The Random Geometric Graph

Bedlewo, June 2023

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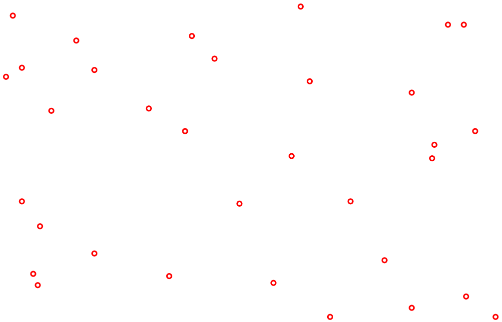
UNIVERSITÄT OSNABRÜCK

Gilbert (or Random Geometric or Distance) Graph

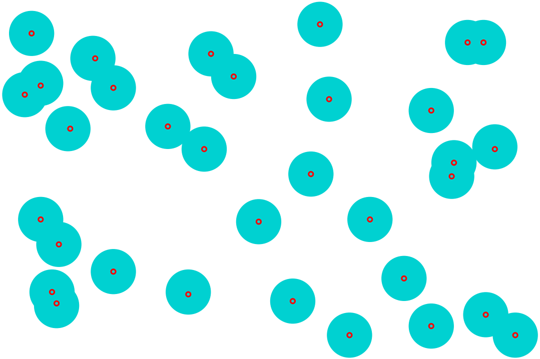
…in $\mathbb{R}^d$



η_t stationary Poisson point process

Gilbert (or Random Geometric or Distance) Graph

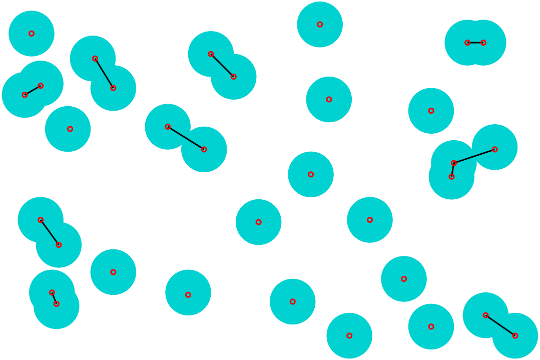
…in $\mathbb{R}^d$



$$\eta_t + B_d(0, \tfrac{1}{2}\delta_t)$$

Gilbert (or Random Geometric or Distance) Graph

...in $\mathbb{R}^d$



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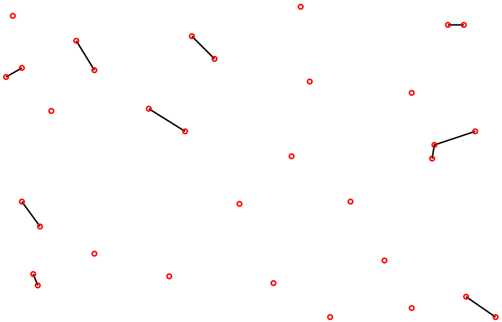
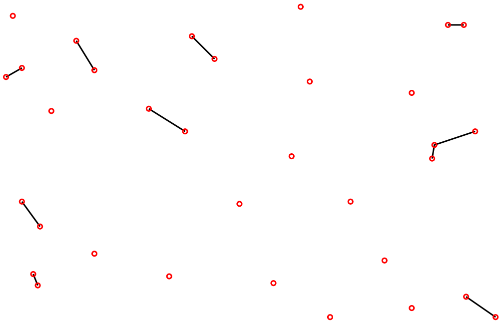


Graph $\mathcal{G}(\eta_t, \delta_t)$

Gilbert (or Random Geometric or Distance) Graph

... in $\mathbb{R}^d$



Graph $\mathcal{G}(\eta_t, \delta_t)$

internet, social networks, wireless connections, smart home,
Erdös-Renyi vs. Gilbert

Expectations

random geometric graph:
$\mathcal{G}(\eta_t, \delta_t) = (\mathcal{F}_0, \mathcal{F}_1) = \big(\eta_t, \{(x, y) \in (\eta \cap W)^2_{t,\neq} \colon \|x - y\| \leq \delta_t\}\big)$

- number of vertices:
  $$f_0 = \infty$$
- number of edges:
  $$f_1 = \infty$$

Expectations

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- number of vertices:
$$f_0 = \quad |\mathcal{F}_0| = \quad |\eta_t \cap W|$$
- number of edges:
$$f_1 = \quad |\mathcal{F}_1| = \quad \tfrac{1}{2} \sum_{(x_1,x_2)\in(\eta_t \cap W)^2_{\neq}} \mathbb{1}(\|x_1 - x_2\| \leq \delta_t)$$

Expectations

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- number of vertices:
$$\mathbb{E} f_0 = \mathbb{E}|\mathcal{F}_0| = \mathbb{E}|\eta_t \cap W| = tV(W)$$
- number of edges:
$$\mathbb{E} f_1 = \mathbb{E}|\mathcal{F}_1| = \mathbb{E}\tfrac{1}{2} \sum_{(x_1,x_2) \in (\eta_t \cap W)^2_{\neq}} \mathbb{1}(\|x_1 - x_2\| \leq \delta_t)$$
$$= \frac{t^2}{2} \int_W \int_W \mathbb{1}(\|x_1 - x_2\| \leq \delta_t)\, dx_2\, dx_1$$

. . . Slivnyak-Mecke formula

Expectations

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$$= \frac{t^2}{2} \int_W \underbrace{\int_W \mathbb{1}(\|x_1 - x_2\| \leq \delta_t)\, dx_2}_{\approx \kappa_d \delta_t^d}\, dx_1$$

…Slivnyak-Mecke formula

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$$= \frac{t^2}{2} V(W)\, \kappa_d\, \delta_t^d + \dots$$

…Slivnyak-Mecke formula

Topology of components in $\mathbb{R}^d$:

$\mathcal{S}$ ... a realizable finite subgraph (isomorphic version):

$$|\mathcal{S} \cap \mathcal{G}| > 0?$$

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Theorem

With probability one, $|\mathcal{S} \cap \mathcal{G}| = \infty$.

. . . Grygierek, Juhnke-K., R., Römer, Röndigs

Topology of components in $\mathbb{R}^d$: The lonely complex . . .

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Percolation

component of $\eta_t \cup \{0\}$ containing 0

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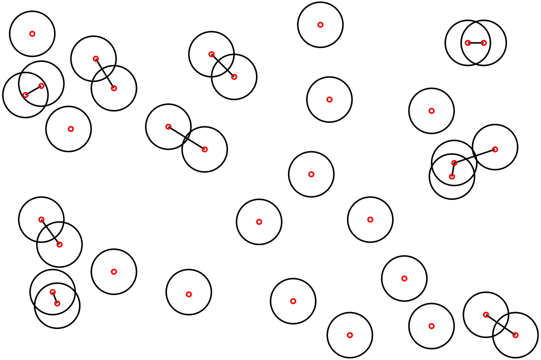

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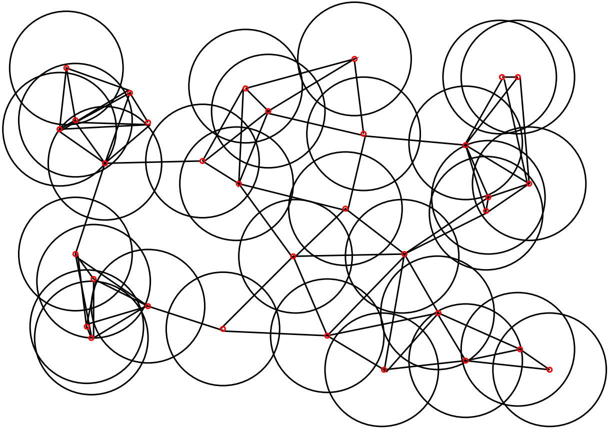

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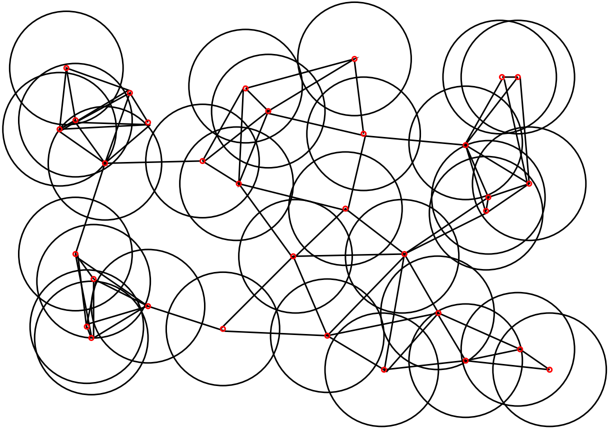
$$v_c = \inf\{\kappa_d t \delta_t^d > 0,\ p_\infty(t) > 0\}$$



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Theorem (Meester and Roy)

Suppose $\kappa_d t \delta_t^d > v_c$. Then with probability one there is precisely one unbounded connected component.

Topology of components in $\mathbb{R}^d$: The lonely complex . . .

$\mathcal{S}$ . . . a realizable finite complex (isomorphic version):

$|\mathcal{S} \cap \mathcal{G}| > 0$?

Theorem

With probability one, $|\mathcal{S} \cap \mathcal{G}| = \infty$.

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. . . and the giant beast

$\mathcal{S}$ . . . a realizable finite complex (isomorphic version):

Theorem

If percolation occurs then with probability one there are infinitely many components of $\mathcal{S}$ linked to the unbounded component via a single edge.

. . . Grygierek, Juhnke-K., R., Römer, Röndigs

Random geometric graph

$$\mathcal{G}(\eta_t, \delta_t) = (\mathcal{F}_0, \mathcal{F}_1) = \big(\eta_t, \{(x, y) \in \eta^2_{t,\neq} : \ \|x - y\| \le \delta_t\}\big)$$

length power functional:

$$L^{(\tau)} = \frac{1}{2} \sum_{(x_1, x_2) \in (\eta_t \cap W)^2_{\neq}} \|x_1 - x_2\|^\tau \mathbb{1}(\|x_1 - x_2\| \le \delta_t)$$

($\tau = 0$ number of edges)

Poisson U-statistic

$$\mathcal{G}(\eta_t, \delta_t) = (\mathcal{F}_0, \mathcal{F}_1) = \big(\eta_t, \{(x, y) \in \eta^2_{t,\neq} : \ \|x - y\| \leq \delta_t\}\big)$$

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Definition: U-statistic

$\eta = \{Z_1, \ldots, Z_n\}$:

$$F(\eta) = \sum_{\eta^k_{\neq}} f(x_1, \ldots, x_k)$$

Poisson U-statistic

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Definition: Poisson U-statistic

η Poisson point process, $f \in L_1(\Omega)$:

$$F(\eta) = \sum_{\eta^k_{\neq}} f(x_1, \dots, x_k)$$

Poisson U-statistic

Malliavin calculus for functions $F(\eta)$ of Poisson point processes:

Wiener-Itô chaos expansion

$F \in L^2(\mathbb{P})$

$$F(\eta) = \sum_{i=0}^{\infty} I_i(f_i)$$

Poisson U-statistic

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finite Wiener-Itô chaos expansion

$F \in L^2(\mathbb{P})$

$$F(\eta) = \sum_{i=0}^{k} I_i(f_i)$$

with kernels $f_i \in L^1 \cap L^2(\Omega)$, iff F is a sum of U-statistics.

R., Schulte

Poisson U-statistic

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$$F(\eta) = \sum_{i=0}^{\infty} I_i(f_i)$$

$$\mathbb{V}F = \sum_{1}^{\infty} i!\|f_i\|^2$$

$\|f_1\|_2^2 \leq \mathbb{V}F$

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$$D_x F(\eta) = F(\eta \cup \{x\}) - F(\eta)$$

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Exchange inequality / Poincaré inquality

$$t \int \left(\mathbb{E} D_x F(\eta)\right)^2 dx \leq \mathbb{V} f(X) \leq t \int \mathbb{E}\left(D_x f(\eta)\right)^2 dx$$

Poisson U-statistic

Central limit theorem

F an absolutely convergent U-statistic of order k. Then

$$d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\mathbb{V}F}}, N\right) \leq 2k^4 \frac{M_4(f)}{\mathbb{V}F}$$

with $N \sim \mathcal{N}(0,1)$.

R., Schulte; Peccati, Sole, Taqqu, Utzet

Poisson U-statistic

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$$M_4(f) = \sum \int_\Omega \cdots \int_\Omega |f(\ldots)f(\ldots)f(\ldots)f(\ldots)| \, d\mu^m(x_1,\ldots,x_m)$$

4th moment theorem

Central limit theorem

$F \in L^4(\mathbb{P})$ a U-statistic of order k with $f \geq 0$. Then

$$d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\mathbb{V}F}}, N\right) \leq C\sqrt{\mathbb{E}\left(\frac{F-\mathbb{E}F}{\sqrt{\mathbb{V}F}}\right)^4 - 3}$$

with $N \sim \mathcal{N}(0,1)$.

Lachieze-Rey, Peccati

Poisson U-statistic

. . . and much more: Bourguin, Lachieze-Rey, Last, Peccati, Penrose, Schulte, Thäle, Trauthwein, Yukich, . . .

Central limit theorem

$F \in L^4(\mathbb{P})$ a Poisson functional (U-statistic):

$$d_{W,K}\left(\frac{F - \mathbb{E}F}{\sqrt{\mathbb{V}F}}, N\right) \leq$$

- localizing / stabilizing / geometric
- $\mathbb{E} \int \cdots \int$ 4th mixed moments of $D_x F, D_{x,y} F, D_{y,z} F \dots dxdy \dots$

Poisson U-statistic

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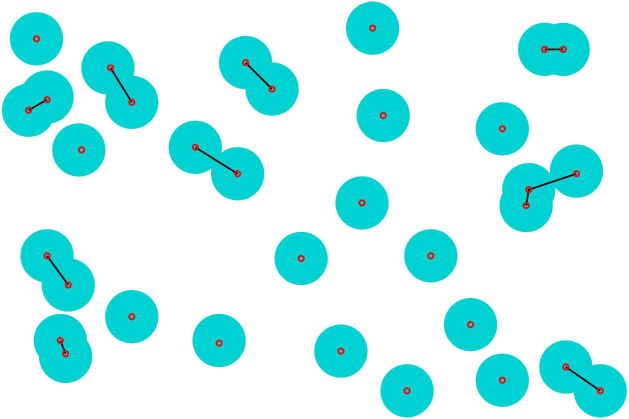
- $\mathbb{E} \int \cdots \int$ 4th mixed moments of $D_x F, D_{x,y} F, D_{y,z} F \dots dxdy \dots$
- $\mathbb{E} \int \cdots \int$ $(2+\epsilon)$-mixed moments of $D_x F, D_{x,y} F \dots dxdy \dots$

Random geometric graph

geometric graph $(\mathcal{F}_0, \mathcal{F}_1) = \big(\eta_t, \{(x,y) \in \eta^2_{t,\neq} : \; \|x - y\| \leq \delta_t\}\big)$



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$$L^{(\tau)} = \frac{1}{2} \sum_{(x_1,x_2)\in(\eta_t\cap W)_{\neq}^2} \|x_1 - x_2\|^{\tau} \mathbb{1}(\|x_1 - x_2\| \leq \delta_t)$$

$$\mathbb{E}L^{(\tau)} = t^2 \frac{1}{2} \int_{W^2} \|x_1 - x_2\|^{\tau} \mathbb{1}(\|x_1 - x_2\| \leq \delta_t) dx_1 dx_2$$

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$$\mathbb{E} L^{(\tau)} \ = \ c t^2 \delta_t^{\tau+d} V(W) + \ldots \qquad \Rightarrow \tau > -d$$

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$$\mathbb{E}L^{(\tau)} = c t^2 \delta_t^{\tau+d} V(W) + \dots \qquad \Rightarrow \tau > -d$$

$$\mathbb{V}L^{(\tau)} = (c_1 t^2 \delta_t^{2\tau+d} + c_2 t^3 \delta_t^{2\tau+2d}) V(W) + \dots \qquad \Rightarrow \tau > -\frac{d}{2}$$

Random geometric graph

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CLT

$$d_{K,W}\left(\frac{L^{(\tau)} - \mathbb{E}L^{(\tau)}}{\sqrt{\mathbb{V}L^{(\tau)}}}, N\right) \leq C\, t^{-\frac{1}{2}} \max\{1, (t\delta_t^d)^{-\frac{1}{2}}\}$$

for $\tau < -\frac{d}{4}$

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Penrose; R., Schulte, Thäle;
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Random geometric graph

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multivariate CLT

$$d_\star\left(\left(\tfrac{L^{(\tau_1)}-\mathbb{E}}{\sigma}, \tfrac{L^{(\tau_2)}-\mathbb{E}}{\sigma}, \dots\right), \mathbf{N}\right) \le C\, t^{-\frac{1}{2}} \max\{1, (t\delta_t^d)^{-\frac{k}{2}}\}$$

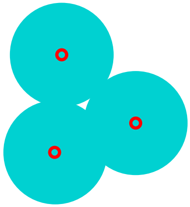
$t\delta_t^d \to \infty :$ $$L_t^{(\tau)} = \frac{d}{\tau + d}\delta_t^\tau L_t^{(0)} + \dots$$

$t\delta_t^d \le C :$ $$L_t^{(\tau)} = \frac{d}{\tau + d}\delta_t^\tau L_t^{(0)} + c\, Z\, \delta_t^\tau \mathbb{V} L_t^{(0)} + \dots$$

Akinwande, R., Römer, Thäle, Schulte, v. Westenholz

Vietoris-Rips complex

$\mathcal{C} = \mathcal{C}_{VR}(\mathcal{G})$ ... Vietoris-Rips complex:
$F = \{x_1, \ldots, x_{k+1}\} \in \mathcal{F}_k(\mathcal{C}_{VR})$ iff $\|x_i, x_j\| \leq \delta_t \forall i, j$



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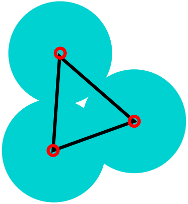

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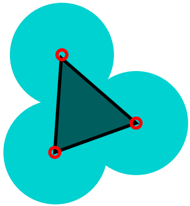

The Cech complex

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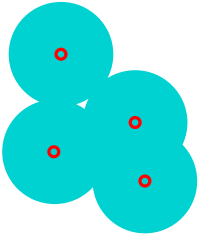

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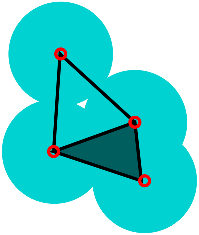

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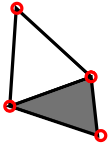

Volume-power of k-dimensional simplices

$V_k^{(\tau)}(\mathcal{C}) = \sum_{S \in \mathcal{F}_k(\mathcal{C})} V_k(S)^\tau$:

$\mathbb{E} V_k^{(\tau)}, \mathbb{V} V_k^{(\tau)}$ ... Slivnyak-Mecke, Poisson U-statistic

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$$d_{K,W}\left(\frac{V_k^{(\tau)} - \mathbb{E} V_k^{(\tau)}}{\sqrt{\mathbb{V} V_k^{(\tau)}}}, N\right) \leq C\, t^{-\frac{1}{2}} \max\{1, (t\delta_t^d)^{-\frac{k}{2}}\}$$

Penrose, R., Schulte, Akinwande

Subcomplex counts

$\mathcal{S}$: set of subcomplexes, $n(\mathcal{C}) = |\mathcal{S} \cap \mathcal{C}(W)|$:

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$$d_W\left(\frac{n - \mathbb{E}n}{\sqrt{\mathbb{V}n}}, N\right) \leq C\, t^{-\frac{1}{2}}$$

Penrose, Lachieze-Rey, Peccati, ...

Betti numbers

$\beta_k(\mathcal{C})$: non-local

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$$d_W\left(\frac{\beta_k - \mathbb{E}\beta_k}{\sqrt{\mathbb{V}\beta_k}}, N\right) \to 0$$

Penrose, Kahle, Meckes, Bobrowski, Skraba, Adler, Yogeshwaran,

Concentration

...for first order U-statistics with $f \geq 0$ ...

$$\mathbb{P}(F - \mathbb{E}F \geq u) \leq e^{-\frac{\mathbb{E}F}{\|f\|_\infty} g\left(\frac{u}{\mathbb{E}F}\right)}$$

$$\text{with } g(u) = (1+u)\ln(1+u) - u, \;\; u \geq 0.$$

Houdre and Privault, Ane and Ledoux
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...for higher order U-statistics with $f \geq 0$?

Concentration

Isop. inequ. for Talagrand's convex distance on PPP

F a nice local U-statistic with $f \geq 0$. Then for all $r \geq 0$,

$$\mathbb{P}(F > \mathbb{M}F + r) \leq 2e^{-\frac{r^2}{4k^2 c_d (r+\mathbb{M}F)^{2-1/k}}}$$
$$\mathbb{P}(F < \mathbb{M}F - r) \leq 2e^{-\frac{r^2}{4k^2 c_d (\mathbb{M}F)^{2-1/k}}}$$

...Lachieze-Rey, R.

Concentration

Log-Sobolev inequality, Herbst argument, ...

F a nice local U-statistic with $f \geq 0$. Then for all $r \geq 0$

$$\mathbb{P}(F \geq \mathbb{E}F + r) \leq e^{-\frac{((\mathbb{E}F+r)^{1/(2k)} - (\mathbb{E}F)^{1/(2k)})^2}{2k^2 c_d}}$$

$$\mathbb{P}(F \leq \mathbb{E}F - r) \leq e^{-\frac{r^2}{2k\mathbb{V}F}}$$

...Bachmann, Peccati

Concentration inequalities

in particular for f_k: for all $r \geq 0$,

$$\mathbb{P}(f_k \geq \mathbb{E}f_k + r) \leq e^{-\frac{((\mathbb{E}f_k+r)^{1/(2k)}-(\mathbb{E}f_k)^{1/(2k)})^2}{2k^2 c_d}}$$

$$\mathbb{P}(f_k \leq \mathbb{E}f_k - r) \leq e^{-\frac{r^2}{2k\mathbb{V}f_k}}$$

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...Bachmann, Peccati, R.

...Betti numbers?

The high-dimensional Gilbert graph

CLT

$W = B^d,\ \delta_d = \frac{1}{d}, \mathbb{E}f_1 \to \infty$:

$$d_W\left(\frac{f_1 - \mathbb{E}f_1}{\sqrt{\mathbb{V}f_1}}, N\right) \to 0 \qquad \text{as } d \to \infty$$

Grygierek, Thäle

CLT for $\|\cdot\|_\infty$

$W = [0,1]^d,\ \delta_d \ll \frac{1}{d}, \mathbb{E}f_k \to \infty$:

$$d_W\left(\frac{f_k - \mathbb{E}f_k}{\sqrt{\mathbb{V}f_k}}, N\right) \to 0 \qquad \text{as } d \to \infty$$

Grygierek

Thank you!