Log-concavity and discrete degrees of freedom

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Solution: degrees of freedom.

Introduction

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Definition

A log-concave sequence p has d degrees of freedom if there exists $\varepsilon > 0$ and linearly independent sequences $q_1, \ldots, q_d \in \mathcal{L}_I$ such that for all $\delta_1, \ldots, \delta_d \in (-\varepsilon, \varepsilon)$ the sequence $p + \delta_1 q_1 + \ldots + \delta_d q_d$ is log-concave.

How it works

Suppose we are given *n* constraints and the sequence *p* having d > n degrees of freedom. Take *d*-dimensional neighbourhood of *p* in \mathcal{L}_I , then by linearity there is a (d - n)-dimensional neighbourhood consisting of sequences satisfying all constraints. In such neighbourhood one can find p_1 , p_2 such that $p \in (p_1, p_2)$. Hence *p* is not extreme.

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Theorem

A log-concave sequence p has at most d + 1 degrees of freedom if and only if it is of the form e^{-V} for V being maximum of at most d arithmetic progressions. Let X be an integer-valued random variable with log-concave distribution p. Denote $M(X) = \max_{n} p(n)$.

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and assuming additionally that X is symmetric

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which is achieved asymptotically for geometric distribution $p(k) = \theta(1-\theta)^k 1_{k \ge 0}$ as $\theta \to 0$. We give a stronger result, namely

$$M(X)^{2} \operatorname{Var}(X) + M(X) \leq 1, \qquad (1)$$

which is optimal for all values of Var(X) and the equality occurs for geometric distributions.

Sketch of the proof:

- 1 LHS is monotone wrt M(X), hence it suffices to maximize convex functional $\Phi(p) = \max_n p(n)$. Constraints are
 - $\Phi_1(p) = \sum_n p(n) = 1$ fixing p to be a probability distribution;

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Developed theory, together with translation and approximation argument, reduces the problem to p being of the form

$$p(n) = Ce^{-\beta_1(n-N)} \mathbb{1}_{[0,N]}(n) + Ce^{-\beta_2(n-N)} \mathbb{1}_{[N+1,L]}(n).$$

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Proof of maximum-variance inequality

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Also one can find a, b such that $n^2 + an + b$ and p(n) - q(n) have opposite sign and

$$\mathbb{E}X^2 - \mathbb{E}Y^2 = \sum_n n^2(p(n) - q(n)) =$$
$$= \sum_n (n^2 + an + b)(p(n) - q(n)) \leq 0.$$

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For Y equality holds in (1), which ends the proof in monotone case.

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Proof of maximum-variance inequality

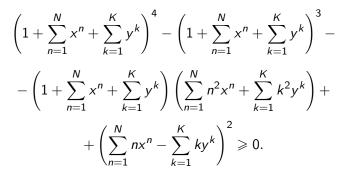
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equivalent form

$$\left(1 + \sum_{n=1}^{N} x^n + \sum_{k=1}^{K} y^k\right)^4 - \left(1 + \sum_{n=1}^{N} x^n + \sum_{k=1}^{K} y^k\right)^3 - \left(1 + \sum_{n=1}^{N} x^n + \sum_{k=1}^{K} y^k\right) \left(\sum_{n=1}^{N} n^2 x^n + \sum_{k=1}^{K} k^2 y^k\right) + \left(\sum_{n=1}^{N} n x^n - \sum_{k=1}^{K} k y^k\right)^2 \ge 0.$$

4 Now it suffices to show that coefficients of the obtained polynomial are nonnegative.

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Theorem (JMNS, 2023)

Let X be an ultra-log-concave random variable with integral mean. Then

$$\mathbb{P}(X = \mathbb{E}X) \ge \mathbb{P}(\mathrm{Poiss}(\mathbb{E}X) = \mathbb{E}X).$$

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