Sections and projections of ℓ_p^n balls

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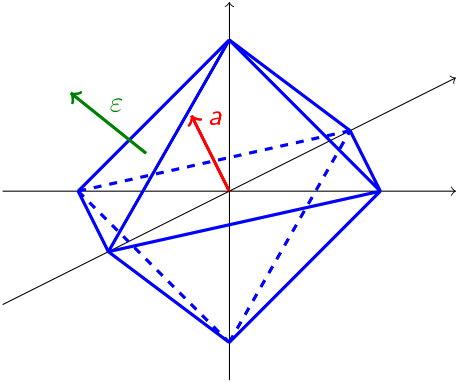
$$H_1 = (1, 0, \ldots, 0)^\perp, \quad H_2 = (1, 1, 0 \ldots, 0)^\perp, \qquad H_n = (1, 1, \ldots, 1)^\perp$$

Projections of B_1^n: if $|a| = 1$ then the contribution from the face with normal vector $\varepsilon \in \{-1, 1\}^n$ is proportional to $|\langle \varepsilon, a \rangle|$.



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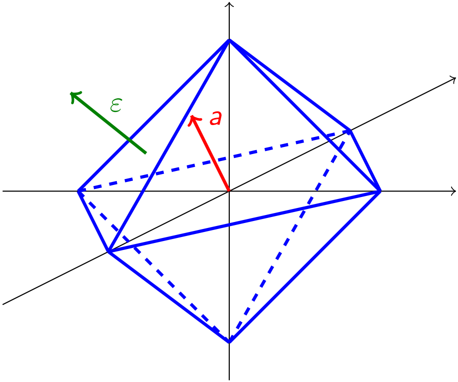

$$|\mathrm{Proj}_{a^\perp}(B_1^n)| = C_n \sum_{\varepsilon \in \{-1,1\}^n} |\langle \varepsilon, a \rangle| = C_n' \, \mathbb{E} \left| \sum_{i=1}^n a_i \varepsilon_i \right|$$

In general the following formula due to Barthe and Naor is true

$$|\mathrm{Proj}_{a^\perp}(B_p^n)| = C_{p,n} \, \mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|, \qquad X_i \sim c_p |t|^{\frac{2-p}{p-1}} e^{-|t|^{\frac{p}{p-1}}}$$

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Therefore we want to find best constants in the following $L_1 - L_2$ Khinchine type inequality:

$$A_{n,p}\left(\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|^2\right)^{1/2} \leq \mathbb{E}\left|\sum_{i=1}^n a_i X_i\right| \leq B_{n,p}\left(\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|^2\right)^{1/2}$$

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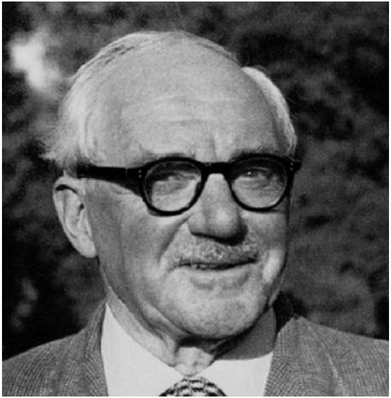
We clearly have $B_{n,1} = 1$ and thus maximal projections of B_1^n are given by H_1.



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- **2002**: Barthe and Naor show that
  - H_1 is the maximizer for $p \in (1, 2)$
  - H_1 is the minimizer for $p > 2$
  - H_n is the maximizer for $p > 2$
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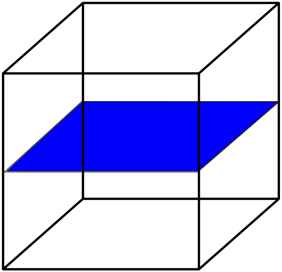

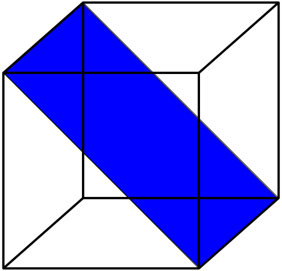

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Dirac delta approximation: if $V \sim f$ then

$$f(0) = \lim_{q \to 1^-} \frac{1-q}{2} \int |s|^{-q} f(s) \mathrm{d}s = \lim_{q \to 1^-} \frac{1-q}{2} \mathbb{E}|V|^{-q}.$$

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Goal: get rid of the limit!

Schechtman & Zinn, Rachev & Rüchendorf: if $Y_1, \ldots, Y_n$ are i.i.d. with densities $c_p e^{-|t|^p}$ and $S = \left(\sum_{i=1}^n |Y_i|^p\right)^{1/p}$ then

$$\frac{Y}{S} \text{ and } S \text{ are independent} \qquad Y = (Y_1, \ldots, Y_n)$$

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$$\mathbb{E}|\langle X, a\rangle|^{-q} = \mathbb{E}U^{-\frac{q}{n}}\mathbb{E}|\langle Y/S, a\rangle|^{-q} = \mathbb{E}U^{-\frac{q}{n}} \cdot \frac{\mathbb{E}S^{-q}}{\mathbb{E}S^{-q}}\mathbb{E}|\langle Y/S, a\rangle|^{-q}$$
$$= \frac{\mathbb{E}U^{-\frac{q}{n}}}{\mathbb{E}S^{-q}}\mathbb{E}|\langle Y, a\rangle|^{-q} = c_{p,q,n}\mathbb{E}|\langle Y, a\rangle|^{-q}$$

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$$= 2c_{p,1,n}\phi(0)\ \mathbb{E}\left|\sum_{i=1}^n a_i^2 R_i^2\right|^{-\frac{1}{2}} \quad \Longrightarrow \quad \begin{array}{l} H_1 - \text{maximum} \\ H_n - \text{minimum} \end{array}$$

What about $p > 2$? We can always write $Y_i \sim R_i U_i$, where $U_i \sim Unif([-1,1])$ and $R_i \sim c_p x^p e^{-x^p}$. Thus

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Archimedes-König-Kwapień formula

$$(1-q)\mathbb{E}\Big|\sum_{i=1}^n x_i U_i\Big|^{-q} = \mathbb{E}\Big|\sum_{i=1}^n x_i \xi_i\Big|^{-q}, \qquad \xi_i \sim Unif(S^2)$$

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Since $\langle \xi_i, \theta \rangle \sim U_i$ and $\mathbb{E}_\theta |\langle v, \theta \rangle|^{-q} = \frac{1}{1-q}|v|^{-q}$ for $v \in \mathbb{R}^3$, we get

$$\frac{1}{1-q}\mathbb{E}\Big|\sum_{i=1}^n x_i \xi_i\Big|^{-q} = \mathbb{E}_{\xi,\theta}\mathbb{E}\Big|\Big\langle \sum_{i=1}^n x_i \xi_i, \theta \Big\rangle\Big|^{-q} = \mathbb{E}\Big|\sum_{i=1}^n x_i U_i\Big|^{-q}$$

We can now evaluate the limit and get the formula

$$A_{n,p}(a) := \frac{|B_p^n \cap a^\perp|}{|B_p^{n-1}|} = \Gamma\left(1 + \frac{1}{p}\right) \mathbb{E}\left|\sum_{i=1}^n a_i R_i \xi_i\right|^{-1}$$

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Proof of Hadwiger-Hensley:

$$\frac{|B_\infty^n \cap a^\perp|}{|B_\infty^{n-1}|} = \mathbb{E}\left|\sum_{i=1}^n a_i \xi_i\right|^{-1} = \mathbb{E}\left(\sum_{i,j=1}^n a_i a_j \langle \xi_i, \xi_j \rangle\right)^{-\frac{1}{2}}$$

$$\geq \left(\sum_{i,j=1}^n a_i a_j \mathbb{E}\langle \xi_i, \xi_j \rangle\right)^{-\frac{1}{2}} = \left(\sum_{i=1}^n a_i^2\right)^{-\frac{1}{2}} = 1.$$

Maximizers for $p > 10^{15}$

Theorem (Chasapis, N., Tkocz, 2022)

For a unit vector a one has

$$\mathbb{E}\left|\sum_{i=1}^{n} a_i \xi_i\right|^{-1} \leq \sqrt{2} - \kappa \left| a - \frac{e_1 + e_2}{\sqrt{2}} \right| \qquad \kappa = 6 \cdot 10^{-5}$$

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Goal: $A_{n,p} := \frac{|B_p^n \cap a^\perp|}{|B_p^{n-1}|} \leq 2^{\frac{1}{2}-\frac{1}{p}}$. Know: $A_{n,\infty} \leq \sqrt{2} - \kappa\left|a - \frac{e_1+e_2}{\sqrt{2}}\right|$.

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Busemann theorem:

$$N(a) := A_{n,\infty}^{-1} = \left(\mathbb{E}\left|\sum_{i=1}^{n} a_i\xi_i\right|^{-1}\right)^{-1} = \frac{|a|}{\left|[-\frac{1}{2},\frac{1}{2}]^n \cap a^\perp\right|} \quad \text{is a norm}$$

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$$|N(a)^{-1} - N(b)^{-1}| = \left|\frac{N(a)-N(b)}{N(a)N(b)}\right| \leq \frac{N(a-b)}{N(a)N(b)} \leq 2\frac{|a-b|}{|a|\cdot|b|}$$

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Proof: We have $A_{n,\infty} = N(a)^{-1}$ and $A_{n,p} = \Gamma(1+\frac{1}{p})\mathbb{E}N(aR)^{-1}$

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Proof:

$$A_{n,p} \leq A_{n,\infty} + \frac{5}{p} \leq \sqrt{2} - \frac{10^5\kappa}{p} + \frac{5}{p} \leq \sqrt{2}\left(1 - \frac{\log 2}{p}\right) \leq 2^{\frac{1}{2}-\frac{1}{p}},$$

where in the last line we use $1 - x \leq e^{-x}$.

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$$\mathbb{E}\left|\sum_{i=1}^{n} a_i R_i \xi_i\right|^{-1} = \mathbb{E}|X+Y|^{-1} = \mathbb{E}\min(|X|^{-1}, |Y|^{-1})$$

$$\leq \mathbb{E}\min(|X|^{-1}, \mathbb{E}|Y|^{-1}) \leq \mathbb{E}\min\left(|X|^{-1}, \frac{C_p}{\sqrt{1-a_1^2-a_2^2}}\right) \leq C_p,$$

where the last line is a delicate 3 page argument.

Thank you!