Comparing moments of real log-concave random variables

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Let X be a real log-concave random variable with non-negative values. From Prékopa-Leindler inequality we know that the function $t \mapsto \mathbb{P}(X \geqslant t)$ is log-concave. Let E denote a random variable with exponential distribution with $\lambda = 1$, $\gamma_p = \|E\|_p = \Gamma(p+1)^{1/p}$.

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Theorem (Barlow, Marshall, Proschan, 1963)

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This, in particular, shows that for symmetric/non-negative log-concave X and $p > q > 0$ we have $\|X\|_p \leqslant \frac{\gamma_p}{\gamma_q}$ and the equality holds for $X = E$.

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$$\leqslant \frac{\gamma_p}{\gamma_q}\|X - X'\|_q + |\mathbb{E}X| \leqslant 2\frac{\gamma_p}{\gamma_q}\|X\|_q + |\mathbb{E}X|.$$

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- $C = 1$ if X symmetric/non-negative,
- $C = 2$ if $\mathbb{E}X = 0$,
- $C = 3$ in general case.

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2. For general case, $p \geqslant q \geqslant 2$ it is $C_0 = e^{W(1/e)} \approx 1.3211$.
3. The ratio is always maximized for a variable of the form $E - t$ for some $t \geqslant 0$.

We consider three linear functionals on the space of measures supported on some interval $[a, b]$ with density g

$$F_1(g) = \int_a^b g(x)dx,$$

$$F_2(g) = \int_a^b |x|^q g(x)dx,$$

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We maximize H with constraints $F_1 = 1$, $F_2 = 1$. By Fradelizi-Guédon theory of degrees of freedom, it is enough to consider densities of the form

$$g(x) = e^{\alpha x+\beta}\mathbb{1}_{[a,b]}(x)$$

Non-negative case already solved, we consider intervals $[-a, b]$, where $a, b > 0$.

Define

$$F_1(a,b,\alpha,\beta) := \int_{-a}^{b} e^{\alpha x+\beta} dx,$$

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Lagrange multipliers:

$$(DH - \lambda_1 DF_1 - \lambda_2 DF_2)(a,b,\alpha,\beta) = 0.$$

Define $r_\lambda(x) = |x|^p - \lambda_2|x|^q - \lambda_1$. The conditions can be written as

$$r_\lambda(a) = 0;\ r_\lambda(b) = 0;\ \int_{-a}^{b} e^{\alpha x} r_\lambda(x)dx = 0;\ \int_{-a}^{b} x e^{\alpha x} r_\lambda(x)dx = 0.$$

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Case $a = b$, $\alpha \neq 0$ is similar. From this the maximizer is $X = E - t$ for some t.

Asymptotics for $\|E-t\|_s$. We have

$$\mathbb{E}|E-t|^s = \int_{-t}^{\infty} e^{-x-t}|x|^s dx = \int_0^t e^{x-t}x^s + e^{-t}\Gamma(s+1).$$

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We note that $(x+y)^{1/s} \approx \max(x^{1/s}, y^{1/s})$. We have

$$\left(e^{-t}\Gamma(s+1)\right)^{1/s} \approx e^{-t/s}\frac{s}{e},$$

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Thus, for large s we have

$$\|E-t\|_s \approx \max(t,\ e^{-t/s}\frac{s}{e}) = \begin{cases} t & \text{for } t \geqslant W(1/e)s \\ e^{-t/s}\frac{s}{e} & \text{for } t \leqslant W(1/e)s. \end{cases}$$

Asymptotic optimal bound Let $p > q$ with q large. We will show that $\frac{\|E-t\|_p}{\|E-t\|_q} \leqslant e^{W(1/e)} \frac{p}{q}$ up to a constant arbitrarily close to 1.

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Case 1. $t \leqslant W(1/e)q$.

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Note: If we take $t = W(1/e)q$, $p >> q$ *we get the lower bound.*

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Case 3. $t \geqslant W(1/e)p$**.**

$$\frac{\|E-t\|_p}{\|E-t\|_q} \approx \frac{t}{t} = 1.$$

Thank you for your attention!