Towards multi-dimensional localisation

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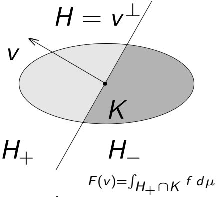

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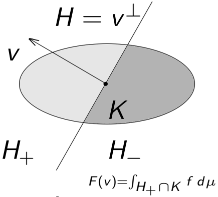

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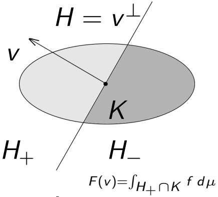

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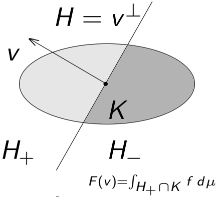

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Motivating example from 1960

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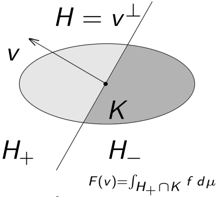

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- It suffices to prove the inequality for $K \cap H_+$ and $K \cap H_-$.
- Reduction can be done as long as $\dim K \geq 2$.
- This procedure produces a measurable partition and conditional measures of μ with respect to the partition are log-concave.
- Therefore, it suffices to prove the inequality in the one-dimensional case.

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- Ohta generalised the technique to Finsler manifolds and Cavalletti and Mondino generalised it to metric measure spaces.

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- The Kantorovich–Rubinstein duality gives us a more useful way to look at the problem. It tells that the minimal cost coincides with

$$\max \left\{ \int_M u f d\mu \middle| u \colon M \to \mathbb{R} \text{ is 1-Lipschitz} \right\}.$$

- Let $v\colon M \to \mathbb{R}$ be a maximiser.
  $\mathcal{T} \subset M$ – is a *transport ray* if it is a maximal set such that $v|_{\mathcal{T}}$ is an isometry.
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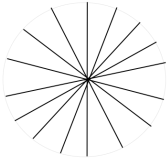

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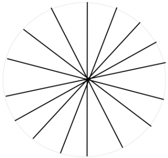

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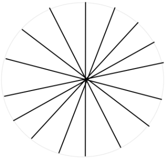

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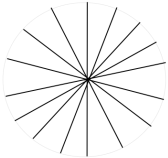

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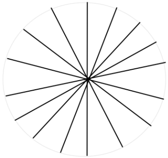
- It follows that $\int_M f d\mu_{\mathcal{T}} = 0$ for ν-almost every $\mathcal{T}$.



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Generalisation to multiple constraints

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- Suppose that for some functions $f_1, \ldots, f_m$ we have $\int_M f_i \, d\mu = 0$ for $i = 1, \ldots, m$. Does there exist a partition of M into m-dimensional pieces for which the related conditional measures would have zero integrals against $f_1, \ldots, f_m$? Would it be possible for the pieces to retain the curvature-dimension properties of (M, d, μ)?

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- Such generalisation was conjectured by Klartag to hold true in Euclidean spaces.

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Conjecture – mass-balance condition

Multi-dimensional case

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- Suppose $\eta \ll \lambda$. It was conjectured by Klartag that $\eta(A) = 0$ for any transport set A.

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- *Let $A \subset \mathbb{R}^n$ and let $v\colon \mathbb{R}^n \to \mathbb{R}$ and $u\colon A \to \mathbb{R}$ be* 1*-Lipschitz functions.*

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Proof.

- Let $\tilde{u}'$ be any 1-Lipschitz extension of u to $\mathbb{R}^n$. Let $\delta = \|u - v\|_{A,\infty}$.

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Using the above theorem, one may show that if v is an optimal potential for $f\,d\mu$, then for any transport set $A \subset M$, $\int_A f\,d\mu = 0$.

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Proof.

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- Take $\tilde{u} = \tilde{u}' \vee (v - \delta) \wedge (v + \delta)$.

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Theorem (Kirszbraun, '34)

Let $X \subset \mathbb{R}^n$ and let $v\colon X \to \mathbb{R}^m$ be a 1-Lipschitz map. Then there exists a 1-Lipschitz extension of v to $\mathbb{R}^n$.

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- This implies that $\lambda(B(x_j, \epsilon) \cap N_{i\epsilon}) \neq 0$ for $i, j = 1, 2, 3$.

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- For $i = 1, 2, 3$ let $N_{i\epsilon}$ be the set of all non-trivial leaves intersecting $B(x_i, \epsilon)$. Then $\nu_\epsilon(N_{i\epsilon}) = 0$.
- This implies that $\lambda(B(x_j, \epsilon) \cap N_{i\epsilon}) \neq 0$ for $i, j = 1, 2, 3$.
- Hence, there exist $(x^\epsilon_{rs}, x^\epsilon_{sr}) \in B(x_r, \epsilon) \times B(x_s, \epsilon)$ such that $\|u_\epsilon(x^\epsilon_{rs}) - u_\epsilon(x^\epsilon_{sr})\| = \|x^\epsilon_{rs} - x^\epsilon_{sr}\|$.

- Then, up to a subsequence, u_ϵ converges locally uniformly to a maximiser u_0 for ν_0.

Mass balance condition (C.)

Multi-dimensional case

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- We infer that for any 1-Lipschitz u

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- By the density of vectors in general position, we infer that for any 1-Lipschitz u and any $w_1, w_2, w_3 \in \mathbb{R}^3$ that sum up to zero there is

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- The same ideas allow to show that similar result for any norm on $\mathbb{R}^n$ and a strictly convex norm on $\mathbb{R}^m$. Also, similar conclusion will follow if we replace $\sup \left\{ \int_{\mathbb{R}^m} \langle u, d\mu \rangle \mid u \colon \mathbb{R}^n \to \mathbb{R}^m \text{ is 1-Lipschitz} \right\}$, by maximisation over any uniformly closed subset of 1-Lipschitz maps.

Optimal transport of vector measures (C.)

Primal problem and duality

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$$\mathcal{I}(\eta) = \sup \left\{ \int_{\mathbb{R}^n} \langle u, d\eta \rangle \middle| u \colon X \to \mathbb{R}^m \text{ is 1-Lipschitz} \right\}.$$

The same holds true for metric spaces (X, d) in lieu of $(\mathbb{R}^n, \|\cdot\|)$. This generalises the result for $m = 1$, where one can take π to be a non-negative measure with $\mathrm{P}_1\pi = \mu_1$ and $\mathrm{P}_2\pi = \mu_2$, $\mu = \mu_1 - \mu_2$.

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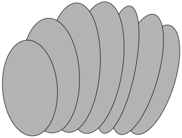

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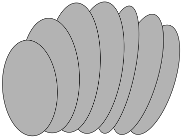

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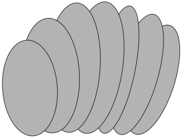

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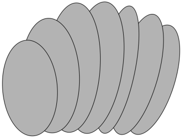

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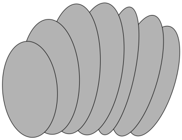

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- The idea of proof of $CD(\kappa, N)$ for conditional measures on leaves builds upon work of Caffarelli, Feldman and McCann.

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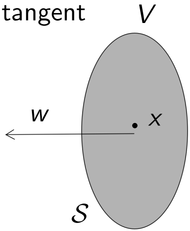

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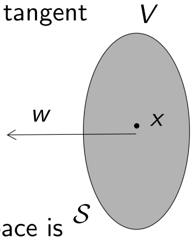

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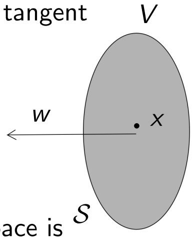

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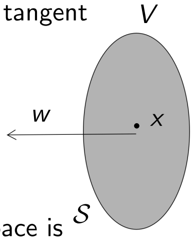

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- If the ghost subspace for $\mathcal{S}$ is trivial then $\eta_{\mathcal{S}} \ll \mathcal{H}_{\dim \mathcal{S}}$.
- If all the ghost subspaces for an optimal potential for a vector measure η are trivial, then there exists an optimal transport and the mass-balance condition holds true.

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i) *for any $A \subset \mathbb{R}^n$ and for any 1-Lipschitz map $u\colon A \to \mathbb{R}^m$ there exists 1-Lipschitz extension $\tilde{u}\colon \mathbb{R}^n \to \mathbb{R}^m$ of u such that*

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ii) v *is affine and* 1-*Lipschitz.*

Thank you for your attention.

C. *Leaves decompositions in Euclidean spaces*, Journal de Mathématiques Pures et Appliquées, 2021
C. *Leaves decompositions in Euclidean spaces II: ghost subspaces*, in preparation, 2023
C. *Optimal transport of vector measures*, Calculus of Variations and Partial Differential Equations, 2021
C. *Conitnuity of extensions of Lipschitz maps*, Israel Journal of Mathematics, 2021
C. *Optimal transport and* 1*-Lipschitz maps*, D.Phil. dissertation, University of Oxford, 2020

Mass balance condition

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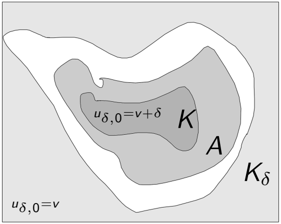
Then it is 1-Lipschitz and within δ-distance to v. We may extend it to M to u_δ, which is 1-Lipschitz and within δ-distance to v.



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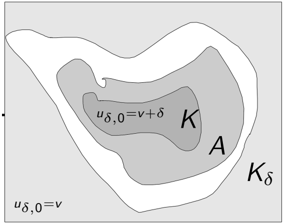

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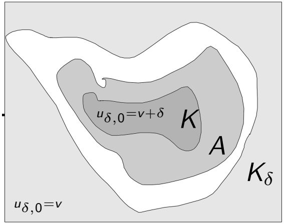
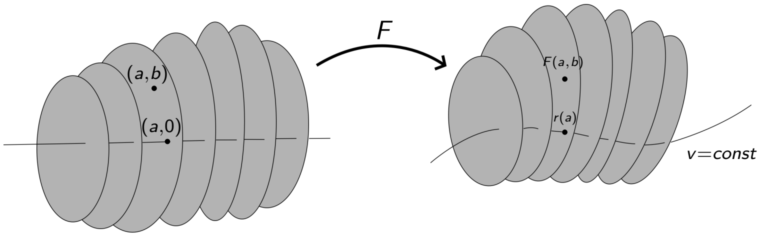
- Moreover $\int_M \frac{v - u_\delta}{\delta} f d\mu \geq 0$. Sending δ to zero, we get $\int_A f d\mu \geq -2\epsilon$. Similarly, $\int_A (-f) d\mu \geq 0$.



Idea of the proof (C.)

- The formula $F(a,b) = r(a) + Dv(r(a))^*(b)$, where $a \in \mathbb{R}^{n-m}$ and $b \in \mathbb{R}^m$, provides a local diffeomorphism, which is linear on the images of leaves of v. Here r is a local parametrisation of a fibre.



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- In the relative interiors of m-dimensional leaves Dv is Lipschitz:

$$2\sigma^2 \|Dv(x_1) - Dv(x_2)\|^2 \leq \|x_1 - x_2\|^2 - \|v(x_1) - v(x_2)\|^2,$$

where $\sigma = \min(\mathrm{d}(x_1, \partial\mathcal{S}(x_1)), \mathrm{d}(x_2, \partial\mathcal{S}(x_2)))$.

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- *Let $A \subset X \subset \mathbb{R}^n$. Suppose that $v\colon X \to \mathbb{R}^m$ is such that for all non-negative $t_1, \dots, t_m$ that sum up to one and all $x, x_1, \dots, x_m \in X$ there is*
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The converse is proven for $m = 1, 2, 3$.

Let $A \subset B \subset \mathbb{R}^n$, $v\colon \mathbb{R}^n \to \mathbb{R}^m$ be 1-Lipschitz. Set

$$d_v(A,B) = \sup\{\|v(x) - v(y)\| | x \in A, y \in B\}.$$

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Rate of continuity

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Theorem (C.)

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Theorem (C.)

- *Let $u\colon A \to \mathbb{R}^m$ be 1-Lipschitz.*
- *Assume that $\|u(x) - v(x)\| \leq \delta$ for $x \in A$.*
- *Then there exists a 1-Lipschitz map $\tilde{u}\colon B \to \mathbb{R}^m$ such that $\tilde{u}(x) = u(x)$ for $x \in A$ and*

$$\|v(x) - \tilde{u}(x)\| \leq \sqrt{\delta^2 + 2\delta d_v(A, B)}$$

for all $x \in B$.

Proof.

- The idea is to bootstrap the Kirszbraun theorem.

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- On $A \times \{0\} \cup B \times \{\epsilon\} \subset \mathbb{R}^{n+1}$ we define w so that $w(x,0) = u(x)$ for $x \in A$ and $w(x,\epsilon) = v(x)$ for $x \in B$.

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- Let $\tilde{w}$ be its 1-Lipschitz extension to $\mathbb{R}^{n+1}$.
- Define $\tilde{u}(x) = \tilde{w}(x,0)$ for $x \in \mathbb{R}^n$.
- Then for $x \in B$, $\|\tilde{u}(x) - v(x)\| = \|\tilde{w}(x,0) - w(x,\epsilon)\| \leq \epsilon$.

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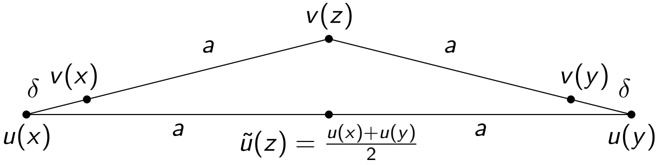
Let $m > 1$. Define u, v as in the picture, $\|x - y\| = 2a$, $z = \frac{x+y}{2}$. Set $A = \{x, y\}$, $B = \{x, y, z\}$.



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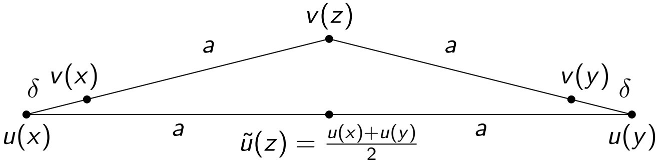

Then, for any 1-Lipschitz extension $\tilde{u}$ of u to B we have

$$\|v(z) - \tilde{u}(z)\| = \sqrt{\delta^2 + 2\delta a}.$$

If $\delta \geq a$, this is equal to $\sqrt{\delta^2 + 2\delta d_v(A, B)}$.