Probabilistic limit theorems induced by the zeros of polynomials

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joint work with:
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High Dimensional Probability
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1) $P_n(z) = a_0 + a_1 z^1 + \ldots + a_n z^n$ polynomial, $\quad a_k \geq 0 \ \forall k$

$\Rightarrow$ defines r.v. X_n by $\quad \mathbb{P}(X_n = k) = a_k / (\sum_{j=0}^{n} a_j)$

2) X_n r.v., values in $\{0, 1, \ldots, n\}$

$\Rightarrow$ probab. gen. fct. $\sum_{k=0}^{n} \mathbb{P}(X_n = k) z^k$ is polynomial

Conclusion: Have natural bijection

$$\{\text{“normed” polynomials, coeff.} \geq 0\} \longleftrightarrow \{\text{bounded } \mathbb{N}_0\text{-valued r.v.s}\}$$

Question: Consider sequences $(P_n)_n \overset{1:1}{\longleftrightarrow} (X_n)_n$. Relations between zeros of P_n and (asymptotic) distributional behaviour of X_n?

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Polynomials and Random Variables

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Situation 1: Assume all roots of $P_n(z)$ are real ($\Rightarrow$ negative)

$\Rightarrow \quad P_n(z) = \prod_{k=1}^{n}(p_k z + q_k)$ for $p_k \in (0,1)$, $q_k = 1 - p_k$

$\Rightarrow \quad X_n = X_{n,1} + \ldots + X_{n,n}$, $X_{n,1}, \ldots, X_{n,n}$ indep., $X_{n,k} \sim \mathrm{Ber}(p_k)$

- $(X_n)_n$ satisfies CLT iff $\mathrm{Var}(X_n) = \sigma_n^2 = \sum_{k=1}^{n} p_k q_k \to \infty$ as $n \to \infty$
- Speed of Convergence $\sim \sigma_n^{-1}$

Example: $\pi \in \mathfrak{S}_n$ permutation has descent at i if $\pi(i) > \pi(i+1)$

$X_n := \#\{$descents of uniform random permutation$\}$

P_n "Eulerian polynomial", only real roots, $\sigma_n^2 \sim n$

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Situation 2: $P_n(z)$ is a Hurwitz polynomial if $\mathrm{Re}(\zeta) < 0$ for all roots ζ

Note: $P_n(\zeta) = 0 \Rightarrow P_n(\bar{\zeta}) = 0 \quad \rightsquigarrow P_n(z) = \prod_j P_{n,j}(z), \quad P_{n,j}(z)$ linear or quadratic with positive coeff. (by Hurwitz prop.)

$\Rightarrow$ corresponds to sum of indep. (unif. bounded) r.v.s again

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all roots of P_n on the left-hand side of the unit circle, $\sigma_n^2 \sim n$

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General polynomials

Situation 3: What about general polynomials?

positive coeff. $\Rightarrow$ factorization into linear and quadratic terms, but possibly with negative coeff. $\rightsquigarrow$ not a sum of indep. r.v.s

Example: inversions of $\pi \in \mathfrak{S}_n$, i. e. tuples $i < j$ with $\pi(i) > \pi(j)$

$X_n := \#\{\text{inversions of uniform random permutation}\}$

$P_n(z) = \prod_{k=1}^{n-1}(1 + \ldots + z^k) = \prod_{k=1}^{n-1} \frac{1-z^{k+1}}{1-z}$ (all roots on unit circle)

Known: $\sigma_n^2 \sim n^3$, but X_n sat. CLT with speed $n^{-1/2} \neq \sigma_n^{-1}$

$\Rightarrow$ something different is happening!

Note: $\operatorname{argmin}\{\zeta : \zeta$ root of $P_n\} = 2\pi/n =: \delta_n$

Observation: rate of convergence agrees with $(\delta_n \sigma_n)^{-1} = n^{-1/2}$

$\Rightarrow$ General principle behind?

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Theorem (Hwang–Zacharovas (2015))

CLT for root-unitary polynomials $P_n(z)$:

$$X_n^* := (X_n - \mathbb{E}(X_n))/\sigma_n \Rightarrow \mathcal{N}(0,1) \qquad \textit{iff} \qquad \mathbb{E}(X_n^*)^4 \to 3$$

Proof based on cumulant bounds

Cumulant bounds imply a lot more probabilistic results than CLTs:

Proposition

Assume Statulevičius condition: $\forall m \geq 3, \forall n \geq 1$

$$|\kappa_{m,n}^*| \equiv |\kappa_m(X_n^*)| \leq \frac{m!}{\Delta_n^{m-2}}, \qquad \textit{some } \Delta_n > 0$$

(i) *CLT & Berry–Esseen bound:*

$\sup_{x\in\mathbb{R}} |\mathbb{P}(X_n^* \geq x) - \mathbb{P}(Z \geq x)| \leq C\Delta_n^{-1}, \quad Z \sim \mathcal{N}(0,1)$

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Proposition (cont.)

(ii) *moderate deviations:* (a_n) *s.th.* $a_n \to \infty$, $a_n = o(\Delta_n)$

$\rightsquigarrow X_n^*/a_n$ *satisfies a MDP with speed* a_n^2 *& rate fct.* $I(x) = x^2/2$

Example: $\lim_{n\to\infty} a_n^{-2} \log \mathbb{P}(X_n^*/a_n \in (x,\infty)) = -x^2/2, \quad x \geq 0$

(iii) *Bernstein-type conc. ineq.:* $\mathbb{P}(X_n^* \geq x) \leq C \exp(-\frac{1}{2}\frac{x^2}{2+x/\Delta_n})$

(iv) *mod-Gaussian convergence: Ass.* $\kappa^*_{m,n} = 0 \ \forall m = 3, \ldots, v-1$, $\kappa^*_{v,n}\Delta_n^{v-2} \to L \in \mathbb{R} \rightsquigarrow \Delta_n^{1-2/v} X_n^*$ *converges in the mod-Gaussian sense with param.* $t_n := \Delta_n^{2(v-2)/v}$ *and limiting fct.* $\Psi(z) := e^{\frac{L}{v!}z^v}$

This means: $\lim_{n\to\infty} \mathbb{E}[e^{zX_n^*}]e^{-t_n\frac{z^2}{2}} = \Psi(z)$ *loc. unif. on* $\mathbb{C}$

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(iii) *Bernstein-type conc. ineq.:* $\mathbb{P}(X_n^* \geq x) \leq C \exp(-\frac{1}{2}\frac{x^2}{2+x/\Delta_n})$

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Important class (C) of root-unitary polynomials:

$$P_n(z) = \prod_{k=1}^n (1 - z^{b_k})/(1 - z^{a_k}), \quad a_k \le b_k$$

- appear in inversions in random permutations (+ generalizations), Gaussian polynomials, q-Catalan numbers etc.
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One further example

$p(n, \ell, j) = \#\{$partitions of j with at most ℓ summands, each $\leq n\}$

Gaussian polynomials

$$G(n, \ell; z) := \sum_{j=0}^{n} p(n, \ell, j) z^j = \prod_{j=1}^{n} (1 - z^{j+\ell})/(1 - z^j)$$

Note: $G(n, \ell; z)$ is polynomial, but not $(1 - z^{j+\ell})/(1 - z^j)$ in gen.

Can calculate: $\sigma_{n,\ell}^2 = (\ell^2 n + \ell n + \ell n^2)/12$, $M_n = n + \ell$

$\Rightarrow \Delta_{n,\ell} = c\sqrt{\ell n(\ell + n + 1)}/(\ell + n) \to \infty$ as $n, \ell \to \infty$

$\Rightarrow (X_{n,\ell})_{n,\ell}$ sat. CLT with speed of convergence $\sim n^{-1/2} + \ell^{-1/2}$

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Note: roots of $\prod_{k=1}^{n}(1-z^{b_k})/(1-z^{a_k})$ are subset of

$$\{\exp(\pm 2\pi ik/b_j), k=1,\dots,\lfloor b_j/2\rfloor, j=1,\dots,n\}$$

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$(X_n)_n$ bounded $\mathbb{N}_0$-valued r.v.s, $(P_n)_n$ probab. gen. fcts. Ass. P_n has no roots in $S(\delta_n)$ for $\delta_n \in (0,\pi)$. Then,

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Comments and comparisons

- includes all previous results (choose any δ_n for Hurwitz pol.)
- Theorem and proof can be found along the lines of the proofs in Michelen–Sahasrabudhe
- our proof for the class (C) is more elementary
- different abs. constant c in $\Delta_n = c\delta_n\sigma_n$:
  $c = \pi\sqrt{7/24}$ for (C) vs. $c = 2^{-3248}$ in general
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Another class of examples

Recall: a polytope $Q \subset \mathbb{R}^N$ is called a lattice polytope if all of its vertices have integer coordinates

Ex.: $Q = W_N = [0,1]^N$ unit cube

Ehrhart's theorem: if Q is a lattice polytope, then

$$E_Q(k) := |\{t \in \mathbb{Z}^N : t \in kQ\}|, \qquad k \in \mathbb{N}_0$$

is the evaluation of a polynomial $E_Q(z)$ of degree N "Ehrhart pol."

Ex.: $E_{W_N}(z) = (z+1)^N \rightsquigarrow$ after renormalization p.g.f. of $\mathrm{Bin}(N, 1/2)$

in general: consider Q Ehrhart positive, i. e. all coefficients of $E_Q(z)$ are positive

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- Q lattice polytope is called CL-polytope if all roots of E_Q lie on $\{z \in \mathbb{C} \colon \Re(z) = -1/2\}$ "critical line"
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$$E_{A_N^\circ}(z) = (z+1)^{N+1} - z^{N+1}, \qquad E_{C_N^\circ}(z) = (z+1)^N + z^N$$

Interpretation in terms of “simple” r.v.s:

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In particular, our asymptotic distributional results hold in these cases.

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