A nonzero analytic solutions of the Generlized Emden-Fowler and Isothermal Sphere equations have $n+2$ singularities located symmetrically with respect to the origin on the rays connecting the origin with all $(n+2)$ roots of -1 in the complex plane.

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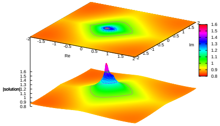
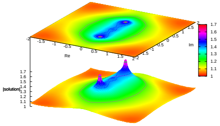
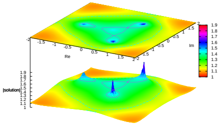


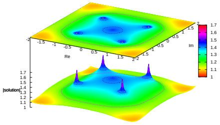
(a) $n = -1$



(b) $n = 0$



(c) $n = 1$



(d) $n = 2$

Figure : $p = 5$ and $u(0) = 1.5$, the Generalized Emden-Fowler solution.

Generalized isothermal sphere equations

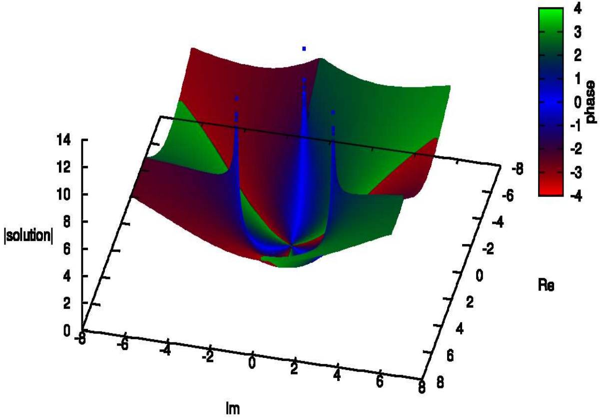


Figure : $u(0) = 0$, $n = 1$

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- C++ code is good for robust (HPC) computations and it is ready for linking with Mathematica by MathLink to provide 'user-friendly' interface.
- The code is flexible - it can be extended by new methods of integration, types of domains, curves (more effective swapping of domain), methods for determining proximity of singularities, etc.

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Bibliography

- R. Conte (editor) **The Painlevé property. One century later** CRM Series in Mathematical Physics, Springer 1999
- A. Goriely, **Integrability and Nonintegrability of Dynamical Systems** , World Scientific (2001)
- C. Hunter, **Series solutions for polytropes and the isothermal sphere**, Mon. Not. R. Astron. Soc. 328 (2001), 839–847.
- R. Kycia, G. Filipuk, **On the singularities of the Emden-Fowler type equations**, Proceedings of ISAAC Conference (2014)
- My homepage `http://www.mimuw.edu.pl/~rkycia/`

Thank You for Your Attention

Backup

Amdahl's law

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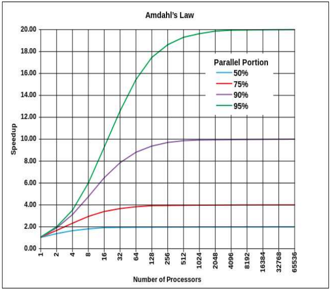
$$S(n) = \frac{T(n)}{T(1)} = \frac{1}{B + \frac{1}{n}(1 - B)}$$

n - no. threads of execution;
$B \in [0;1]$ - the fraction of the algorithm that is strictly serial;
$T(n)$ - time of execution of n threads;



see, $http://en.wikipedia.org/wiki/Amdahl\%27s\_law$