

Riesz Energy Problems and Integral Identities

Unexpected phenomena for equilibrium measures

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High Dimensional Probability

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Plan

Motivation

Riesz Energy Problems

Findings and Open Questions

Integral Identities and Special Functions ($\lambda \in [0, 1]$)

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Integral Identities and Special Functions ($\lambda \in [0, 1]$)

$$\int_0^1 ((\lambda + r) \log(\lambda + r) - (\lambda - r) \log|\lambda - r|) \frac{r \, dr}{\sqrt{1 - r^2}} = \frac{\pi}{2} \left(\lambda^2 + \frac{1}{2} - \log 2 \right)$$

$$\int_0^1 K \left(\frac{4\lambda r}{(r + \lambda)^2} \right) \frac{(\lambda - r)r \, dr}{\sqrt{1 - r^2}} = \frac{\pi^2}{8} \left(\frac{3\lambda^2}{2} - 1 \right)$$

$$\int_0^1 {}_2F_1 \left(\frac{d-1}{2}, \frac{d-3}{2}; d-1; \frac{4\lambda r}{(\lambda + r)^2} \right) \frac{(\lambda + r)^{3-d} r^{d-1}}{\sqrt{1 - r^2}} \, dr = \frac{\pi}{4} \left(\left(\frac{3}{d} - 1 \right) \lambda^2 + 1 \right)$$

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Mathematica...

$$\begin{aligned}
 & \int_0^1 (\lambda + r) \log(\lambda + r) \frac{r \, dr}{\sqrt{1 - r^2}} \\
 &= \frac{{}_3F_2\left(\frac{1}{2}, 1, 1; \frac{3}{2}, \frac{5}{2}; \frac{1}{\lambda^2}\right)}{3\lambda} \\
 &\quad - \pi \frac{{}_3F_2\left(1, 1, \frac{3}{2}; 2, 3; \frac{1}{\lambda^2}\right)}{32\lambda^2} \\
 &\quad\quad + \frac{(4\lambda + \pi) \log(\lambda) + \pi}{4}
 \end{aligned}$$

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Marcel Riesz



MARCEL RIESZ

(1886, Győr, Hungary – 1969, Lund, Sweden)

- Intégrales de Riemann–Liouville et potentiels
Acta Sci. Math. Szeged 9 (1938): 1–42, 116–118

Riesz original problem (1931, 1938)

- Riesz s -kernel in \mathbb{R}^d , $-2 < s < d$

$$K_s = \begin{cases} \frac{1}{s |\cdot|^s} & \text{if } s \neq 0 \\ -\log |\cdot| & \text{if } s = 0 \end{cases}$$

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$$I_s(\mu) := \iint K_s(x - y) d\mu(x) d\mu(y) = \int (K_s * \mu) d\mu$$

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- Equilibrium Measure on ball $B_R := \{x \in \mathbb{R}^d : |x| \leq R\}$

$$\mu_{\text{eq}} = \arg \min_{\substack{\mu \\ \text{supp}(\mu) \subset B_R}} I_s(\mu)$$

Marcel Riesz original problem ($d \geq 2$)

■ Equilibrium Measure on B_R

$$\mu_{\text{eq}} = \begin{cases} \sigma_R & \text{if } -2 < s \leq d-2 \\ \frac{\Gamma(1 + \frac{s}{2})}{R^s \pi^{\frac{d}{2}} \Gamma(1 + \frac{s-d}{2})} \frac{\mathbb{1}_{B_R}}{(R^2 - |\mathbf{x}|^2)^{\frac{d-s}{2}}} d\mathbf{x} & \text{if } d-2 < s < d \end{cases}$$

(σ_R is uniform law on sphere of radius R)

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$$\int_{|y| \leq R} \frac{|x-y|^{-s}}{(R^2 - |y|^2)^{\frac{d-s}{2}}} dy = \frac{\pi^{\frac{d}{2}+1}}{\Gamma(\frac{d}{2}) \sin(\frac{\pi}{2}(d-s))}, \quad x \in B_R$$

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- ▶ Dyda – Kuznetsov – Kwaśnicki (Constr. Approx. 2017)

About Mellin transform and Riesz kernels

- Reciprocal pair of integral transform of $f : (0, +\infty) \rightarrow \mathbb{R}$

$$\mathcal{M}f(z) := \int_0^{\infty} f(x)x^{z-1}dx, \quad \alpha < \Re z < \beta,$$

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- Riesz potential of radial function : if $f(x) = \varphi(|x|^2)$ then

$$(K_s * f)(x) = \psi(|x|^2) \quad \text{where} \quad \mathcal{M}\psi(z) = \eta_{d,s}(z)\mathcal{M}(\varphi)\left(z + \frac{d-s}{2}\right)$$

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- More on <https://djalil.chafai.net/blog/2022/08/26/mellin-transform-and-riesz-potentials/>

External Field Equilibrium Problem

- Energy with external field V on \mathbb{R}^d

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- Frostman or Euler – Lagrange characterization of μ_{eq}

$$K_s * \mu + V \begin{cases} = c & \text{q.e. on } \text{supp}(\mu) \\ \geq c & \text{q.e. outside } \text{supp}(\mu) \end{cases}$$

Coulomb case : $s = d - 2$

- Laplace fundamental solution

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- If $V = |\cdot|^\alpha$, $\alpha > 0$, then

$$\mu_{\text{eq}} = \alpha(\alpha + d - 2) |\cdot|^{\alpha-2} \mathbb{1}_{B_R} dx$$

$$\text{(with } R = (\frac{1}{\alpha})^{\frac{1}{\alpha+d-2}})$$

Continuous case $-2 < s < 0$ and $\alpha = 2$

Equilibrium problem :

$$\arg \min_{\mu} \left\{ - \iint |x - y|^{s|} d\mu(x) d\mu(y) + 2 \int |x|^2 d\mu(x) \right\}$$

- Arises in steepest descent for halftoning functionals

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- Hertrich – Gräf – Beinert – Steidl arXiv:2211.01804
*Wasserstein Steepest Descent Flows
of Discrepancies with Riesz Kernels*

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*Wasserstein Steepest Descent Flows
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- Explicit formulas for μ_{eq}

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Iterated Coulomb case : $s = d - 2n$, $n = 1, 2, 3, \dots$

Proposition (CSW 2022)

If $s = d - 2n$ then, restricted to the interior of μ_{eq} ,

$$\mu_{\text{eq}} = \frac{\Delta^n V}{c_d C_{d,n}} \quad \text{where} \quad C_{d,n} := (-1)^{n-1} (2n-2)!!$$

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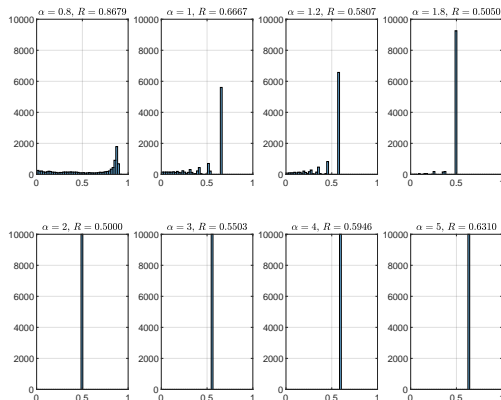
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- Key : $-\Delta K_u = c_{d,u} K_{u+2}$, $c_{d,u} := d - 2 - u$
- In particular : if $s = d - 4$ and $V = |\cdot|^\alpha$, $\alpha \geq 2$, then $C_{d,2} < 0$ while $\Delta V = \alpha(\alpha + d - 2) |\cdot|^{\alpha-2} \geq 0$ and thus μ_{eq} is necessarily singular!

Condensation phenomenon for $s = d - 4$?



Point norms histograms $|x_j|, 1 \leq j \leq N = 10^4$ of a discrete approximation of μ_{eq} , $d = 4$, $s = d - 4 = 0$, $V = |\cdot|^\alpha$, $\alpha > 0$.

Theorem (CSW arXiv 2022 : $V = \gamma |\cdot|^\alpha$, $\gamma > 0$, $\alpha > 0$)

- *Let $d \geq 4$ and $s = d - 4 \geq 0$.*

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■ Let $d \geq 4$ and $s = d - 4 \geq 0$.

▶ If $\alpha \geq 2$ then $\mu_{\text{eq}} = \sigma_R$ where

$$R = \left(\frac{2}{(s+4)\alpha\gamma} \right)^{\frac{1}{\alpha+s}}$$

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■ Let $d = 3$ and $s = d - 4 = -1$ (non-singular kernel!).

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▶ If $\alpha > 1$, then μ_{eq} is as above

Theorem (CSW JMAA 2022 : $V = \gamma |\cdot|^2, \gamma > 0$)

■ If $s = d - 3$ and $\alpha = 2$ then

$$\mu_{\text{eq}} = \frac{\Gamma(\frac{s+4}{2})}{\pi^{\frac{s+4}{2}} R^{s+2}} \frac{1_{B_R}}{\sqrt{R^2 - |\cdot|^2}} dx$$

where

$$R = \left(\frac{\sqrt{\pi} \Gamma(\frac{s+4}{2})}{4\gamma \Gamma(\frac{s+5}{2})} \right)^{\frac{1}{s+2}}$$

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- Initial motivation : $(d, s, \alpha) = (3, 0, 2)$ and Gumbel at edge.

About the proofs

- Methods

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■ Challenges

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- ▶ Non-locality of fractional Laplacian

Link : obstacle problems (Sylvia Serfaty)

- Fundamental solution (fractional Laplacian) :

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- Variational : convex duality for energies

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Random commuting matrices

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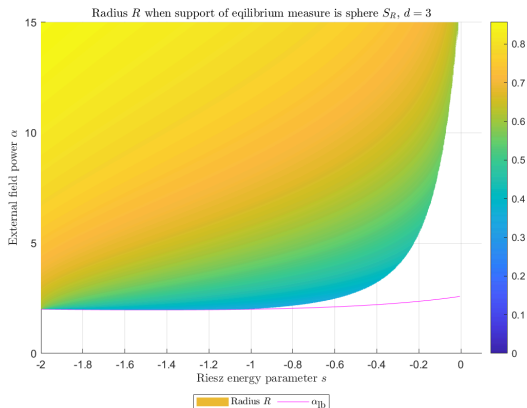
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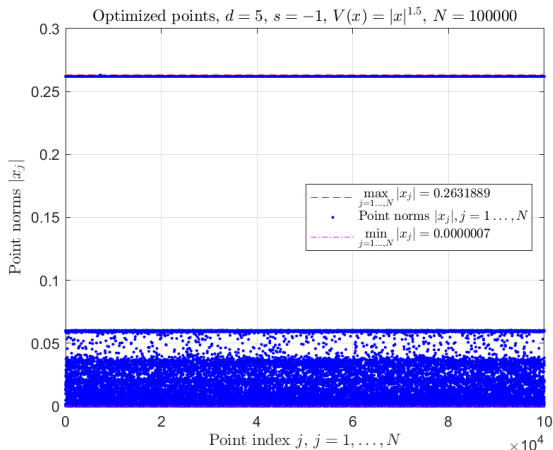
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- Other norms in kernel and external field

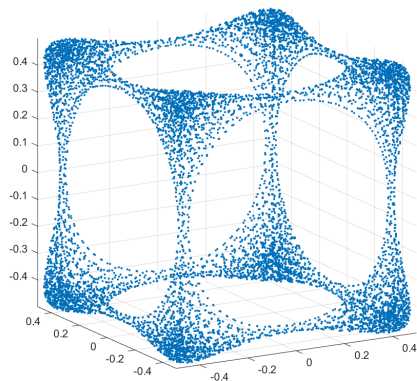


A plot of the radius R when the support of the equilibrium measure is the sphere S_R for $d = 3$ and various s and α .



Point norms $|x_j|$, $1 \leq j \leq N = 10^5$,
 $d = 5$, $s = d - 6 = -1$, $V = |\cdot|^{3/2}$.

More than 70% on outer boundary!



Projection on $x_1 = 0$ of a numerical approximation of μ_{eq} ,

$$N = 10^4, \quad d = 4, \quad s = d - 4 = 0, \quad V = |\cdot|_4^4,$$

$$|x|_p := (|x_1|^p + \dots + |x_d|^p)^{1/p}.$$

Naoum Samoilovitch Landkof



(1915, Kharkov, Russian Empire – 2004, Israel)

- Foundations of Modern Potential Theory
Grundlehren der mathematischen Wissenschaften 180
Springer 1972 (translated from Russian, Moscow 1966)