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Log-concavity and discrete degrees of freedom

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General setting

We say that a sequence p defined on an interval $I \subset \mathbb{Z}$ is log-concave if it is of the form e^{-V} for some convex V, equivalently $p(n)^2 \geqslant p(n-1)p(n+1)$ for all n and the support is an interval. Denote the set of log-concave sequences by $\mathcal{L}_I$.

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Solution: degrees of freedom.

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Definition

A log-concave sequence p has d degrees of freedom if there exists $\varepsilon > 0$ and linearly independent sequences $q_1, \ldots, q_d \in \mathcal{L}_I$ such that for all $\delta_1, \ldots, \delta_d \in (-\varepsilon, \varepsilon)$ the sequence $p + \delta_1 q_1 + \ldots + \delta_d q_d$ is log-concave.

Discrete degrees of freedom

How it works

Suppose we are given n constraints and the sequence p having $d > n$ degrees of freedom. Take d-dimensional neighbourhood of p in $\mathcal{L}_I$, then by linearity there is a $(d-n)$-dimensional neighbourhood consisting of sequences satisfying all constraints. In such neighbourhood one can find p_1, p_2 such that $p \in (p_1, p_2)$. Hence p is not extreme.

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Theorem

A log-concave sequence p has at most $d+1$ degrees of freedom if and only if it is of the form e^{-V} for V being maximum of at most d arithmetic progressions.

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In 2021 Bobkov, Marsiglietti and Melbourne proved that

$$\frac{1}{\sqrt{1+12\mathrm{Var}(X)}} \leqslant M(X) \leqslant \frac{1}{\sqrt{\frac{1}{4}+\mathrm{Var}(X)}}$$

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which is achieved asymptotically for geometric distribution $p(k) = \theta(1-\theta)^k 1_{k \geqslant 0}$ as $\theta \to 0$. We give a stronger result, namely

$$M(X)^2\mathrm{Var}(X) + M(X) \leqslant 1, \qquad (1)$$

which is optimal for all values of $\mathrm{Var}(X)$ and the equality occurs for geometric distributions.

Sketch of the proof:

1 LHS is monotone wrt $M(X)$, hence it suffices to maximize convex functional $\Phi(p) = \max_n p(n)$. Constraints are

- $\Phi_1(p) = \sum_n p(n) = 1$ fixing p to be a probability distribution;
- $\Phi_2(p) = \sum_n np(n)$, $\Phi_3(p) = \sum_n n^2 p(n)$ fixing the variance.

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Developed theory, together with translation and approximation argument, reduces the problem to p being of the form

$$p(n) = Ce^{-\beta_1(n-N)} 1_{[0,N]}(n) + Ce^{-\beta_2(n-N)} 1_{[N+1,L]}(n).$$

2 Assume p is monotone. For simplicity we allow sequences supported on $\{0, 1, \ldots\}$ in this case. Assume p is non-increasing and consider a variable Y with geometric distribution q of the same mean. Then $p - q$ changes sign exactly twice with sign pattern $(-, +, -)$.

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Also one can find a, b such that $n^2 + an + b$ and $p(n) - q(n)$ have opposite sign and

$$\mathbb{E}X^2 - \mathbb{E}Y^2 = \sum_n n^2(p(n) - q(n)) =$$

$$= \sum_n (n^2 + an + b)(p(n) - q(n)) \leqslant 0.$$

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For Y equality holds in (1), which ends the proof in monotone case.

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By algebraic transformations we obtain inequality (1) in the equivalent form

$$\left(1+\sum_{n=1}^{N} x^n + \sum_{k=1}^{K} y^k\right)^4 - \left(1+\sum_{n=1}^{N} x^n + \sum_{k=1}^{K} y^k\right)^3 -$$
$$-\left(1+\sum_{n=1}^{N} x^n + \sum_{k=1}^{K} y^k\right)\left(\sum_{n=1}^{N} n^2 x^n + \sum_{k=1}^{K} k^2 y^k\right) +$$
$$+\left(\sum_{n=1}^{N} n x^n - \sum_{k=1}^{K} k y^k\right)^2 \geqslant 0.$$

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$$+ \left(\sum_{n=1}^{N} n x^n - \sum_{k=1}^{K} k y^k\right)^2 \geqslant 0.$$

4 Now it suffices to show that coefficients of the obtained polynomial are nonnegative.

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Theorem (JMNS, 2023)

Let X be an ultra-log-concave random variable with integral mean. Then

$$\mathbb{P}(X = \mathbb{E}X) \geqslant \mathbb{P}(\mathrm{Poiss}(\mathbb{E}X) = \mathbb{E}X).$$