Norms of structured random matrices

Michał Strzelecki

Based on joint work
with R. Adamczak, J. Prochno, and M. Strzelecka.

Będlewo, June 12, 2023

- $1 \le p, q \le \infty$,
- a deterministic real $m \times n$ matrix $A = (a_{ij})_{i \le m, j \le n}$,
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$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \ge \mathbb{E} \max_j \| G_A e_j \|_q = \mathbb{E} \max_j \| (a_{ij} g_{ij})_i \|_q$$

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where (a^*_{ij}) *is obtained by permuting the rows of* A *so that* $\max_j |a^*_{1j}| \geq \ldots \geq \max_j |a^*_{nj}|$.

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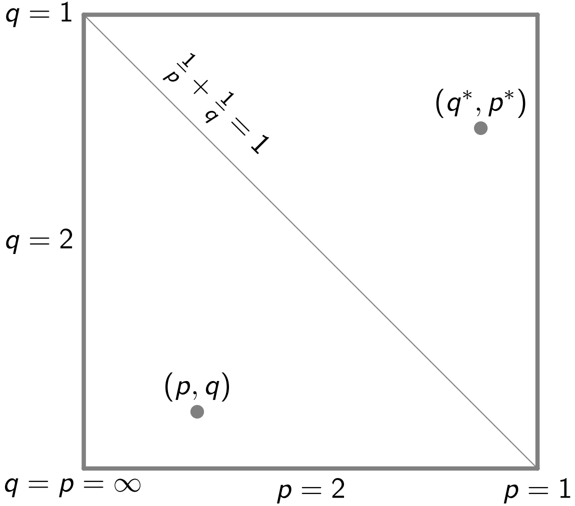

$$\|G_A \colon \ell_p^n \to \ell_q^m\| = \|(G_A)^T \colon \ell_{q^*}^m \to \ell_{p^*}^n\| = \|(G^T)_{A^T} \colon \ell_{q^*}^m \to \ell_{p^*}^n\|$$

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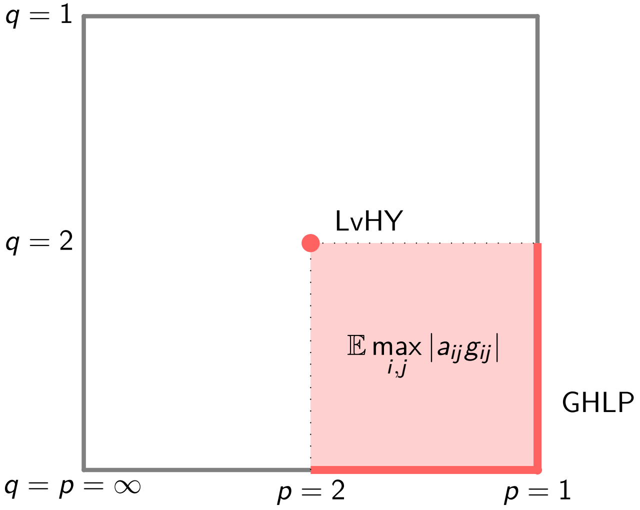

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Proof. $g_{ij} \sim \varepsilon_{ij}|g_{ij}|$ + the contraction principle:

$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \gtrsim \mathbb{E}\|(a_{ij}\varepsilon_{ij}) \colon \ell_p^n \to \ell_q^m\|$$

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$$\begin{aligned}
\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| &\gtrsim \mathbb{E}\|(a_{ij}\varepsilon_{ij})\colon \ell_p^n \to \ell_q^m\| \\
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&= \sup_{z\in B_{p/2}^n}\Big\|\Big(\sum_{j=1}^n a_{ij}^2 z_j\Big)_{i=1}^m\Big\|_{q/2}^{1/2} = \|A\circ A\colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2}.
\end{aligned}$$

Lower bounds:

$$\mathbb{E}\max_j \|(a_{ij}g_{ij})_i\|_q + \mathbb{E}\max_i \|(a_{ij}g_{ij})_j\|_{p^*} \quad \textbf{vs.} \quad \|A\circ A\colon \ell^n_{p/2}\to \ell^m_{q/2}\|^{1/2} + \|(A\circ A)^T\colon \ell^m_{q^*/2}\to \ell^n_{p^*/2}\|^{1/2}$$

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Case $q < p$

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On the other hand,

$$\|\,\mathrm{Id}\colon B^n_{p/2}\to\ell^m_{q/2}\|^{1/2} = \sup_{x\in B^n_{p/2}}\Big(\sum_{j\le n}|x_j|^{q/2}\Big)^{1/q}$$

$$= \Big(\sup_{y\in B^n_{p/q}}\sum_{j\le n}|y_j|\Big)^{1/q} = \big(n^{1/(p/q)^*}\big)^{1/q} \gg \sqrt{\ln n}.$$

Theorem (Latała–van Handel–Youssef, 2018)

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Conjecture (APSS, 2023)

For all $1 \le p, q \le \infty$,

$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \overset{?}{\asymp}_{p,q} \|A \circ A\colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2} + \|(A \circ A)^T\colon \ell_{q^*/2}^m \to \ell_{p^*/2}^n\|^{1/2} + \text{[see fig.]}$$

$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} D_1 + D_2 + \dots$$

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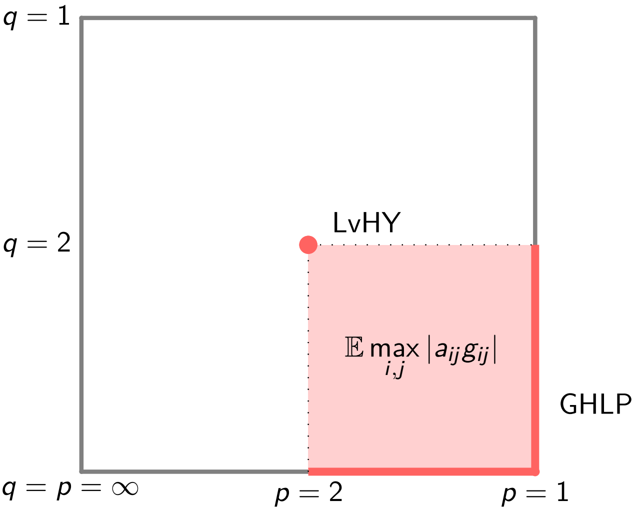

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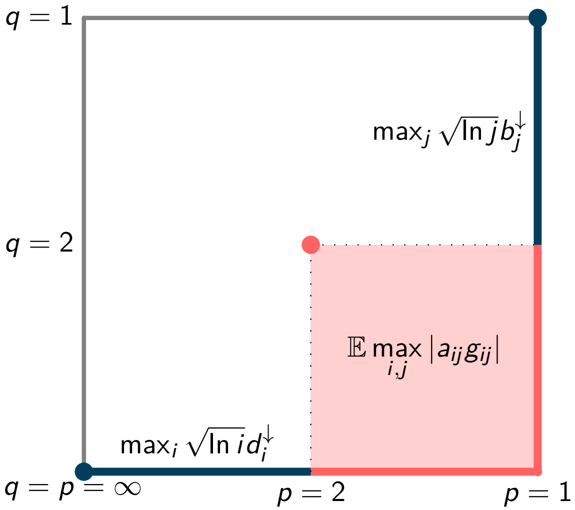

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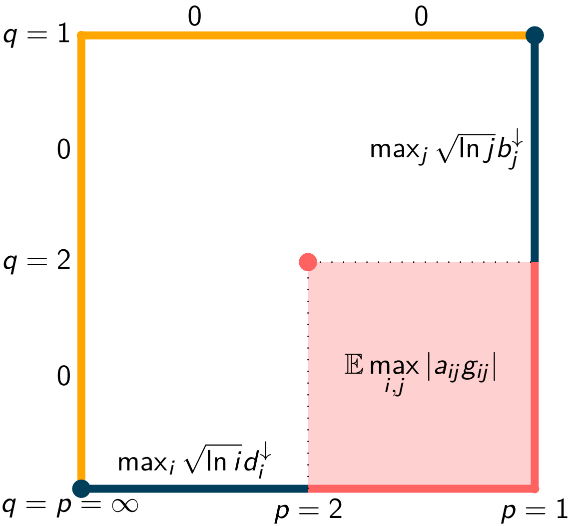

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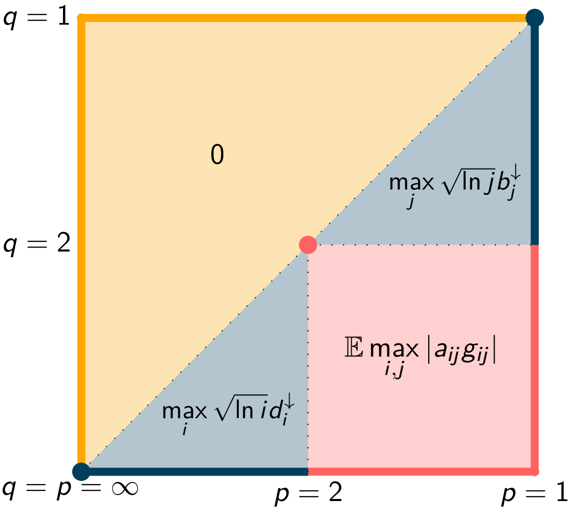

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Conjecture (APSS, 2023)

$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \asymp_{p,q} D_1 + D_2 + \begin{cases} \mathbb{E} \max_{i\le m, j\le n} |a_{ij} g_{ij}| & \text{if } p \le 2 \le q, \\ \max_{j\le n} \sqrt{\ln(j+1)} b_j^{\downarrow} & \text{if } p \le q \le 2, \\ \max_{i\le m} \sqrt{\ln(i+1)} d_i^{\downarrow} & \text{if } 2 \le p \le q, \\ 0 & \text{if } q < p. \end{cases}$$

Theorem (APSS, 2023)

$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \lesssim (\ln n)^{1/p^*} (\ln m)^{1/q} \Big[\sqrt{\ln(mn)} D_1 + \sqrt{\ln n} D_2 \Big].$$

$$D_1 = \| A \circ A \colon \ell_{p/2}^n \to \ell_{q/2}^m \|^{1/2} \qquad b_j = \| (a_{ij})_{i\le m} \|_{2q/(2-q)}$$

$$D_2 = \| (A \circ A)^T \colon \ell_{q^*/2}^m \to \ell_{p^*/2}^n \|^{1/2} \qquad d_i = \| (a_{ij})_{j\le n} \|_{2p/(p-2)}$$

Lower bounds: $\|A \circ A \colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2} + \dots$

vs. $\mathbb{E} \max_j \|(a_{ij} g_{ij})_i\|_q + \mathbb{E} \max_i \|(a_{ij} g_{ij})_j\|_{p^*}$

Case $2 < p \le q$

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$$D_1 = \|(1)_{ij} \colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2} = \sup_{x \in B_{p/2}^n} \Big(\sum_{i \leq m} \Big| \sum_{j \leq n} x_j \Big|^{q/2} \Big)^{1/q}$$

$$= m^{1/q} \sup_{x \in B_{p/2}^n} \sqrt{\Big| \sum_{j \leq n} x_j \Big|} = m^{1/q} \sqrt{n^{1/(p/2)^*}} = m^{1/q} n^{\frac{1}{2(p/2)^*}}.$$

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On the other hand,

$$\max_{j \le n} \|(a_{ij})_i\|_q + \max_{i \le m} \|(a_{ij})_j\|_{p^*} = m^{1/q} + n^{1/p*} \sim m^{1/q} \ll D_1,$$

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Theorem (APSS, 2023)

Assume that $1 \le p, q \le \infty$. *Then,*

$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \lesssim (\ln n)^{1/p^*}(\ln m)^{1/q}\Big[\sqrt{\ln(mn)}D_1 + \sqrt{\ln n}D_2\Big].$$

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Corollary

Assume that $K, L > 0$, $r \in (0,2]$, $1 \le p, q \le \infty$, *and* $X = (X_{ij})_{i \le m, j \le n}$ *has independent mean-zero entries satisfying*

$$\mathbb{P}(|X_{ij}| \ge t) \le Ke^{-t^r/L} \qquad \text{for all } t \ge 0, i \le m, j \le n.$$

Then

$$\mathbb{E}\|X_A \colon \ell_p^n \to \ell_q^m\| \lesssim_{r,K,L} (\ln n)^{1/p^*}(\ln m)^{1/q}\ln(mn)^{\frac{1}{r}-\frac{1}{2}} \Big[\sqrt{\ln(mn)}D_1 + \sqrt{\ln n}D_2\Big].$$

Lemma

Assume that $1 \le p \le \infty$, $n \in \mathbb{N}$, *and define the convex set*

$$L := \operatorname{conv}\Big\{ \frac{1}{|J|^{1/p}} \big(\varepsilon_j \mathbf{1}_{\{j \in J\}}\big)_{j=1}^n : \qquad J \subset \{1, \dots, n\},\ J \neq \emptyset, \\ (\varepsilon_j)_{j=1}^n \in \{-1, 1\}^n \Big\}.$$

Then $B_p^n \subset \ln(en)^{1/p^*} L$.

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Rough sketch of the proof of the Theorem.

$$\| G_A \colon \ell_p^n \to \ell_q^m \| = \sup_{x\in B_p^n} \sup_{y \in B_{q^*}^m} \sum_{i=1}^m \sum_{j=1}^n y_i a_{ij} g_{ij} x_j$$

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$$=_{\text{logs}} \mathbb{E} \max_{\substack{k \le m \\ l \le n}} \frac{1}{k^{1/q^*} l^{1/p}} \max_{\substack{I \subset [m] \\ |I| = k}} \max_{\substack{J \subset [n] \\ |J| = l}} \sup_{x \in B_\infty^n} \sup_{y \in B_\infty^m} \sum_{i \in I} \sum_{j \in J} y_i a_{ij} g_{ij} x_j$$

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Using symmetrization and the contraction principle one proves:

$$\mathbb{E} \sup_{I,J} \sup_{x \in B_\infty^n} \sup_{y \in B_\infty^m} \sum_{i \in I, j \in J} y_i a_{ij} g_{ij} x_j$$

$$\leq \sqrt{\frac{2}{\pi}} \sup_{I,J} \sum_{i \in I} \sqrt{\sum_{j \in J} a_{ij}^2} + 2\mathbb{E} \sup_{I,J} \sup_{x \in B_\infty^n} \sum_{j \in J} \sum_{i \in I} a_{ij} g_{ij} x_j .$$

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Use Slepian's lemma and compare $\mathbb{E} \sup_{I,J} X_{I,J} \le \mathbb{E} \sup_{I,J} Y_{I,J}$, where $Y_{I,J} = \sum_{i \in I} g_i \sqrt{\sum_{j \in J} a_{ij}^2} + \sum_{j \in J} \widetilde{g}_j \sqrt{\sum_{i \in I} a_{ij}^2}$.

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