# Random (Beta) Polytopes

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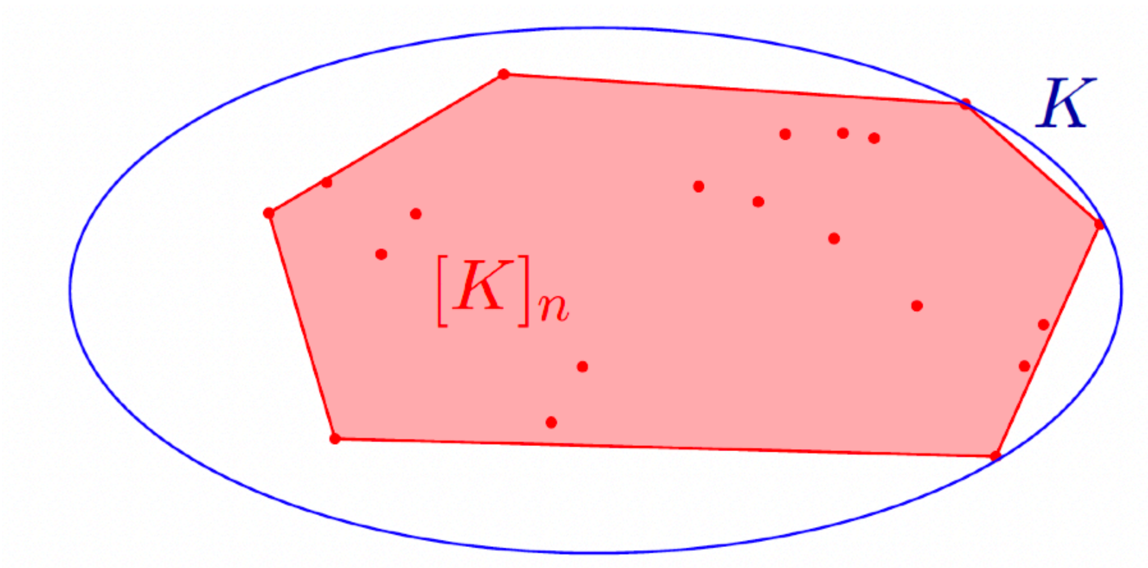
Based on joint works with Florian Besau, Thomas Godland, Anna Gusakova, Zakhar Kabluchko, Olexander Marynych, Matthias Reitzner, Daniel Rosen, Carsten Schütt, Daniel Temesvari, Elisabeth Werner

# Classical random polytopes

- $K \subset \mathbb{R}^d$ a convex body (compact, convex & non-empty interior)




- Two classical models: For $n \in \mathbb{N}$ let
  - $X_1, \ldots, X_n$ be uniformly distributed in the interior of $K$

    Random polytope $[K]_n = \operatorname{conv}\{X_1, \ldots, X_n\}$

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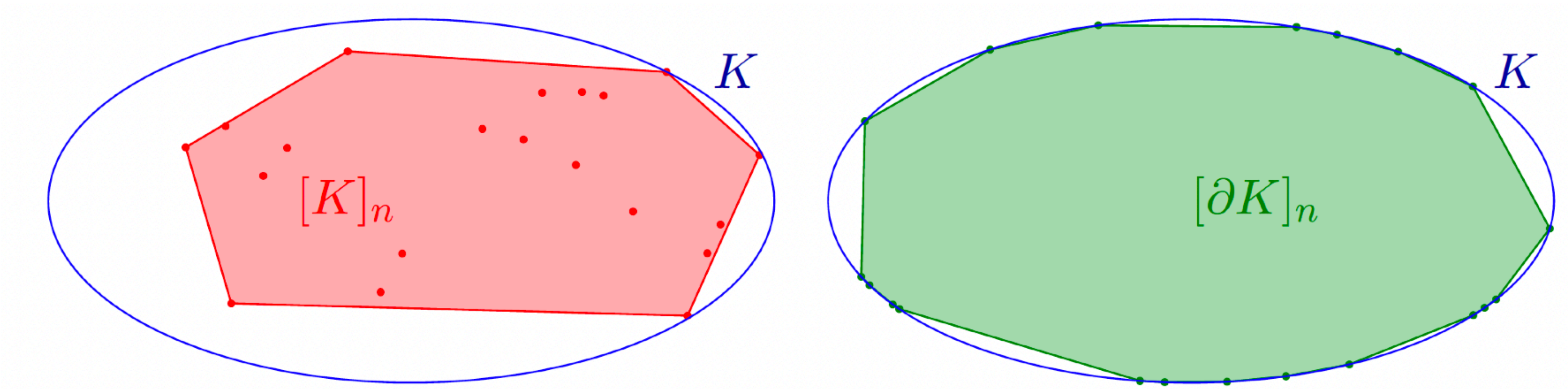


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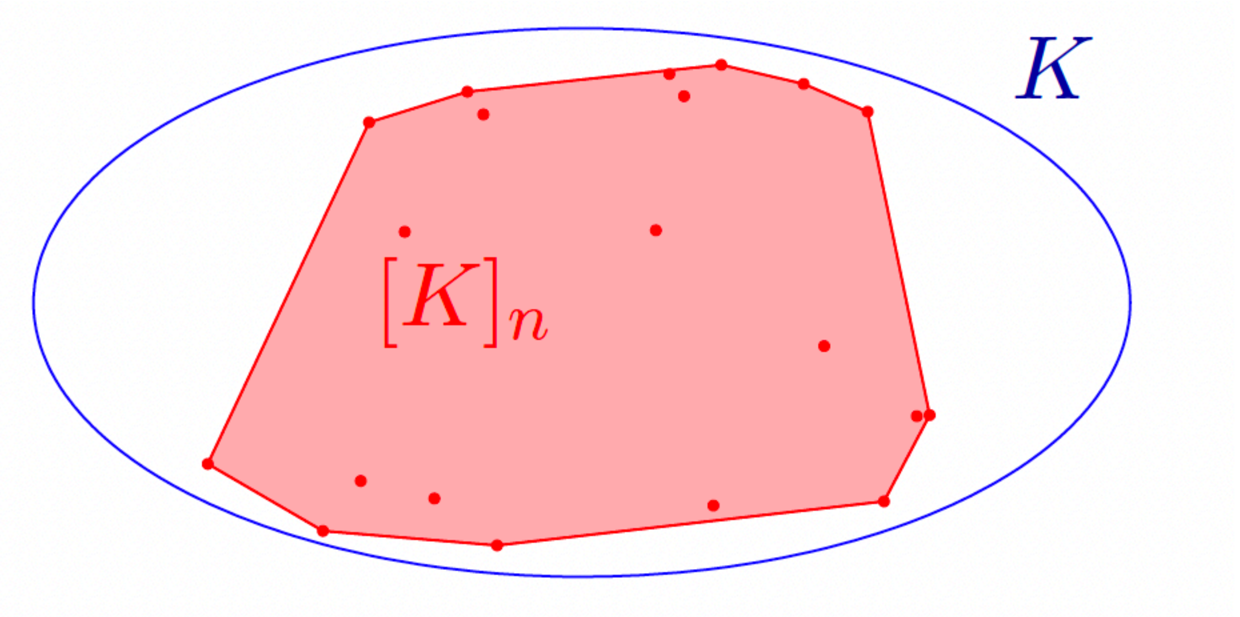
      Random inscribed polytope $[\partial K]_n = \mathrm{conv}\{X_1, \ldots, X_n\}$

**Theorem**

$$\mathbb{E}\mathrm{Vol}([K]_n) = \mathrm{Vol}(K) - c(d)\mathrm{as}(K)\left(\frac{\mathrm{Vol}(K)}{n}\right)^{\frac{2}{d+1}}(1 + o_n(1))$$

- Rényi & Sulanke (1963)
  $d = 2$, $K$ smooth enough
- Schneider & Wieacker (1980)
  $d \geq 2$, $K$ smooth enough
- Bárány (1982)
  $d \geq 2$, $K$ of class $C^3_+$
- Schütt (1994), Böröczky, Fodor, Hug (2010)
  $d \geq 2$, $K$ general



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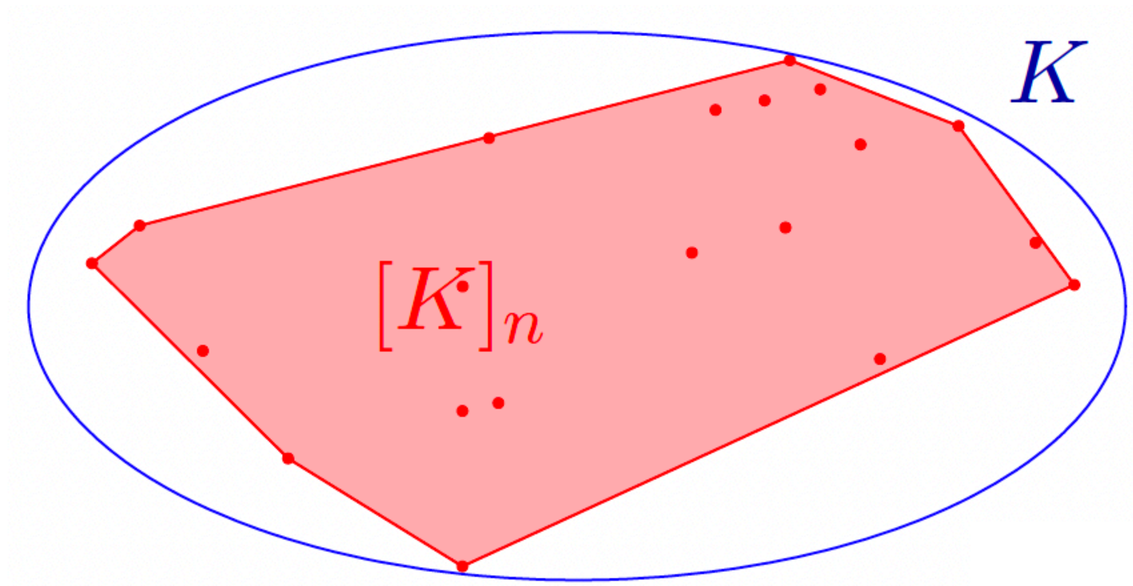
- The constant $\mathrm{as}(K)$ is the affine surface area of $K$:

$$\mathrm{as}(K) = \int_{\partial K} H_{d-1}(K;x)^{\frac{1}{d+1}}\,\mu_K(\mathrm{dx}).$$




- Blaschke (1923), Santaló (1949)
  $d = 2, d = 3$, $K$ smooth enough

- Leichtweiss (1988), Schütt & Werner (1990)
  $d \geq 2$, general $K$

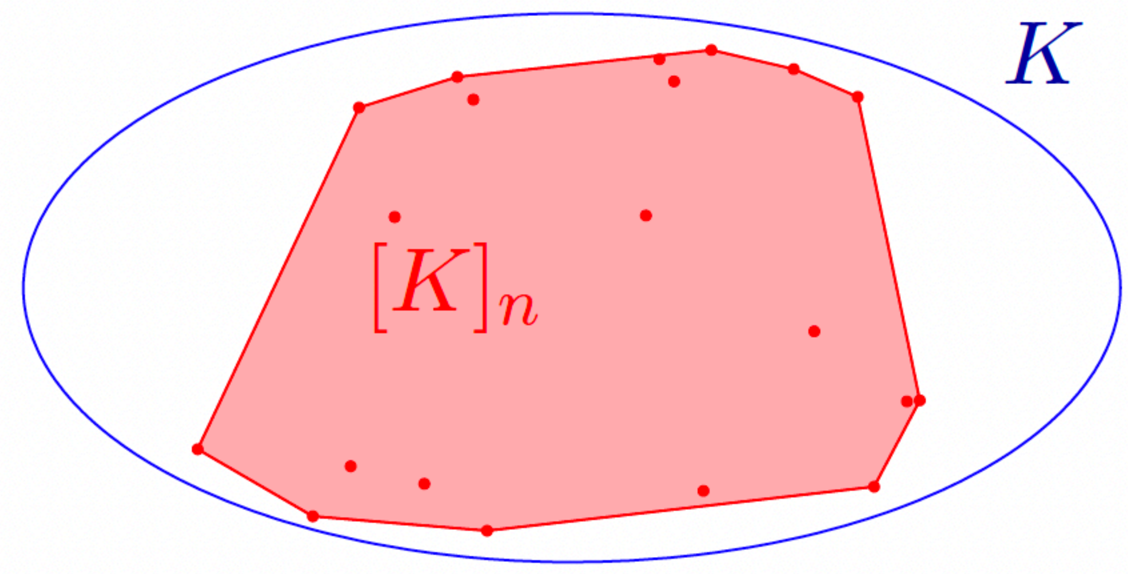
- Ludwig & Reitzner (1999)
  characterization

**Theorem** (Reitzner 2003)

Suppose $K$ is of class $C^2_+$. Then

$$c_1(K)n^{-1-\frac{2}{d+1}} \le \operatorname{Var}\operatorname{Vol}([K]_n) \le c_2(K)n^{-1-\frac{2}{d+1}}$$

- Küfer (1994)
  upper bound if $K = B^d$
- Buchta (2005)
  lower bound $d = 2$, $K$ smooth enough
- Calka & Yukich (2014)
  asymptotics, $K$ smooth enough



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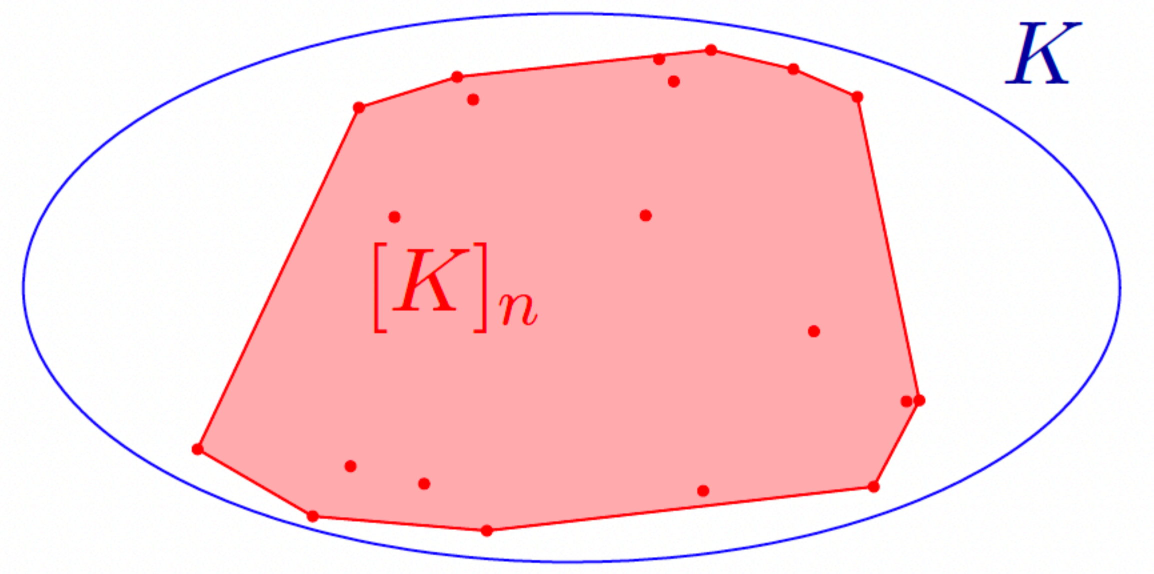


**Theorem** (Reitzner 2005)

Suppose $K$ is of class $C^2_+$. Then $\dfrac{\operatorname{Vol}([K]_n) - \mathbb{E}\operatorname{Vol}([K]_n)}{\sqrt{\operatorname{Var}\operatorname{Vol}([K]_n)}} \xrightarrow{D} Z \sim \mathcal{N}(0,1)$

**Theorem**

$$\mathbb{E}\mathrm{Vol}([\partial K]_n) = \mathrm{Vol}(K) - \tilde{c}(d)\Omega(K)\left(\frac{\mu_K(K)}{n}\right)^{\frac{2}{d-1}}(1 + o_n(1))$$

Buchta, Müller & Tichy (1985): $K$ of class $C^3_+$

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Suppose $K$ is of class $C^2_+$. Then

$$c_1(K)n^{-1-\frac{4}{d-1}} \leq \mathrm{Var}\,\mathrm{Vol}([\partial K]_n) \leq c_2(K)n^{-1-\frac{4}{d-1}}$$

Richardson, Vu, Wu (2008)

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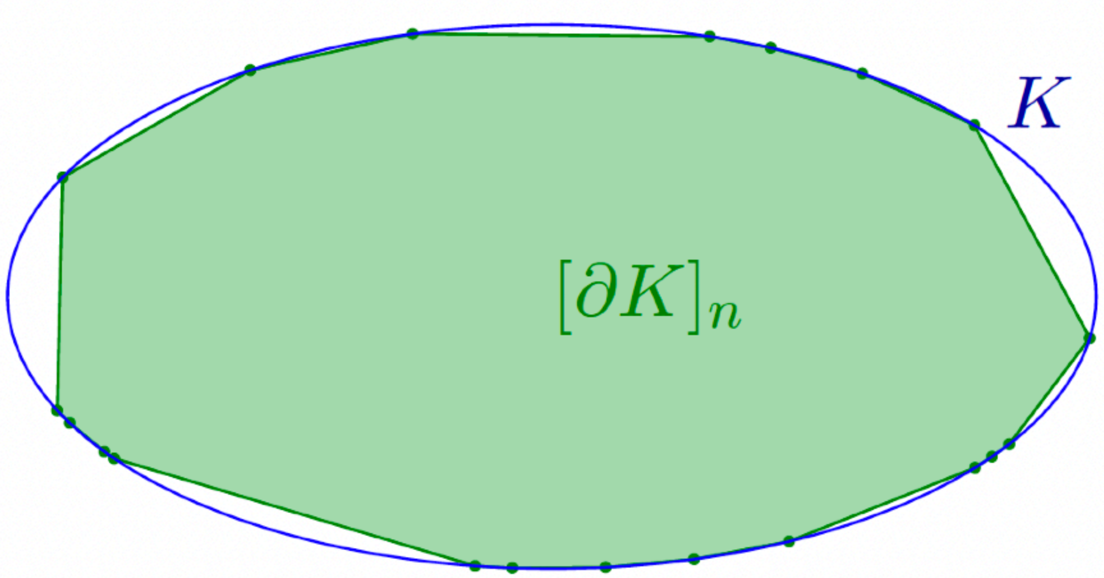
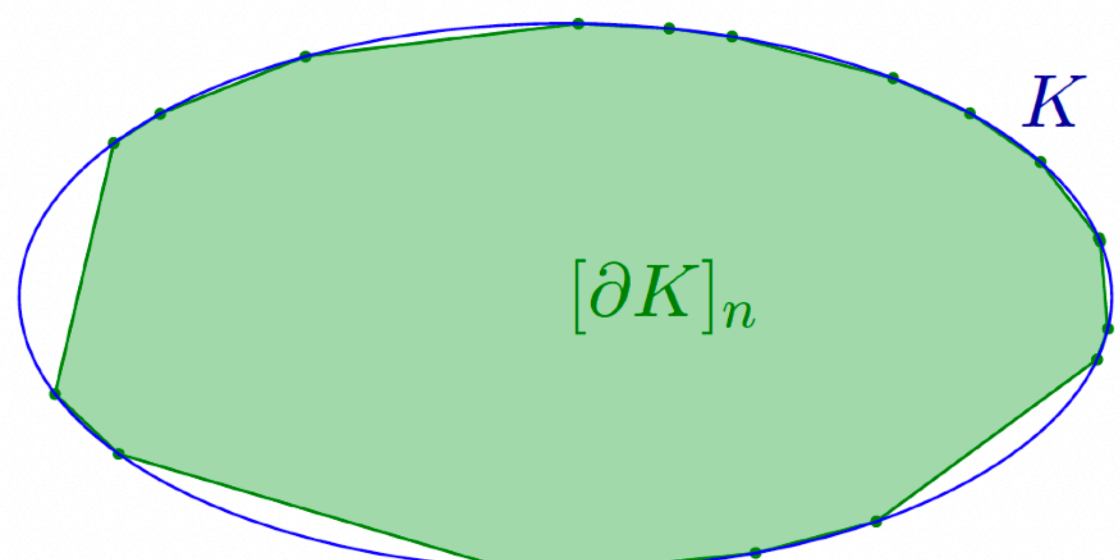
T. (2018)



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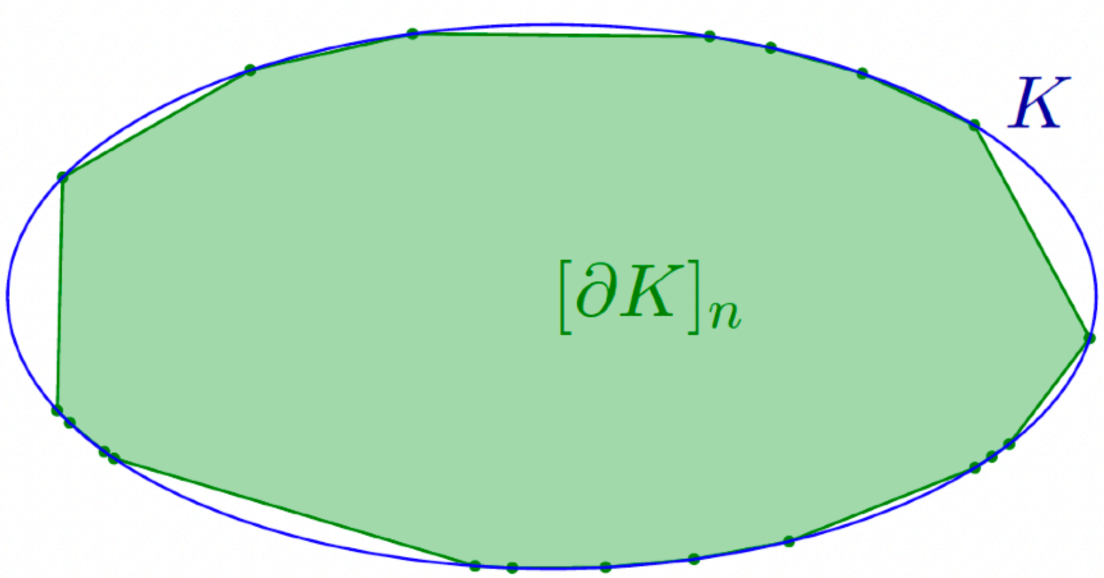
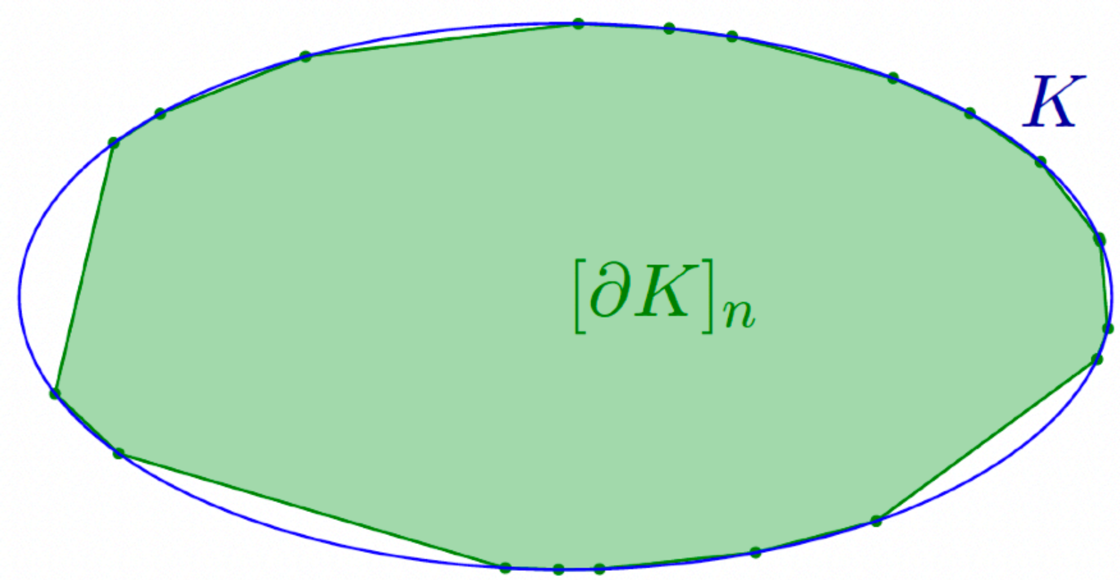
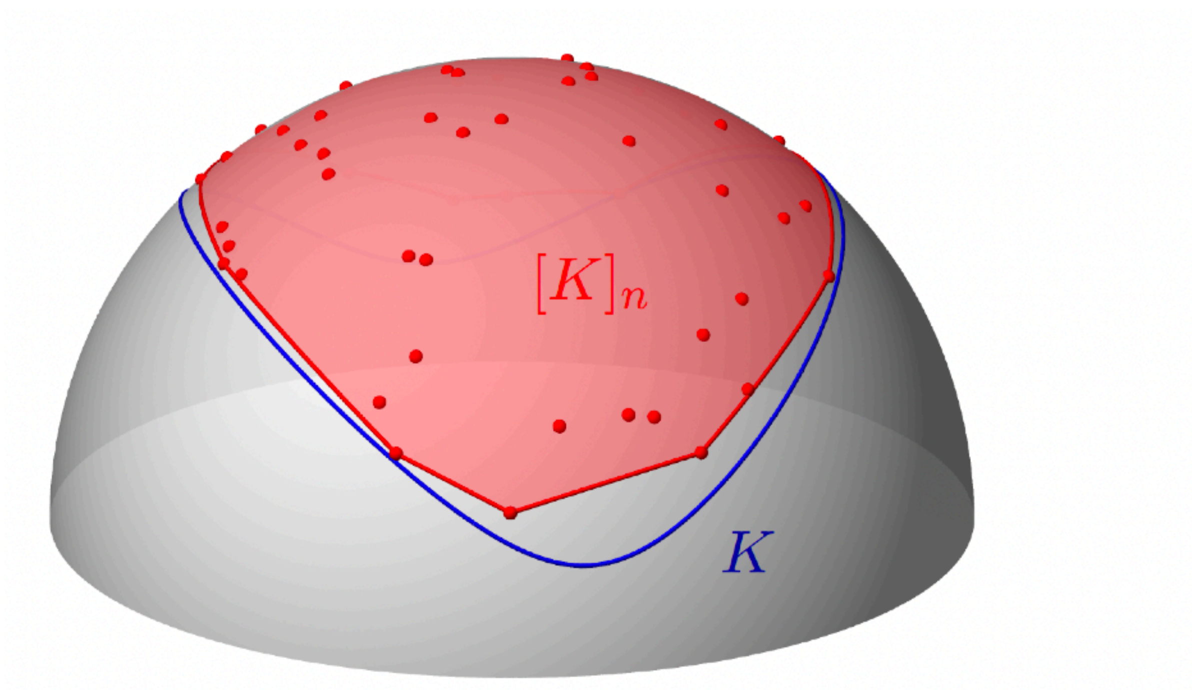


# Random polytopes on the sphere

- $K \subset \mathbb{S}^d$ a spherical convex body (geodesically convex, contained in open hemisphere)

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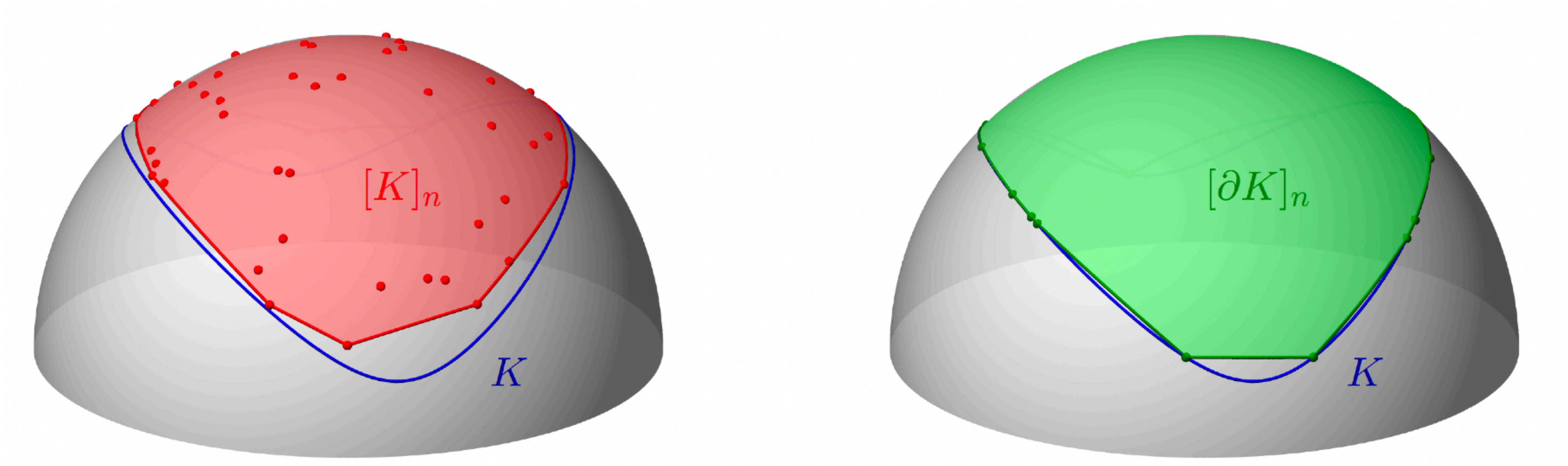


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Besau, Ludwig, Werner (2018)

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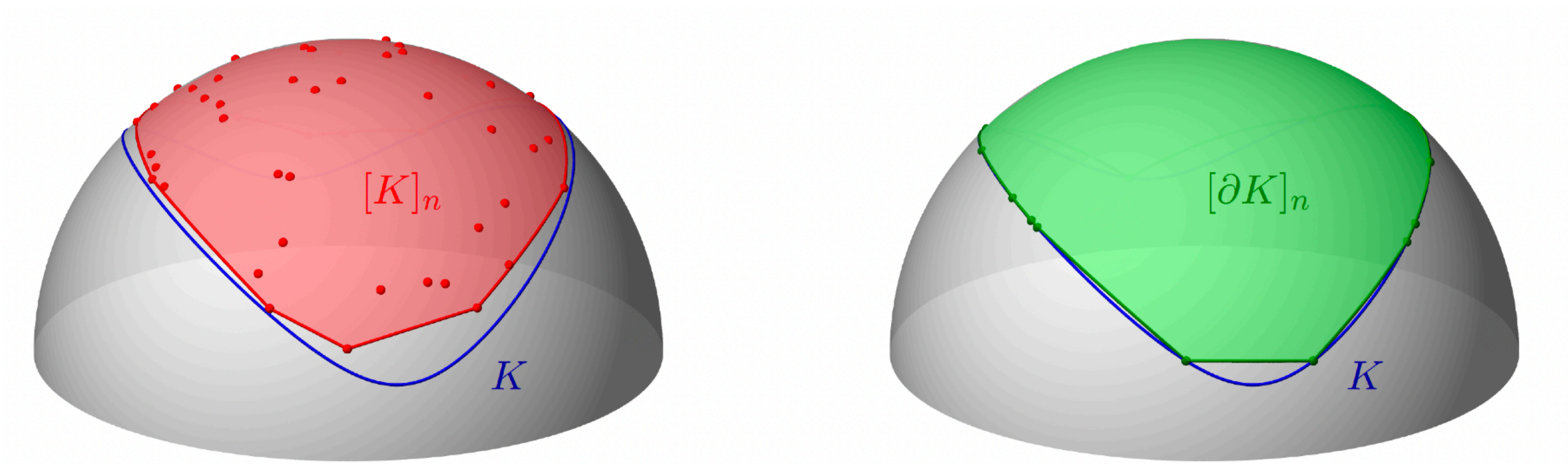
Follows from Böröczky, Fodor, Hug (2013) and Besau, Ludwig, Werner (2018)



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Suppose $K$ is of class $C^2_+$. Then

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Besau & Thäle (2020)

$$\frac{\operatorname{Vol}_s([K]_n) - \mathbb{E}\operatorname{Vol}_s([K]_n)}{\sqrt{\operatorname{Var} \operatorname{Vol}_s([K]_n)}} \xrightarrow{D} Z \sim \mathcal{N}(0,1)$$

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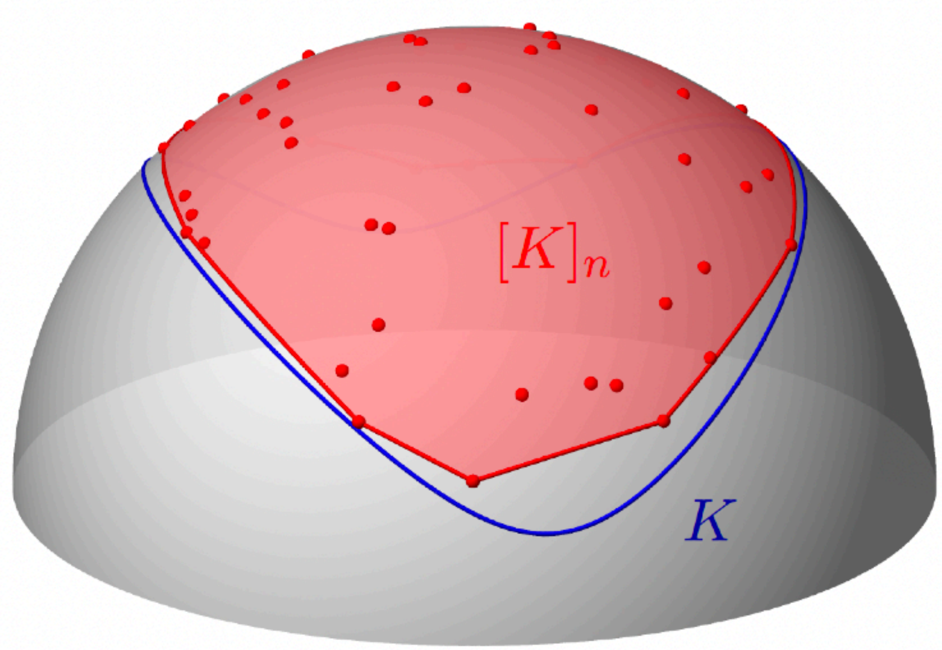


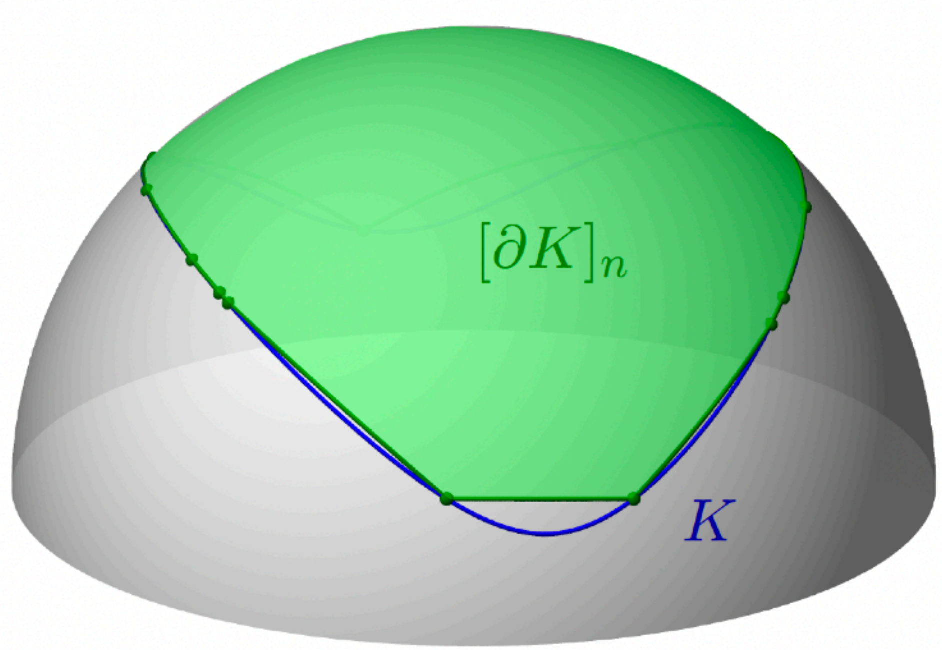

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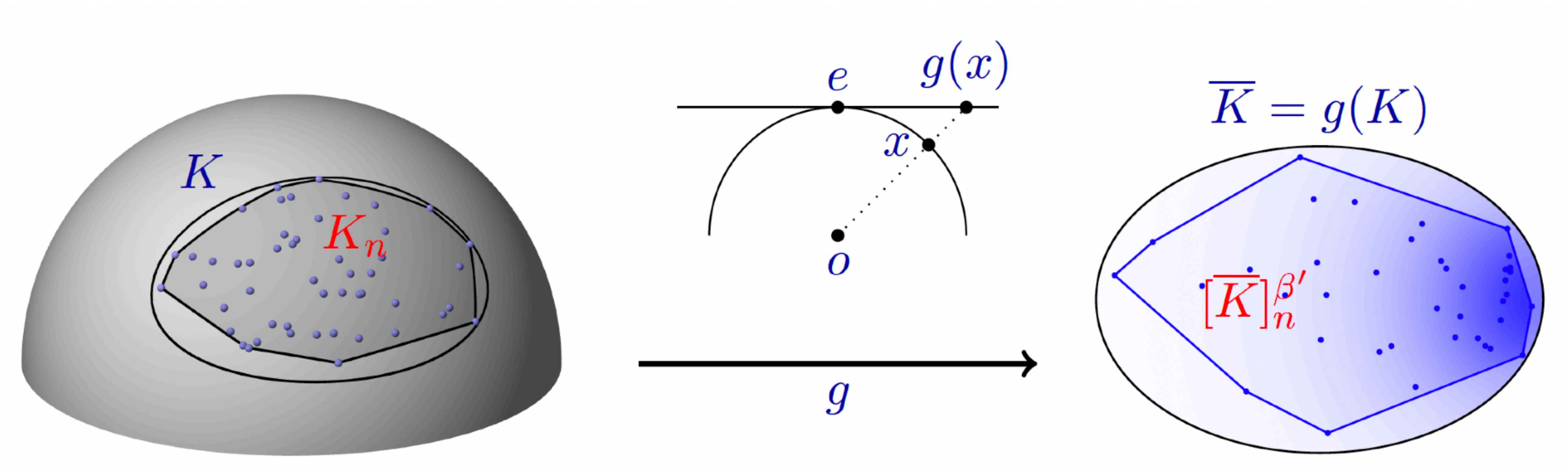
- The proof for asymptotic normality relies on a version of Stein's method due to Chatterjee (2008) as presented by Lachiéze-Rey & Peccati (2017),
- geometric estimates involving weighted floating bodies and weighted surface bodies,
- and a projection argument reducing the problem to $\mathbb{R}^d$.

## The projection argument




$X_i \sim \mathrm{Vol}_s|_K$

$[K]_n = \mathrm{conv}_s\{X_1, \ldots, X_n\}$

$\mathrm{Vol}_s([K]_n)$

$\bar{X}_i \sim (g\sharp\mathrm{Vol}_s)|_{\bar{K}}$

$[\bar{K}]_n = \mathrm{conv}\{\bar{X}_1, \ldots, \bar{X}_n\}$

$$\mathrm{Vol}_\varphi([\bar{K}]_n) = \int_{[\bar{K}]_n} \varphi(x)\,\mathrm{d}x$$

$\varphi$ = density of $g\sharp\mathrm{Vol}_s$

## Stein's method

$K \subset \mathbb{R}^d$ a convex body, $X_1, \ldots, X_n$ independent random points in $K$ distributed according to some *nice* density $\psi$. Consider the weighted volume $\mathrm{Vol}_\varphi([K]_n^\psi)$ where $\varphi$ is another *nice* density function on $K$.

$$d_{\mathrm{Wass}}(V_n, Z) \leq \frac{c}{\mathrm{Var}\,\mathrm{Vol}_\varphi([K]_n^\psi)}\Big(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4\Big)$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are terms only involving the first and second order difference operator applied to the weighted volume functional:

$$D_1\mathrm{Vol}_\varphi([K]_n^\psi) = \mathrm{Vol}_\varphi(\mathrm{conv}\{X_1, \ldots, X_n\}) - \mathrm{Vol}_\varphi(\mathrm{conv}\{X_2, \ldots, X_n\})$$

$$D_{1,2}\mathrm{Vol}_\varphi([K]_n^\psi) = D_2(D_1\mathrm{Vol}_\varphi([K]_n^\psi))$$

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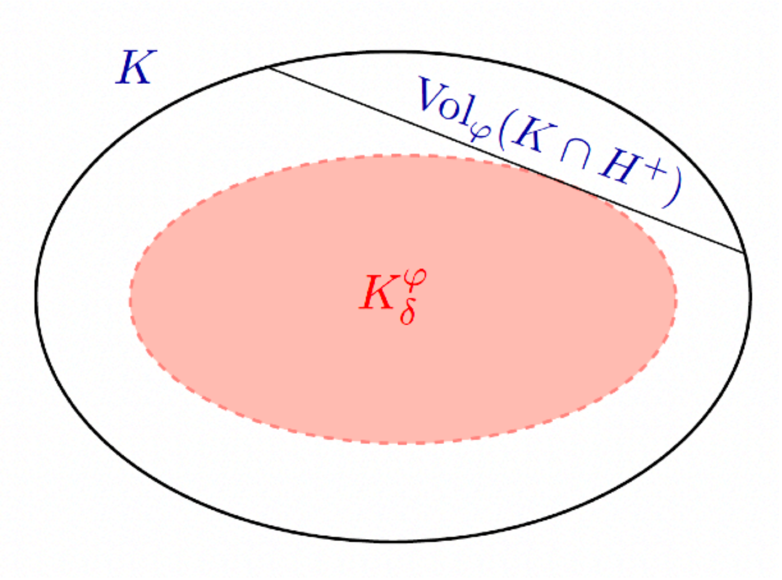
Especially $\gamma_3 \asymp \mathbb{E}\,|\,D_1\mathrm{Vol}_\varphi([K]_n^\psi)\,|^3$ and $\gamma_4 \asymp \mathbb{E}\,|\,D_1\mathrm{Vol}_\varphi([K]_n^\psi)\,|^4$.

## Weighted floating bodies

The weighted floating body of $K$ is defined by

$$K_\delta^\varphi = \bigcap \{H^- : \mathrm{Vol}_\varphi(K \cap H^+) \leq \delta\}$$

Schütt & Werner (1990), Werner (2002)



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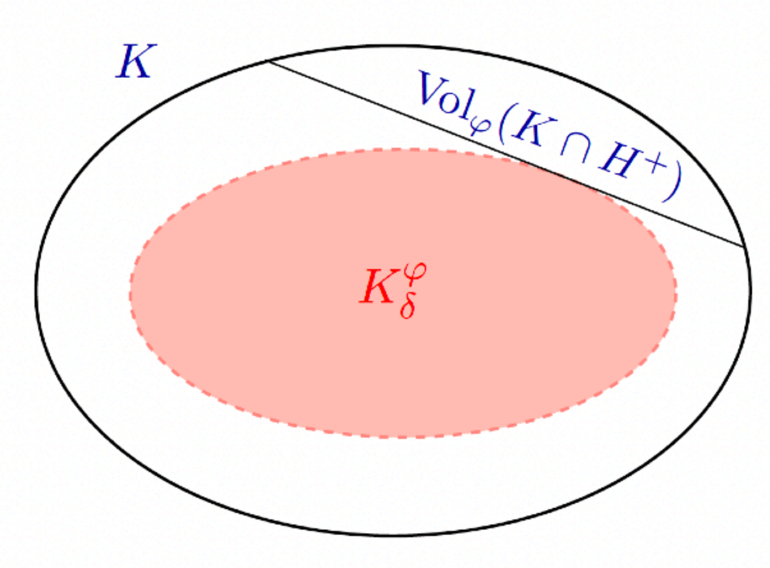
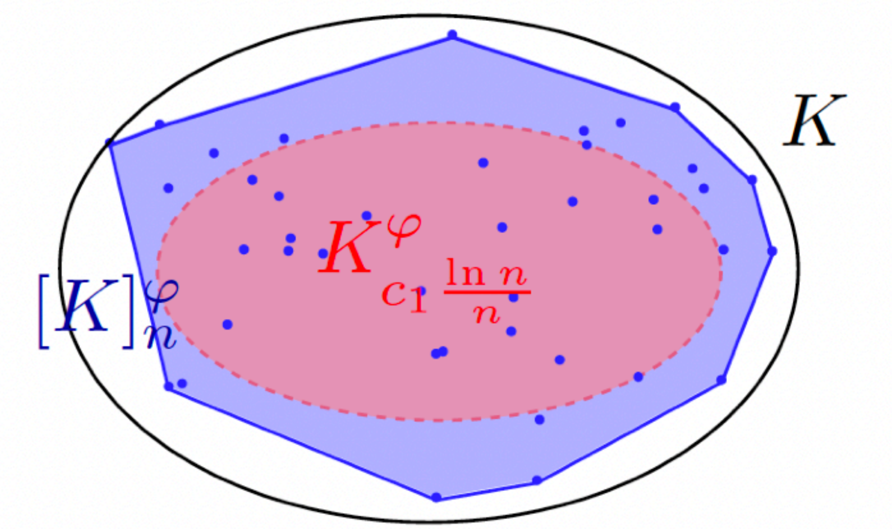



$[K]_n^\psi$ contains $K^\varphi_{c_1 \frac{\log n}{n}}$ with probability $1 - c_2 n^{-7}$.

Vu (2005)



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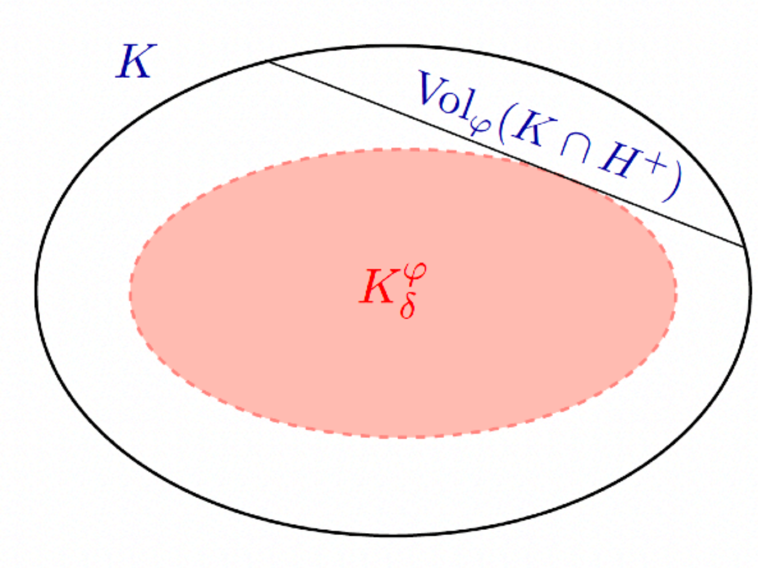


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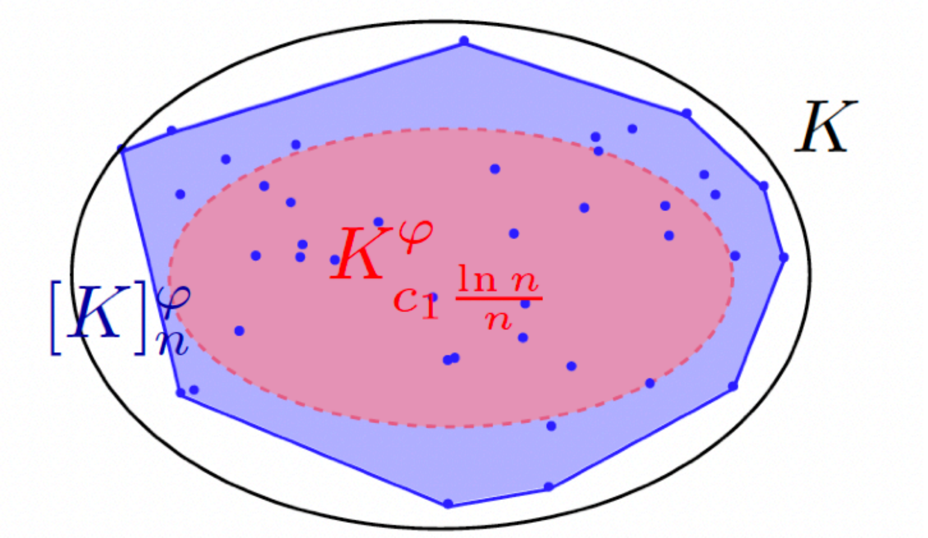
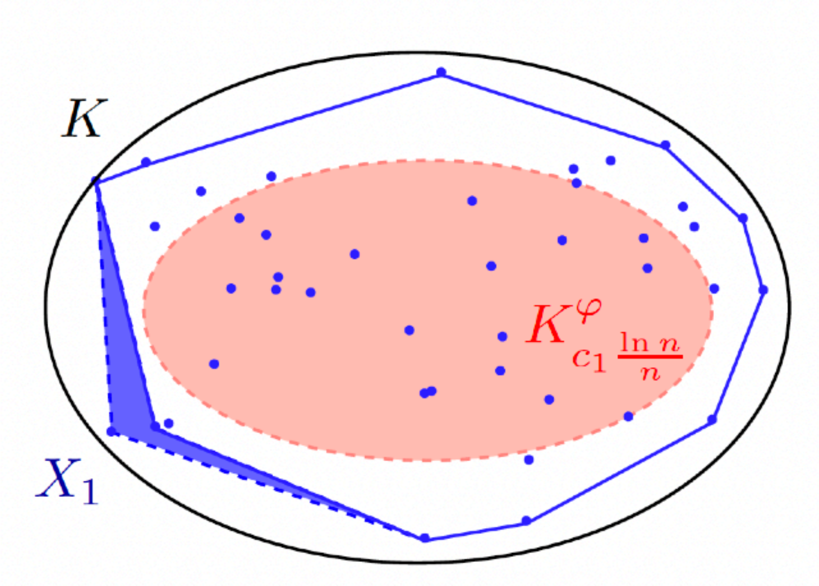



$$\gamma_4 = \mathbb{E}\,|\,D_1 \mathrm{Vol}_\varphi([K]_n^\psi)\,|^4$$



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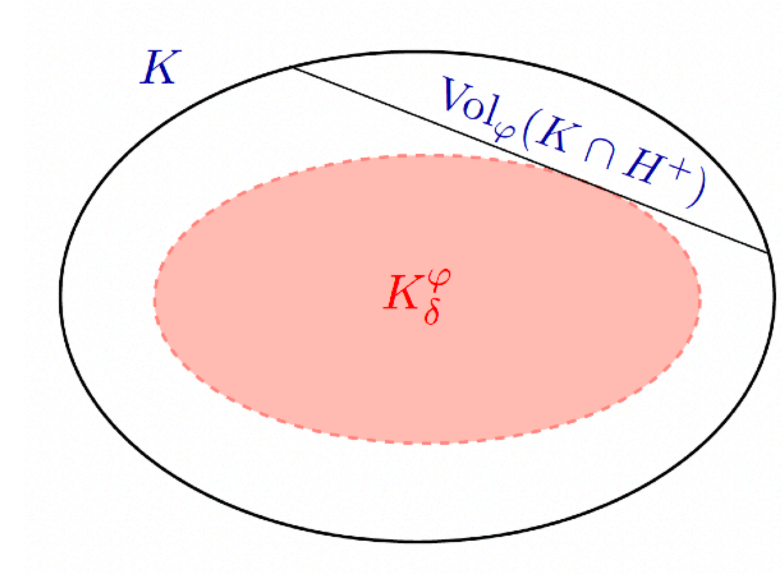


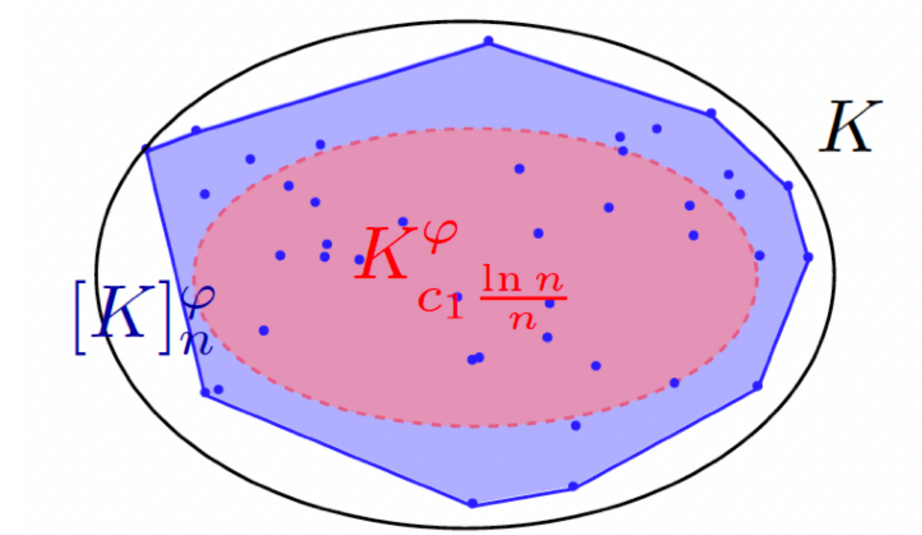


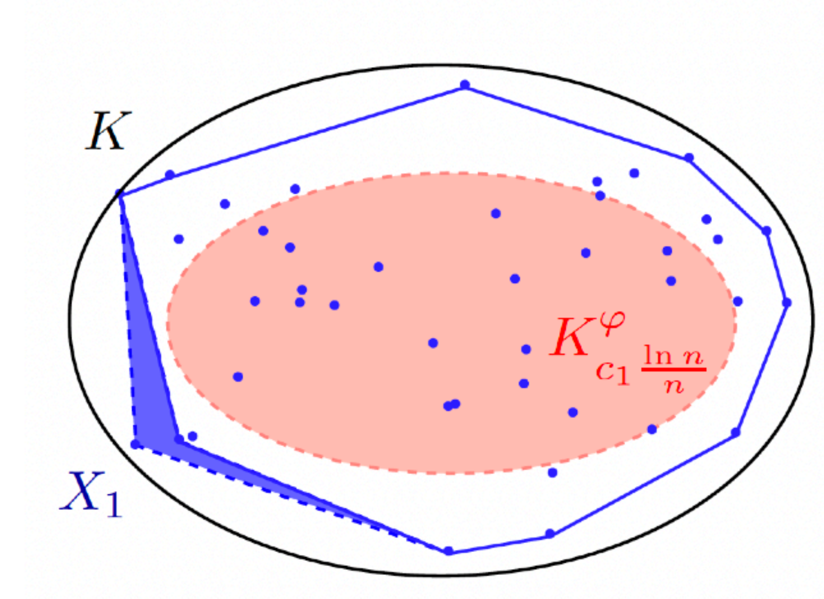

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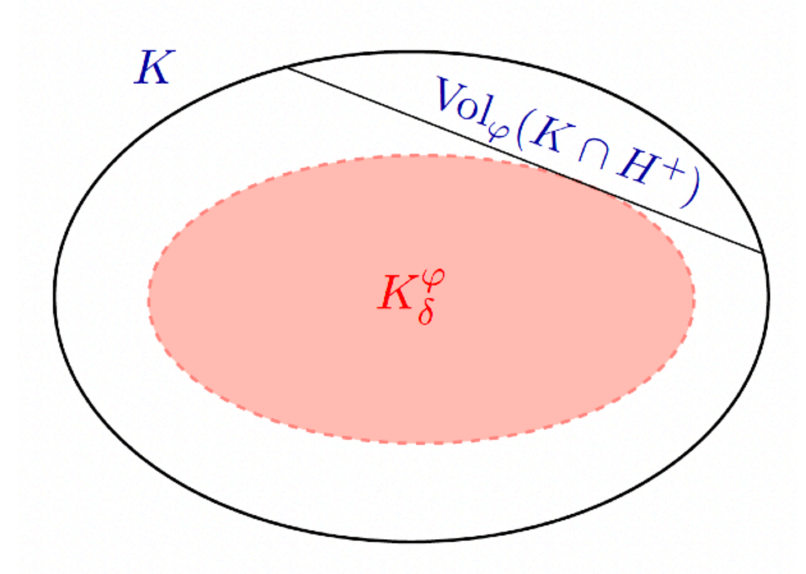


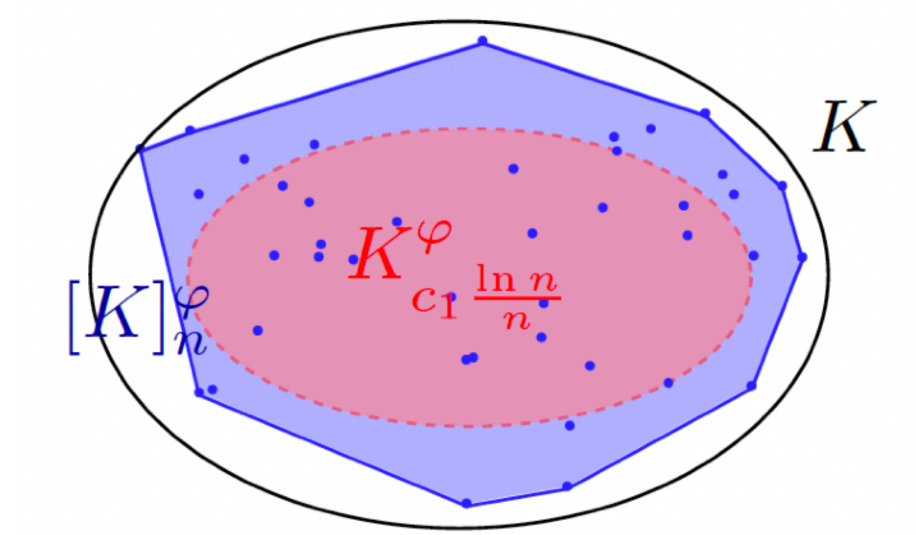


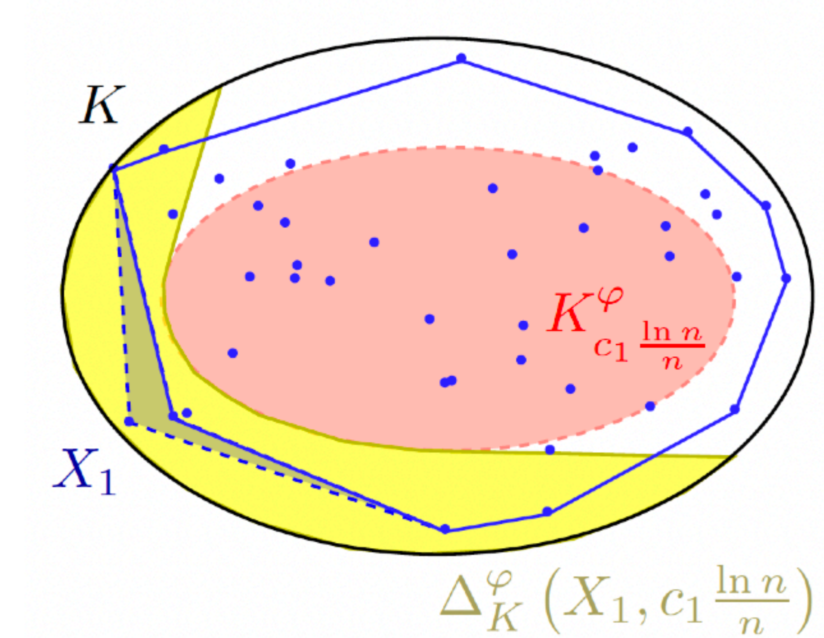

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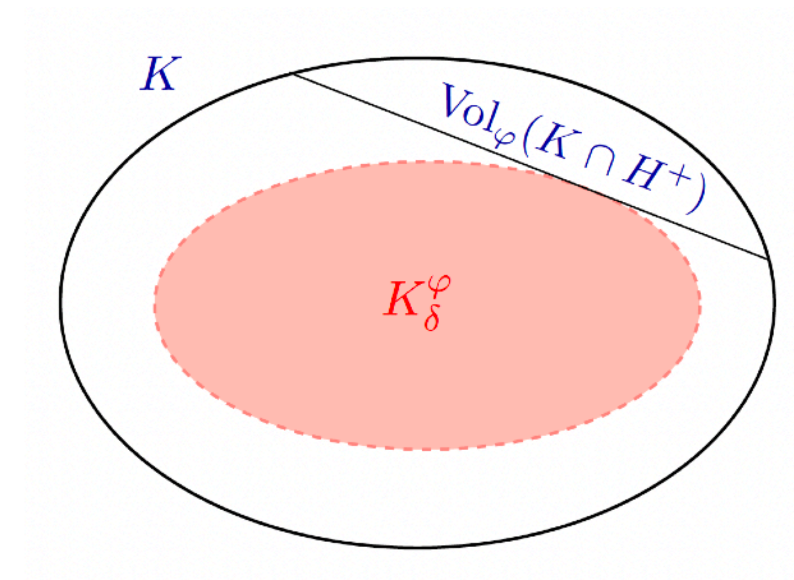


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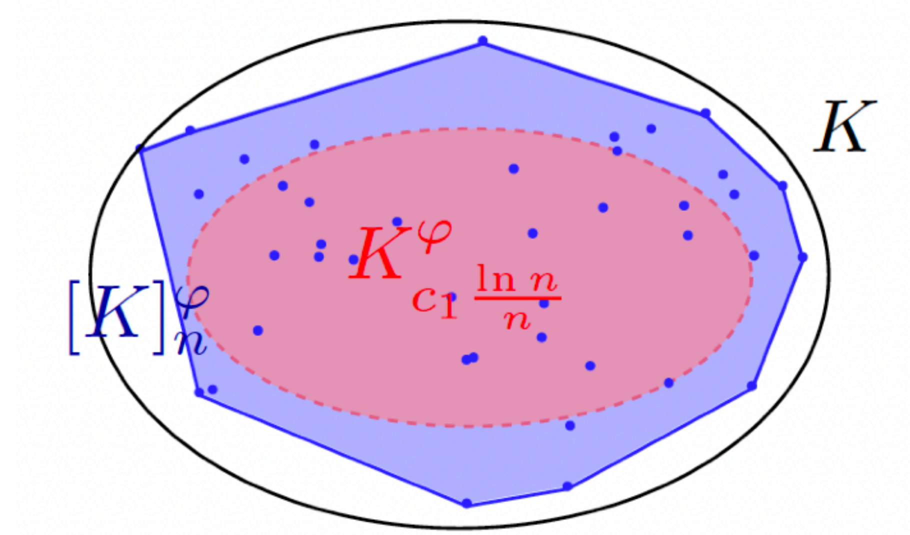


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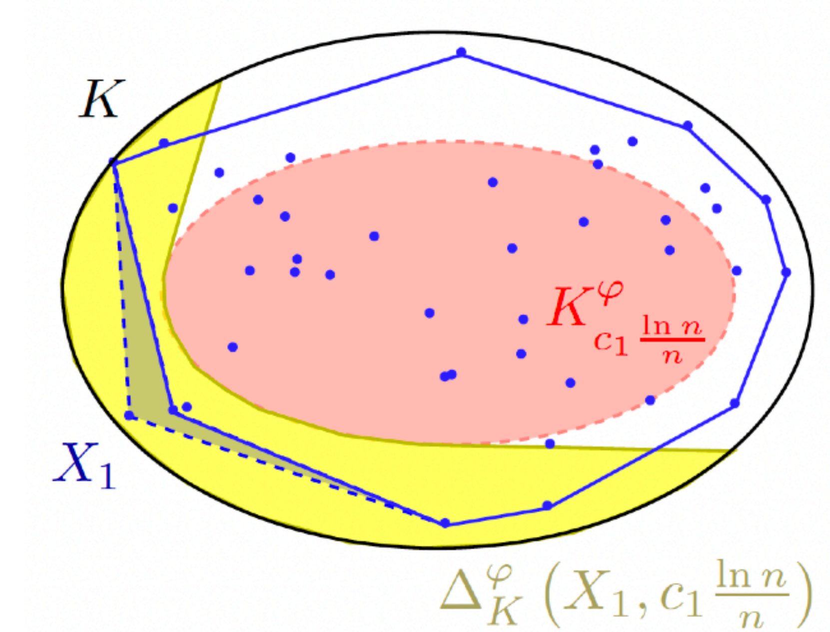
$$\le \mathbb{E}\mathrm{Vol}_\psi\Big(\Delta_K^\varphi\Big(X_1, c_1 \frac{\log n}{n}\Big)\Big)^4 + c_4 n^{-7}$$



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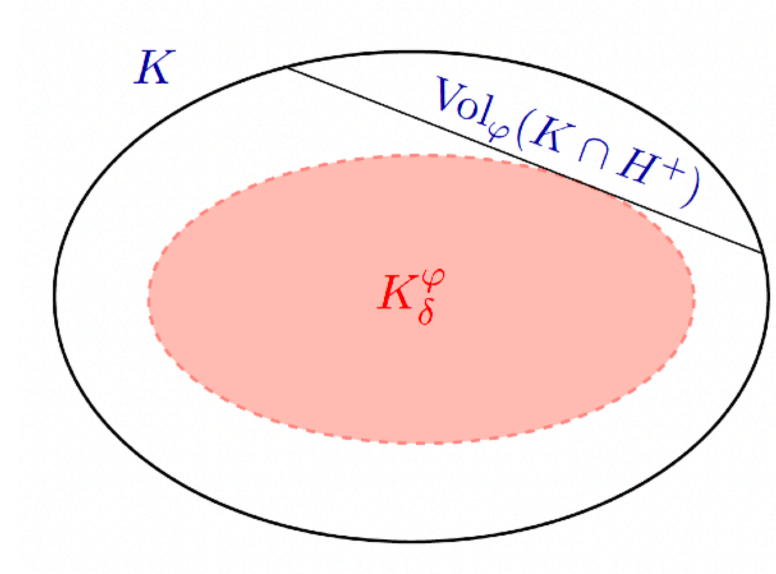


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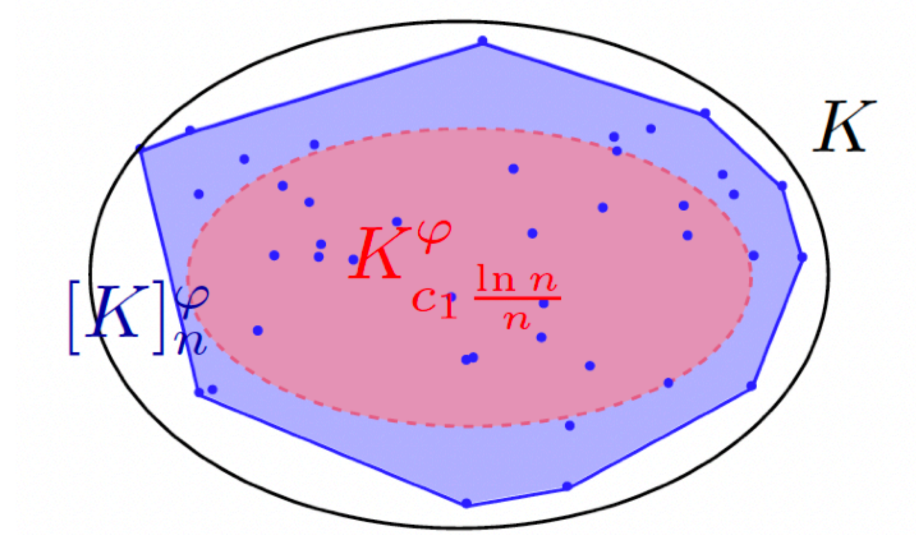
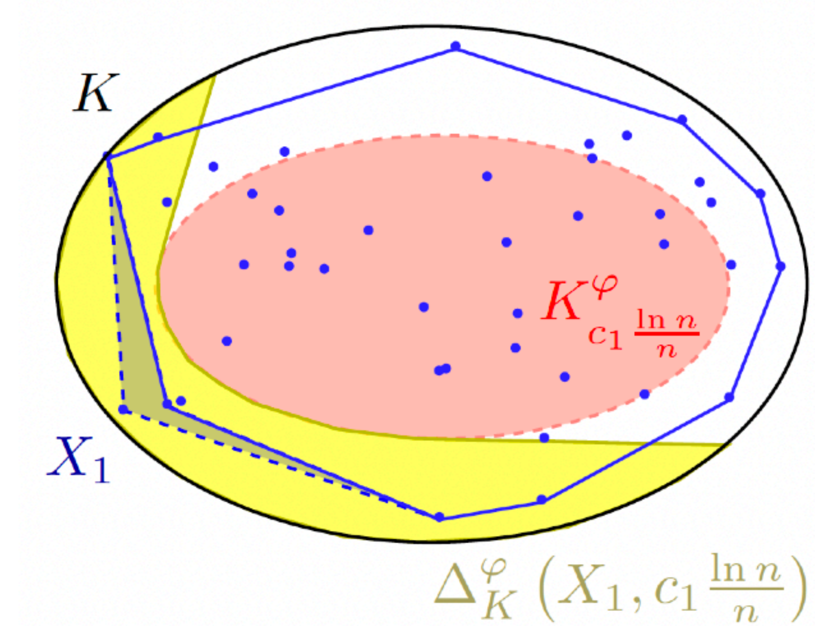



$$\begin{aligned}
\gamma_4 &\asymp \mathbb{E}\,|\,D_1 \mathrm{Vol}_\varphi([K]_n^\psi)\,|^4 \\
&\le \mathbb{E}[\,|\,D_1 \mathrm{Vol}_\varphi([K]_n^\psi)\,|^4\,|\,A_n] + c_3 \mathbb{P}(A_n^c) \\
&\le \mathbb{E}\mathrm{Vol}_\psi\Big(\Delta_K^\varphi\Big(X_1, c_1 \frac{\log n}{n}\Big)\Big)^4 + c_4 n^{-7} \\
&\le c_5 \Big(\frac{\log n}{n}\Big)^4 \mathrm{Vol}_\psi(K \backslash K^\varphi_{c_1 \frac{\log n}{n}}) + c_4 n^{-7}
\end{aligned}$$

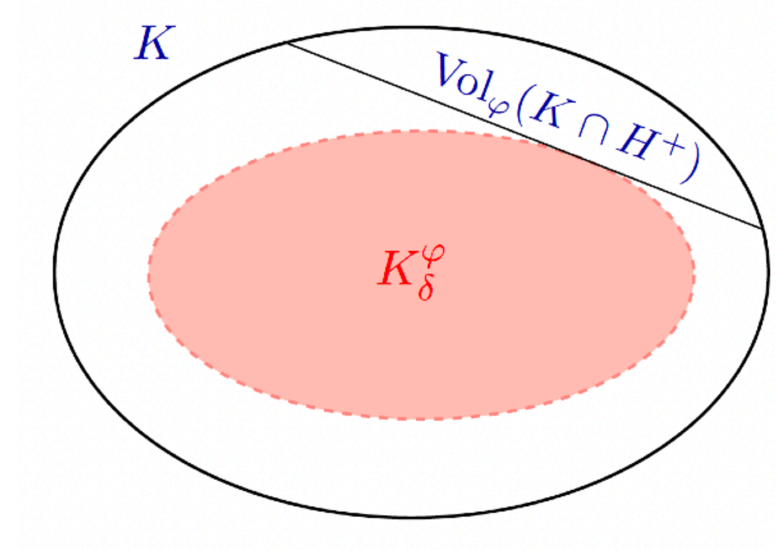


## Weighted floating bodies

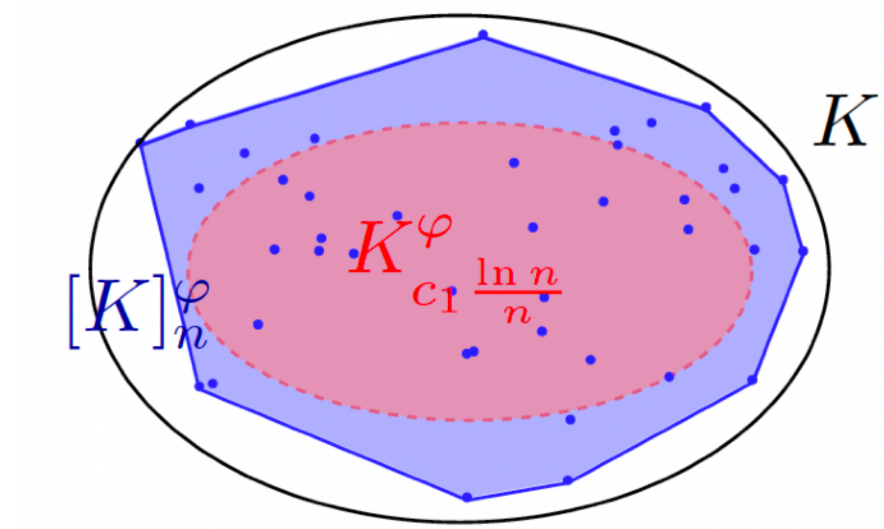
The weighted floating body of $K$ is defined by

$$K_\delta^\varphi = \bigcap \{H^- : \mathrm{Vol}_\varphi(K \cap H^+) \leq \delta\}$$

Schütt & Werner (1990), Werner (2002)




$[K]_n^\psi$ contains $K^\varphi_{c_1 \frac{\log n}{n}}$ with probability $1 - c_2 n^{-7}$. Vu (2005)



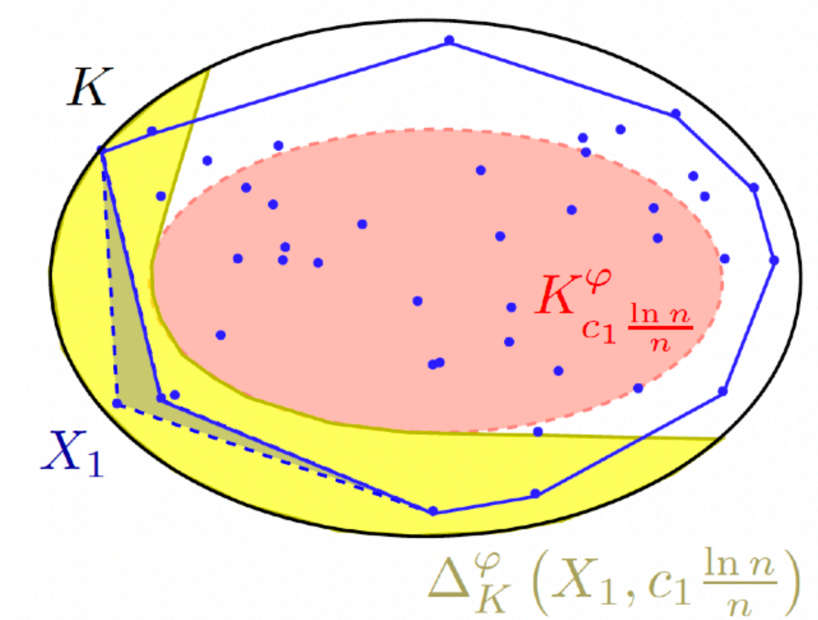

$$\gamma_4 \asymp \mathbb{E}\,|\,D_1 \mathrm{Vol}_\varphi([K]_n^\psi)\,|^4$$

$$\leq \mathbb{E}[\,|\,D_1 \mathrm{Vol}_\varphi([K]_n^\psi)\,|^4\,|\,A_n] + c_3 \mathbb{P}(A_n^c)$$

$$\leq \mathbb{E}\mathrm{Vol}_\psi\left(\Delta_K^\varphi\left(X_1, c_1 \frac{\log n}{n}\right)\right)^4 + c_4 n^{-7}$$

$$\leq c_5 \left(\frac{\log n}{n}\right)^4 \mathrm{Vol}_\psi(K \backslash K^\varphi_{c_1 \frac{\log n}{n}}) + c_4 n^{-7}$$

$$\leq c_6 \left(\frac{\log n}{n}\right)^{4 + \frac{2}{d+1}}$$



# Random polytopes in hyperbolic space

**Theorem**

Suppose $K$ is of class $C^2_+$. Then

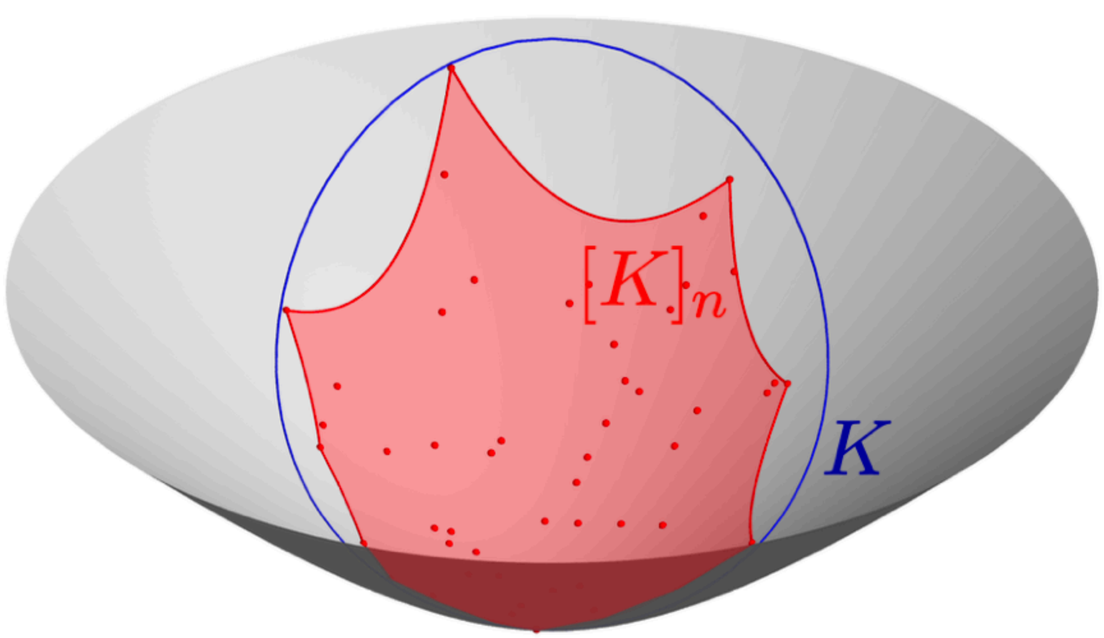
$$\mathbb{E}\mathrm{Vol}_h([K]_n) = \mathrm{Vol}_h(K) - c(d)\mathrm{as}_h(K)\left(\frac{\mathrm{Vol}_h(K)}{n}\right)^{\frac{2}{d+1}}(1 + o_n(1))$$

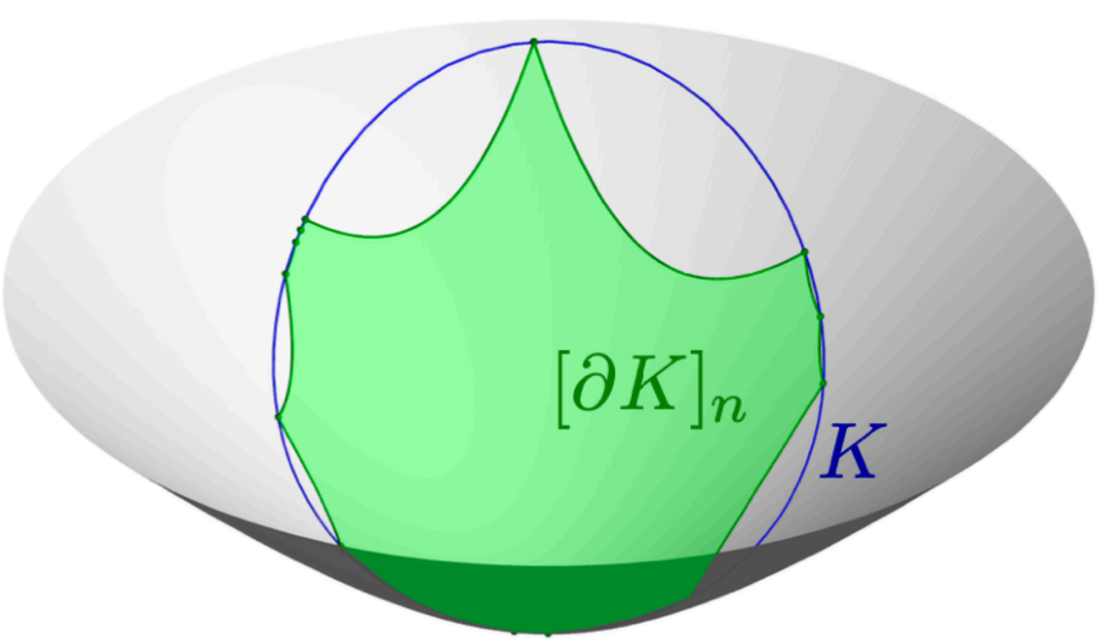
$$\mathrm{Var}\,\mathrm{Vol}_h([K]_n) \geq c(K)n^{-1-\frac{2}{d+1}}$$

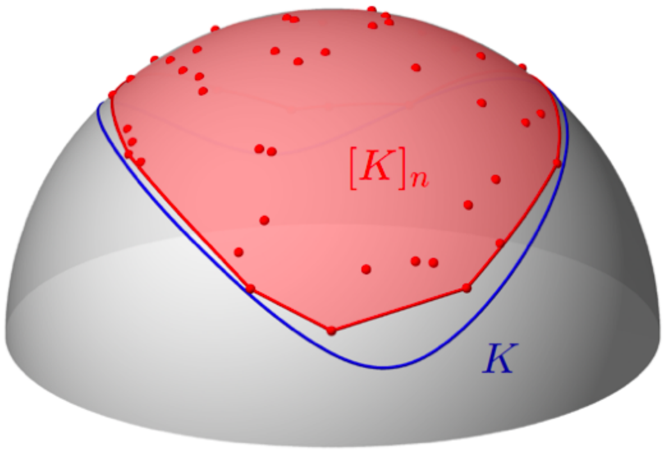
$$\frac{\mathrm{Vol}_h([K]_n) - \mathbb{E}\mathrm{Vol}_h([K]_n)}{\sqrt{\mathrm{Var}\,\mathrm{Vol}_h([K]_n)}} \xrightarrow{D} Z \sim \mathcal{N}(0,1)$$

Besau, Ludwig, Werner (2018)

Besau & Thäle (2020)

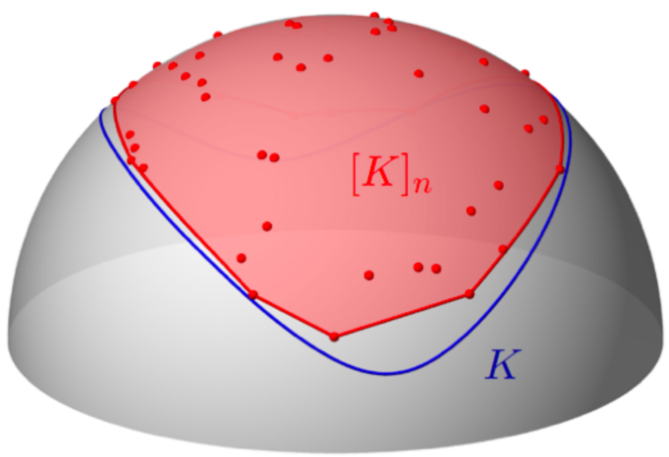





Suppose $K$ is of class $C^2_+$.

$$\mathbb{E}\mathrm{Vol}_s([K]_n) = \mathrm{Vol}_s(K) - c(d,K)\left(\frac{\mathrm{Vol}_s(K)}{n}\right)^{\frac{2}{d+1}}(1 + o_n(1))$$

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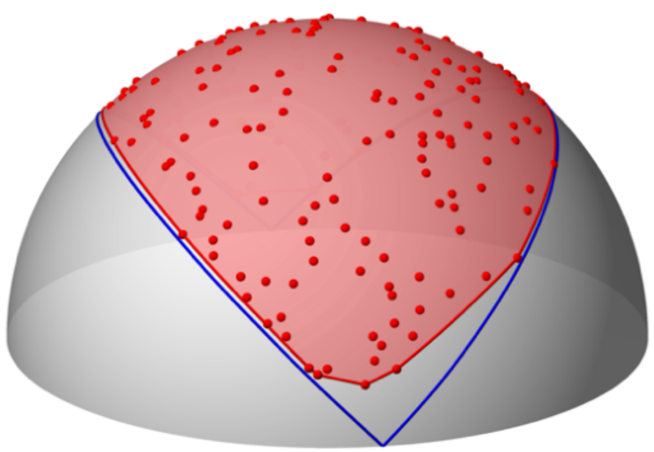
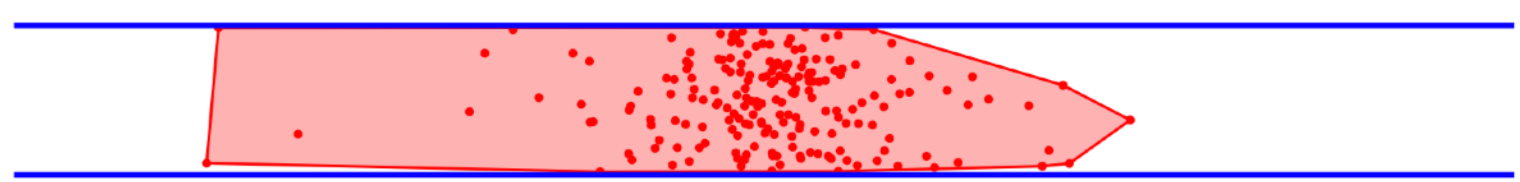


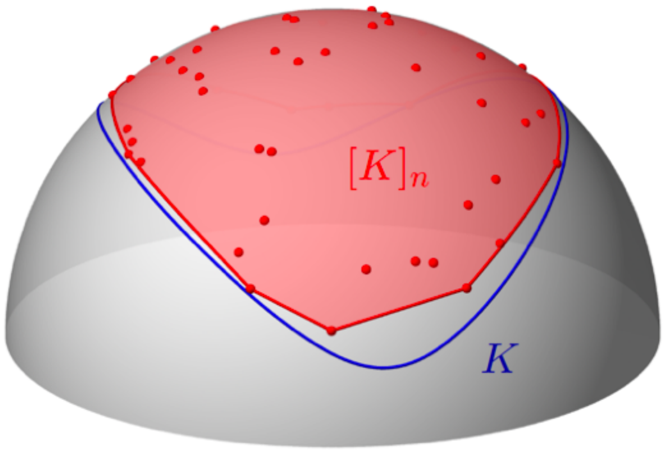
… random polytopes in a spherical wedge

$$\mathbb{E}\mathrm{Vol}_s([K]_n) = \mathrm{Vol}_s(K) - c(d)\frac{\log n}{n}(1 + o_n(1))$$

Besau, Gusakova, Reitzner, Schütt, T. & Werner (2022)

In this case $g(K) = \text{a strip}$ and $g\sharp\mathrm{Vol}_s$ has some density.



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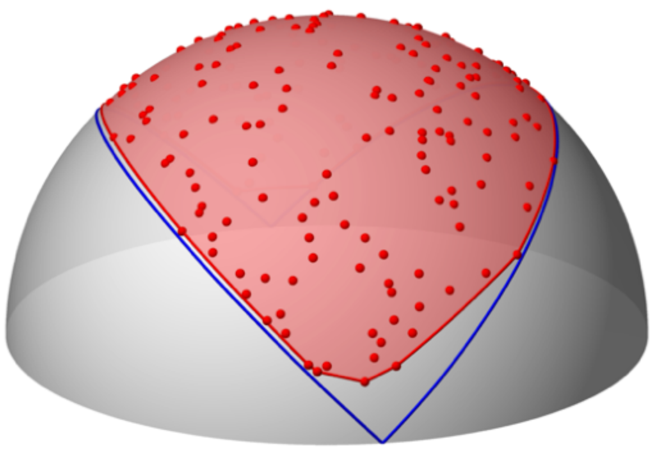

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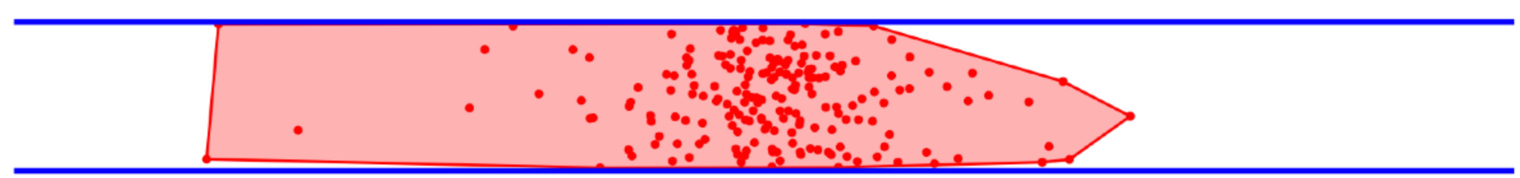

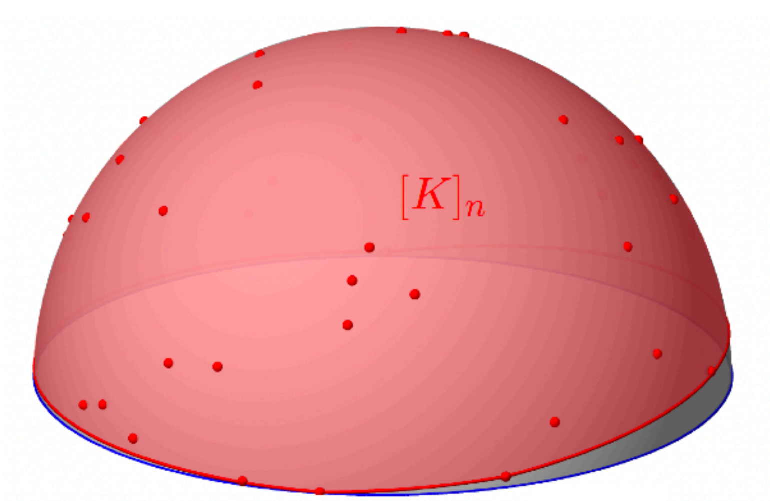


Suppose $K$ is the half-sphere.

$$\mathbb{E}\mathrm{Vol}_s([K]_n) = \mathrm{Vol}_s(K) - c(d)n^{-1}(1+o_n(1))$$

Bárány, Hug, Reitzner & Schneider (2017)

In this case $g(K) = \mathbb{R}^d$ and $g\sharp\mathrm{Vol}_s$ has density $\propto (1-\|x\|^2)^{-\frac{d+1}{2}}$.

- Beta distribution in $\mathbb{R}^d$ : $f_{d,\beta}(x) = c_{d,\beta}(1 - \|x\|^2)^\beta \qquad (\|x\| \leq 1, \beta > -1)$
- Beta polytope: $P_{n,d}^\beta = [X_1, \ldots, X_n]$

  $X_1, \ldots, X_n$ iid with density $f_{d,\beta}$

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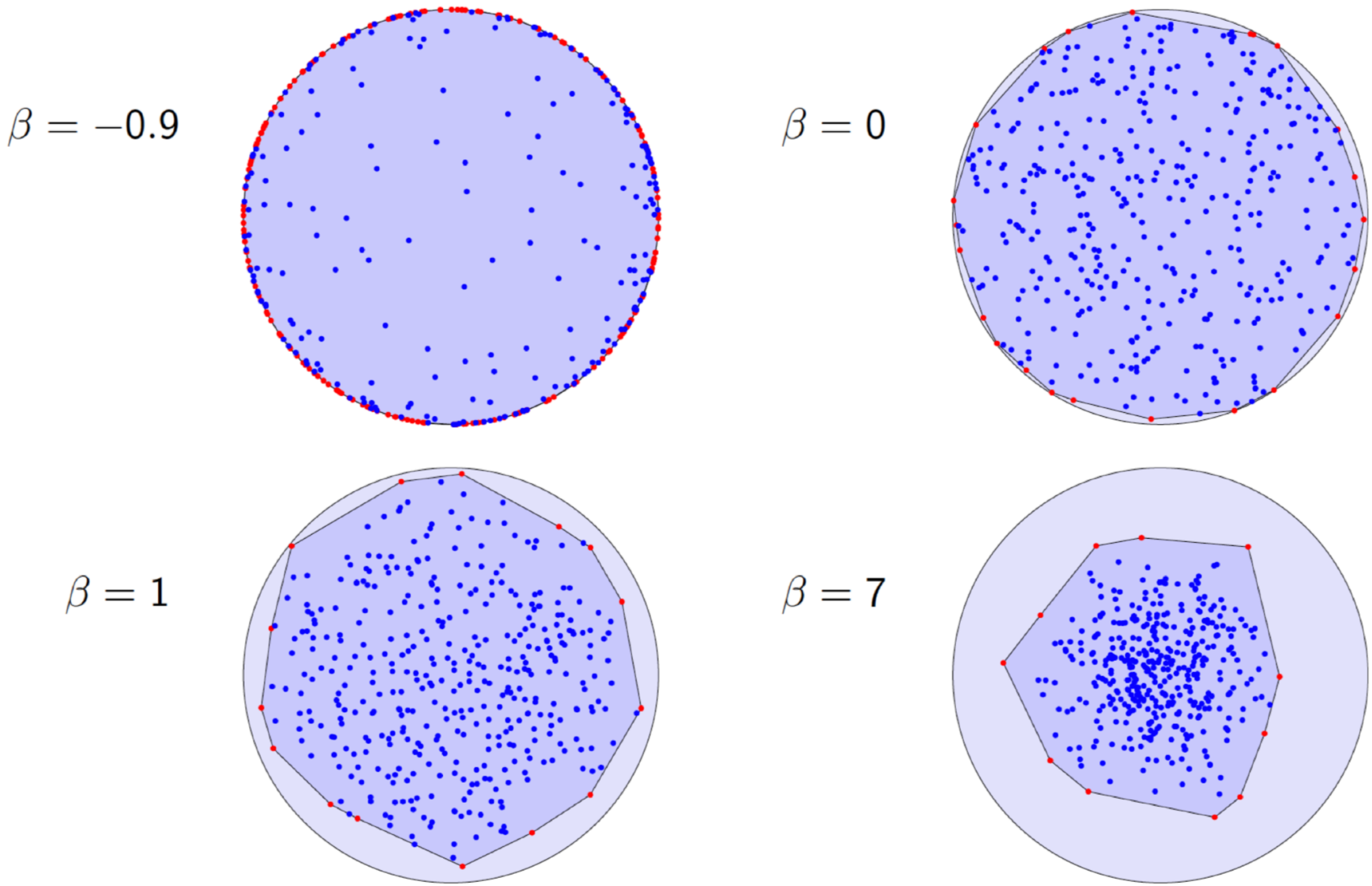

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- Special cases: $\beta \to -1$ uniform distribution on $\mathbb{S}^{d-1}$

  There are weak sub-sequential limits by Helly's compactness theorem.
  All sub-sequential limits must be concentrated on $\mathbb{S}^{d-1}$.
  All sub-sequential limits must be rotationally invariant.

  $\beta = 0$ uniform distribution on $\mathbb{B}^d$

  $\beta \to \infty$ Gaussian distribution on $\mathbb{R}^d$

$$X \sim f_{d,\beta} \Longrightarrow \sqrt{2\beta}\, X \sim c_{d,\beta}\left(1 - \frac{\|x\|^2}{2\beta}\right)^\beta \longrightarrow ce^{-\|x\|^2/2}$$

  Now, apply Scheffé's lemma.

- Beta distribution in $\mathbb{R}^d$ : $f_{d,\beta}(x) = c_{d,\beta}(1 - \|x\|^2)^\beta \qquad (\|x\| \le 1, \beta > -1)$

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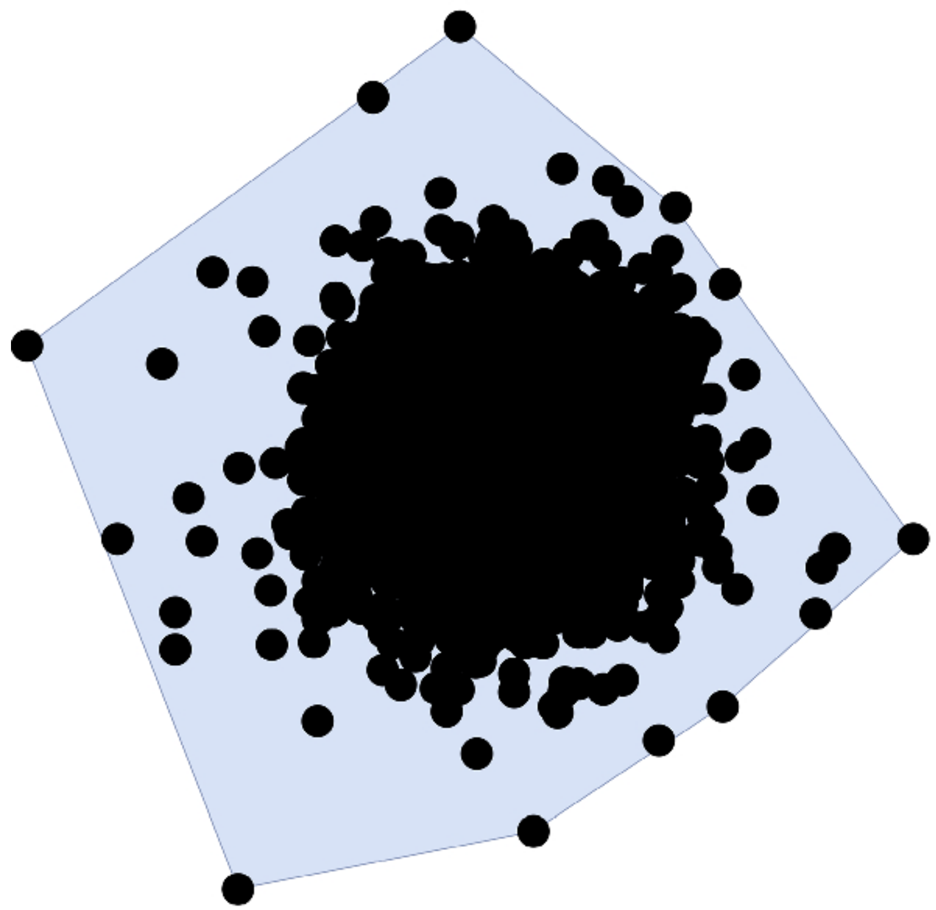

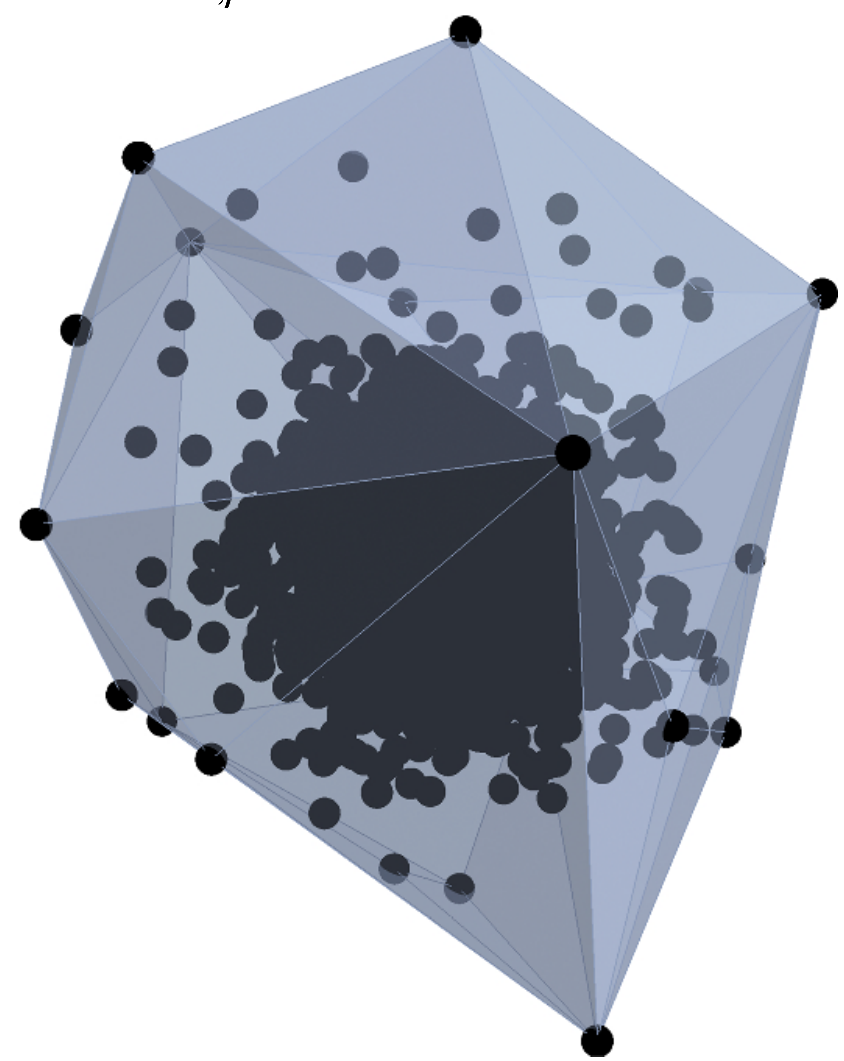

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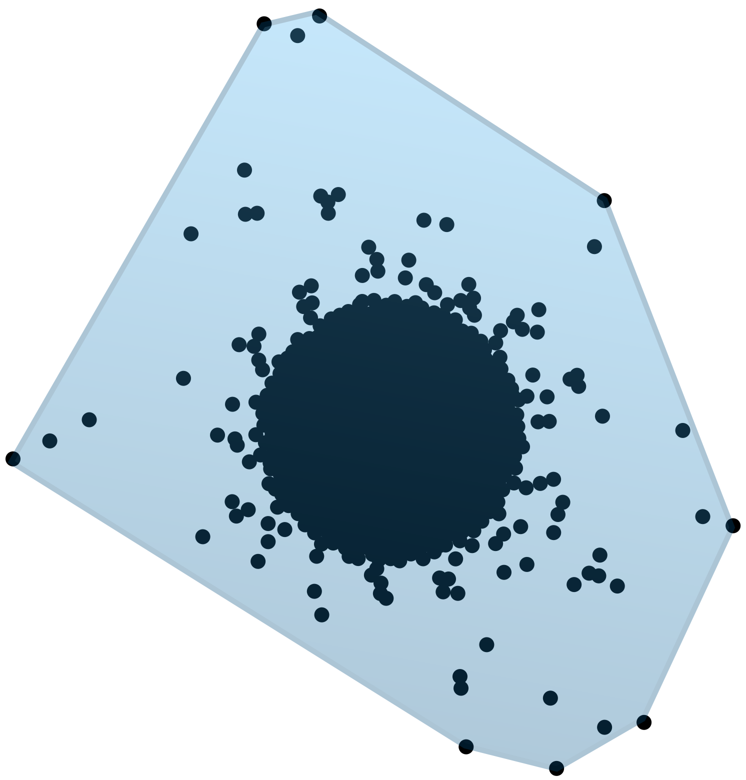
- Poisson polytope:

$$\eta = \mathrm{PPP}\left(\frac{\mathrm{d}x}{\|x\|^{d+\alpha}}\right), \qquad \alpha > 0$$

$$\Pi_{d,\alpha} = [\eta]$$

Poisson polytopes are rescaled limits of beta prime polytopes:

$$n^{-1}\tilde{P}_{n,d}^\beta \longrightarrow \Pi_{d,2\beta-d}$$



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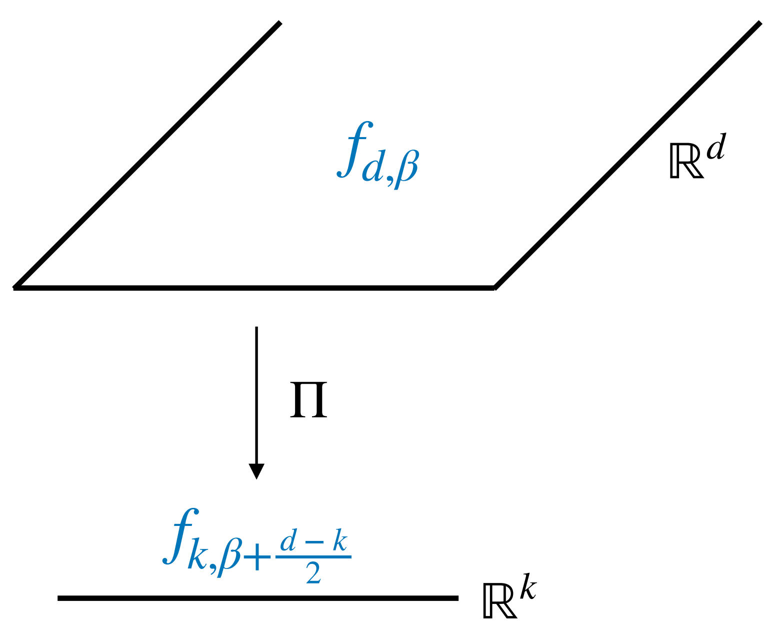
- Key properties:

Projection invariance



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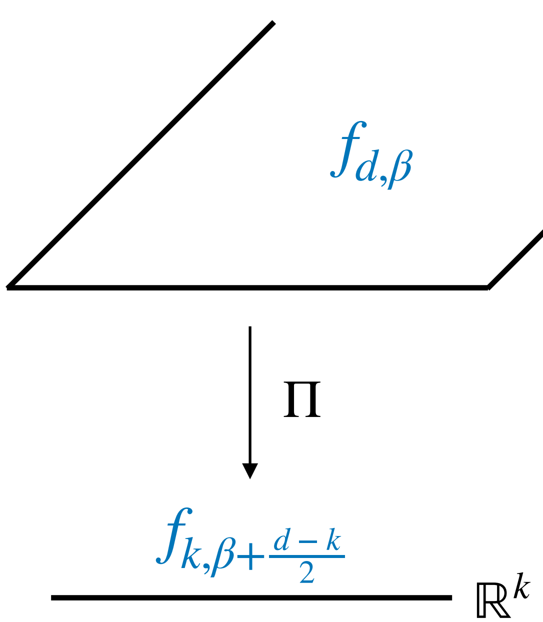
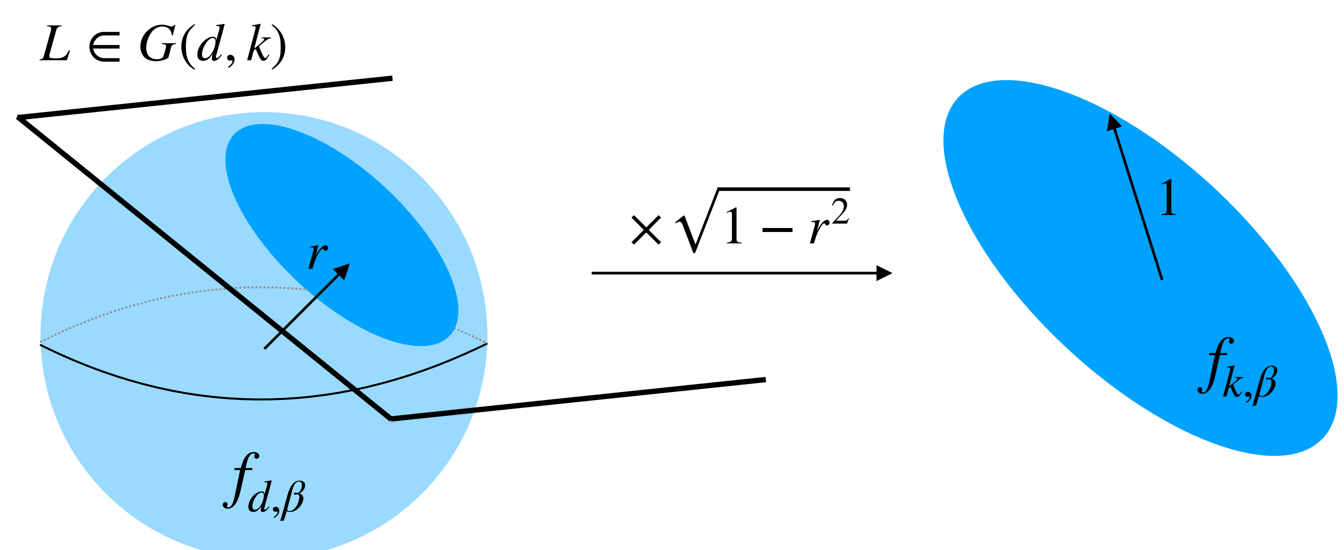



Slice invariance



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$\mathbb{E} f_k(P^\beta_{n,d}), \quad \mathbb{E} f_k(\tilde{P}^\beta_{n,d})$ — Explicit formulas

$\mathbb{E} V_k(P^\beta_{n,d}), \quad \mathbb{E} V_k(\tilde{P}^\beta_{n,d})$ — Asymptotic expansions

Angles of tangent cones — Threshold phenomena

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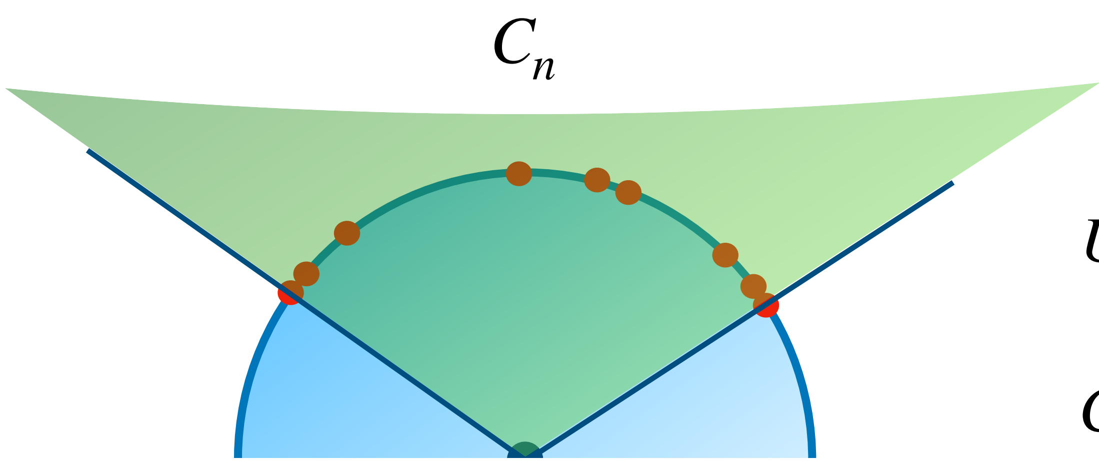
**Application to stochastic geometry models**

# Application 1: Random cones in a half-sphere

Kabluchko, Marynych, Temesvari & T. (2019)




$$\mathbb{S}^d_+ := \{(x_0, x_1, \ldots, x_d) \in \mathbb{R}^{d+1} : x_0^2 + x_1^2 + \ldots + x_d^2 = 1, x_0 \geq 0\}$$

upper half-sphere in $\mathbb{R}^{d+1}$

$U_1, U_2, \ldots$ iid uniform random points on $\mathbb{S}^d_+$

$$C_n := \mathrm{pos}(U_1, \ldots, U_n) = \left\{ \sum_{i=1}^{n} \lambda_i U_i : \lambda_i \geq 0 \right\}$$

random cone generated by $U_1, \ldots, U_n$

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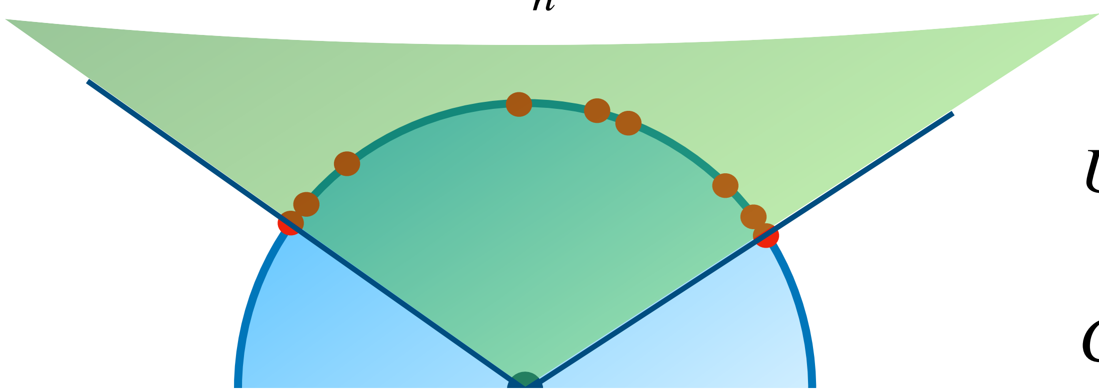


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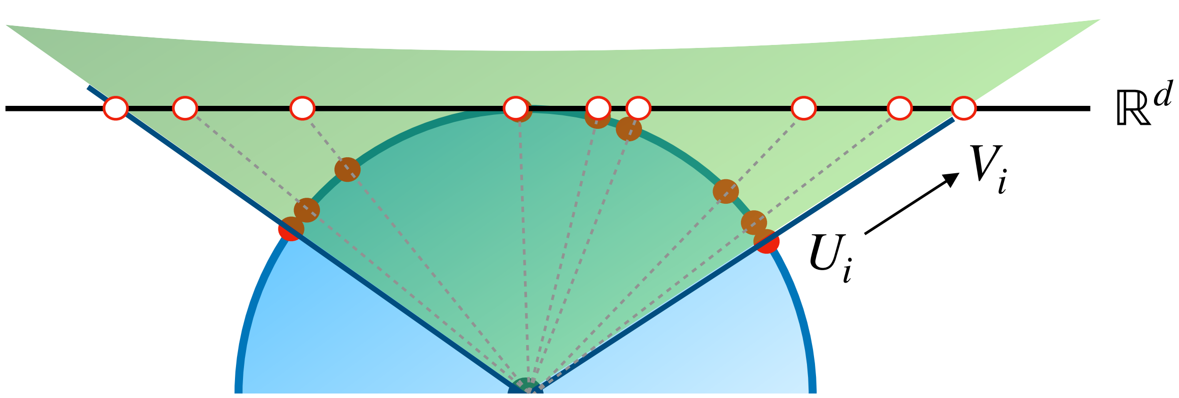


$$(x_0, x_1, \ldots, x_d) \mapsto \left(\frac{x_1}{x_0}, \ldots, \frac{x_d}{x_0}\right)$$

$$U_i \mapsto V_i$$

Density of $V_i$ : $\dfrac{c_d}{(1 + \|x\|^2)^{\frac{d+1}{2}}} = \tilde{f}_{d,\frac{d+1}{2}}$

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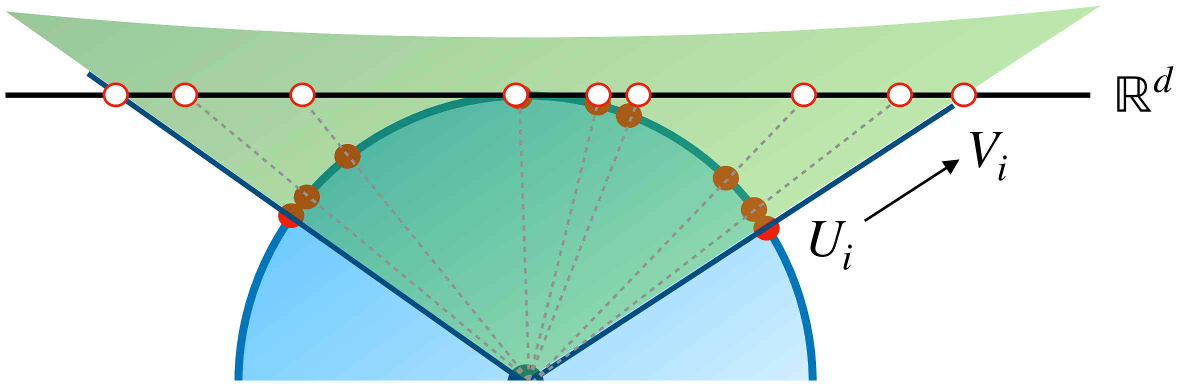


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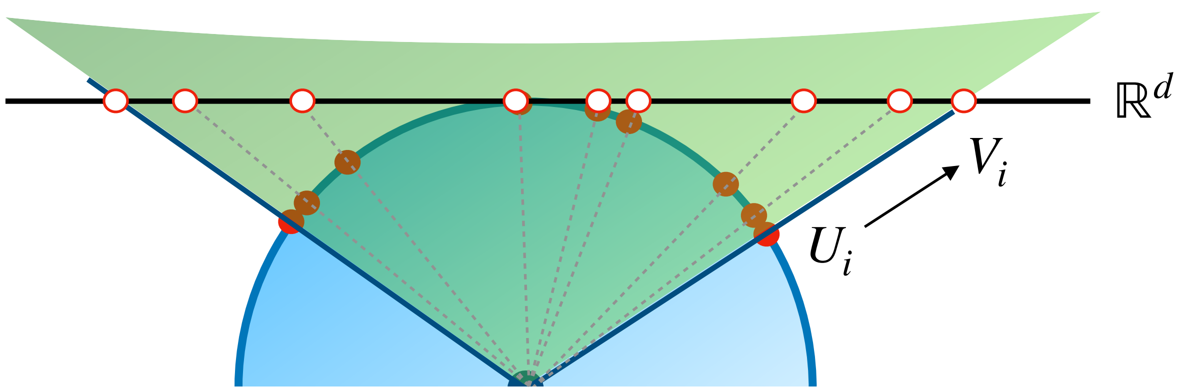


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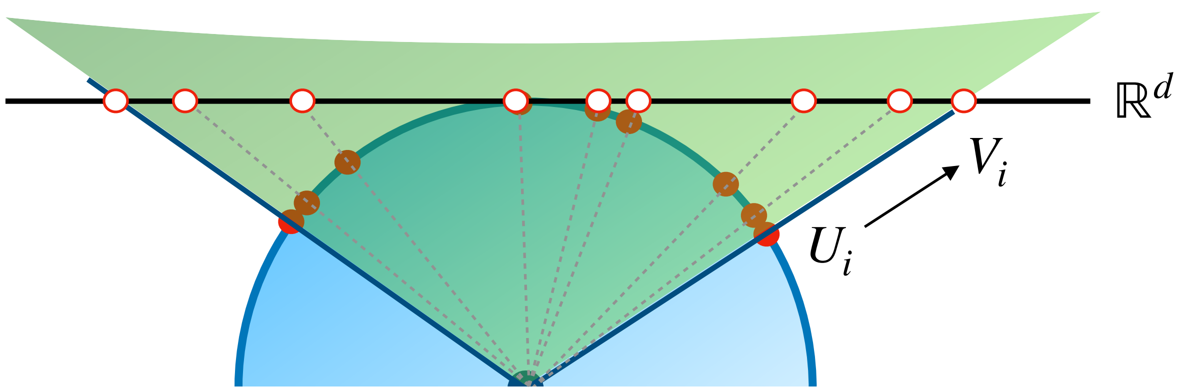
Density of $\dfrac{V_i}{n}$ : $\quad c_d \dfrac{n^d}{(1+\|nx\|^2)^{\frac{d+1}{2}}} = c_d \dfrac{n^d}{n^{d+1}(\frac{1}{n^2}+\|x\|^2)^{\frac{d+1}{2}}} \sim \dfrac{1}{n} \dfrac{c_d}{\|x\|^{d+1}}$

## Application 1: Random cones in a half-sphere

Kabluchko, Marynych, Temesvari & T. (2019)

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Take $B \subset \mathbb{R}^d \backslash \{0\}$ : $\displaystyle\sum_{i=1}^{n} \mathbf{1}\left\{\frac{V_i}{n} \in B\right\} \sim \mathrm{Bin}\left(n, \frac{c_d}{n}\int_B \frac{dx}{\|x\|^{d+1}}\right) \xrightarrow{d} \mathrm{Po}\left(c_d \int_B \frac{dx}{\|x\|^{d+1}}\right)$

# Application 1: Random cones in a half-sphere

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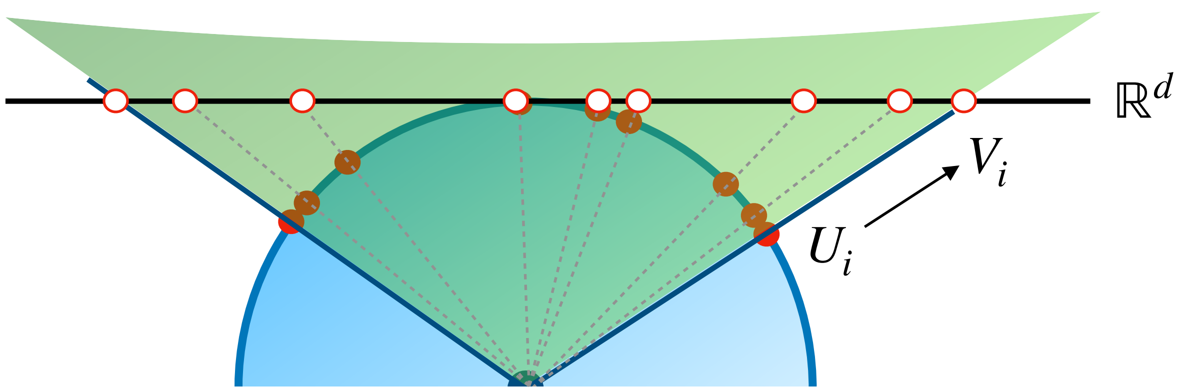


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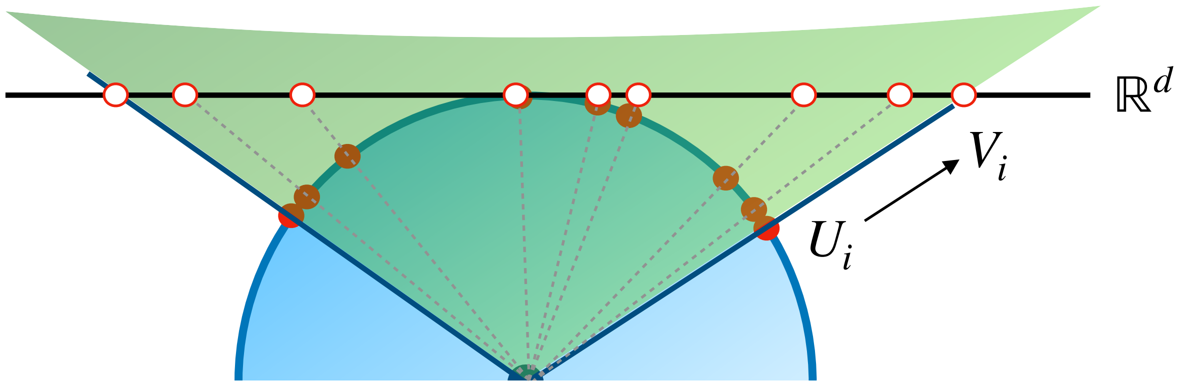
Density of $\dfrac{V_i}{n}$ : $c_d \dfrac{n^d}{(1+\|nx\|^2)^{\frac{d+1}{2}}} = c_d \dfrac{n^d}{n^{d+1}(\frac{1}{n^2}+\|x\|^2)^{\frac{d+1}{2}}} \sim \dfrac{1}{n}\dfrac{c_d}{\|x\|^{d+1}}$

Take $B \subset \mathbb{R}^d \backslash \{0\}$ : $\displaystyle\sum_{i=1}^{n} \mathbf{1}\left\{\frac{V_i}{n} \in B\right\} \sim \mathrm{Bin}\left(n, \frac{c_d}{n}\int_B \frac{dx}{\|x\|^{d+1}}\right) \xrightarrow{d} \mathrm{Po}\left(c_d \int_B \frac{dx}{\|x\|^{d+1}}\right)$

$$\Longrightarrow \sum_{i=1}^{n} \delta_{\frac{V_i}{n}} \xrightarrow{w} \mathrm{PPP}\left(\frac{c_d\, dx}{\|x\|^{d+1}}\right)$$

# Application 1: Random cones in a half-sphere

Kabluchko, Marynych, Temesvari & T. (2019)




$$(x_0, x_1, \dots, x_d) \mapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_d}{x_0}\right)$$

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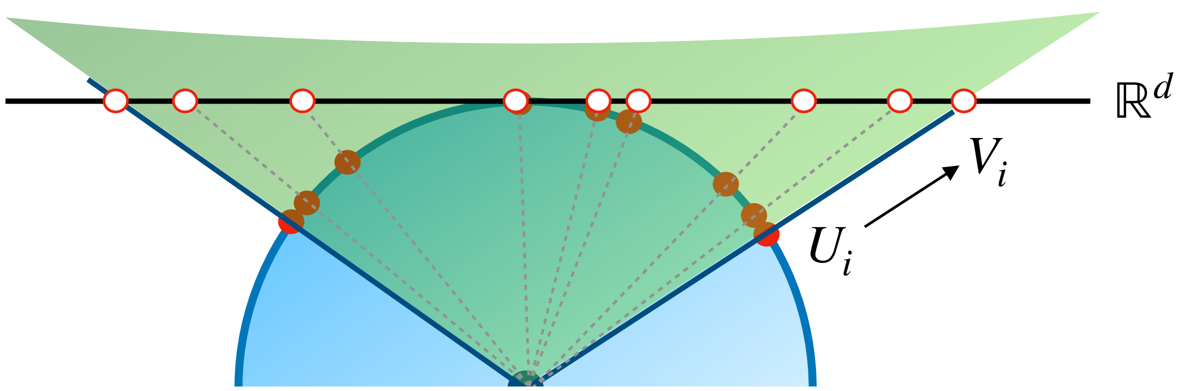


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Poisson polytope!

$$\Longrightarrow \mathbb{E}f_{k+1}(C_n) = \mathbb{E}f_k(n^{-1}[V_1, \ldots, V_n]) \longrightarrow \mathbb{E}f_k(\Pi_{d,1})$$

Example $d = 2$: $$\mathbb{E}f_0(C_n) \to \mathbb{E}f_0(\Pi_{d,1}) = \frac{\pi^2}{2}$$

# Application 2: Typical Voronoi cells on the sphere

Kabluchko & T. (2021)

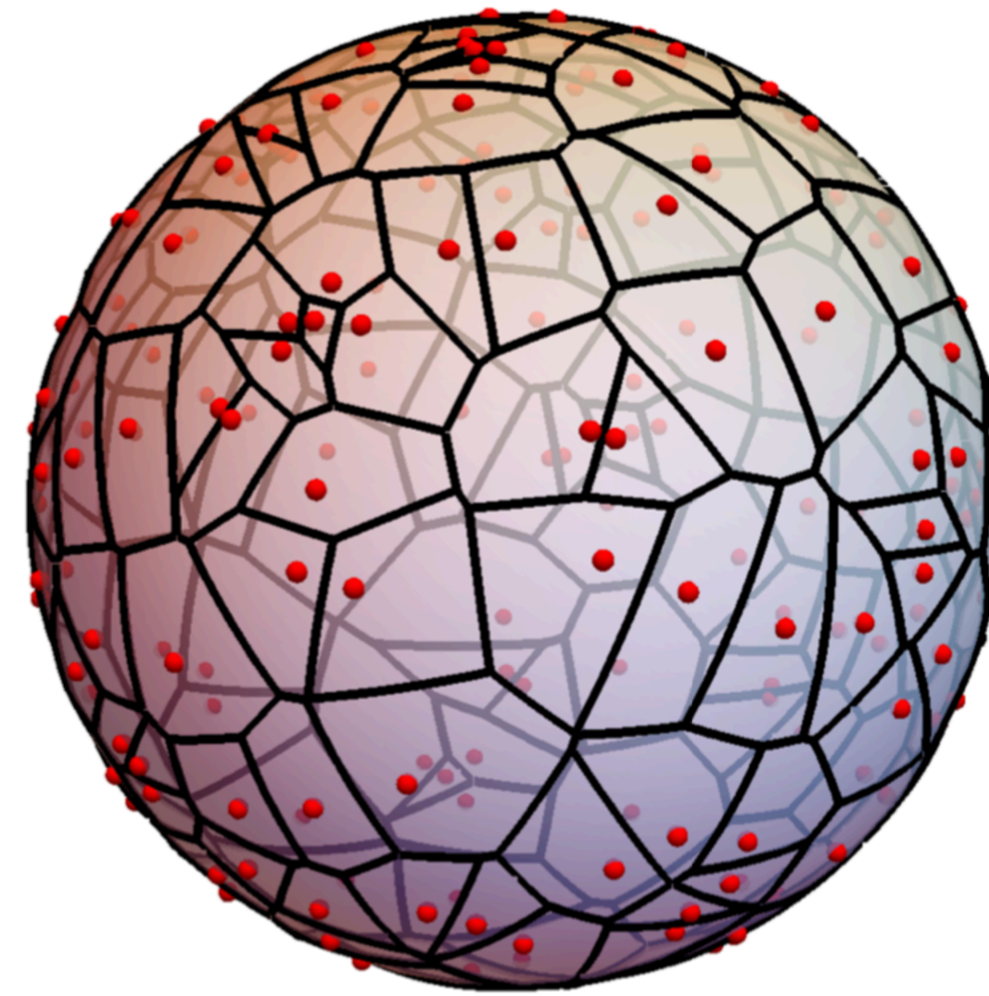
$X_1, \ldots, X_n$ iid uniform random points on $\mathbb{S}^d$

$C_i = \{z \in \mathbb{S}^d : d_g(z, X_i) \leq d_g(z, X_j) \ \forall j\}$

Voronoi cell of $X_i$

$\{C_1, \ldots, C_n\}$ Voronoi tessellation

$V_n$ Typical Voronoi cell = cell picked uniformly at random



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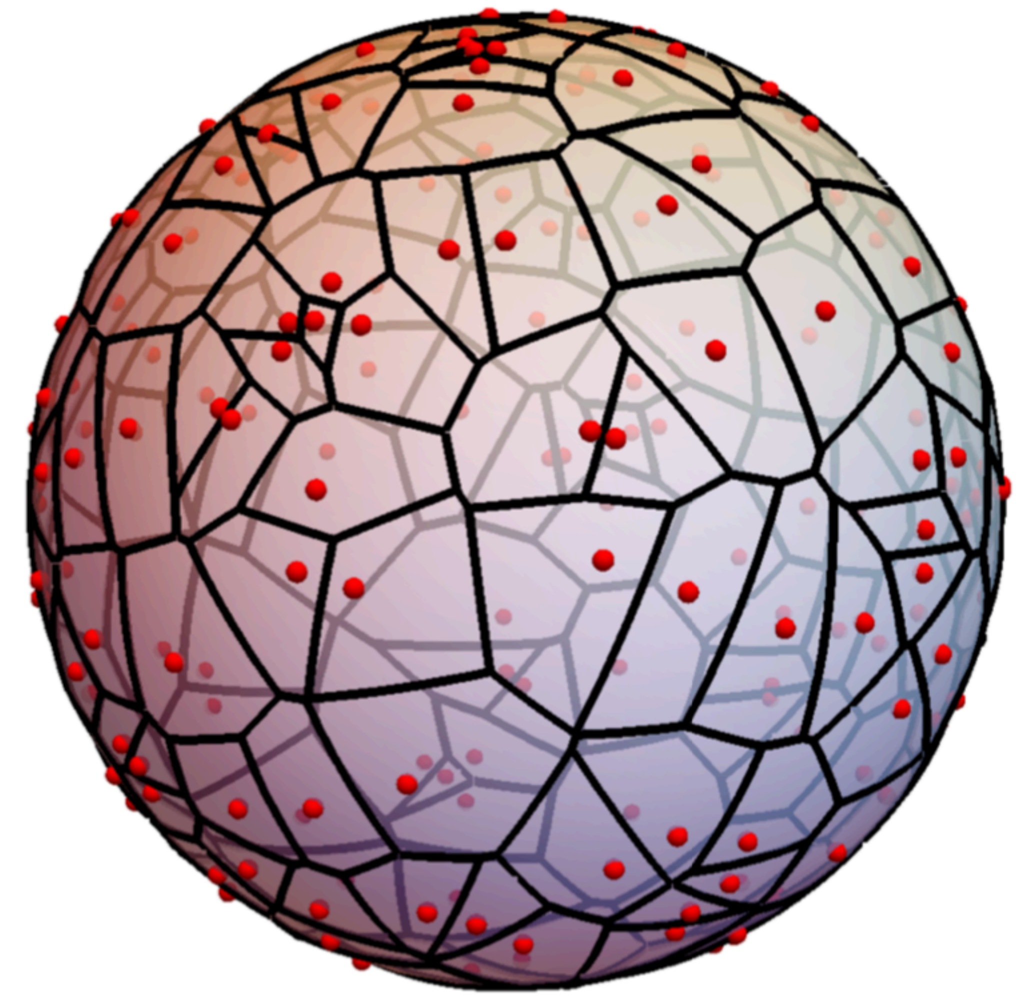

Theorem: $\mathbb{E} f_k(V_n)$ can be computed explicitly!

More precisely: $f_k(V_{n+1}) \stackrel{d}{=} f_{d-k-1}(\tilde{P}^d_{n,d})$

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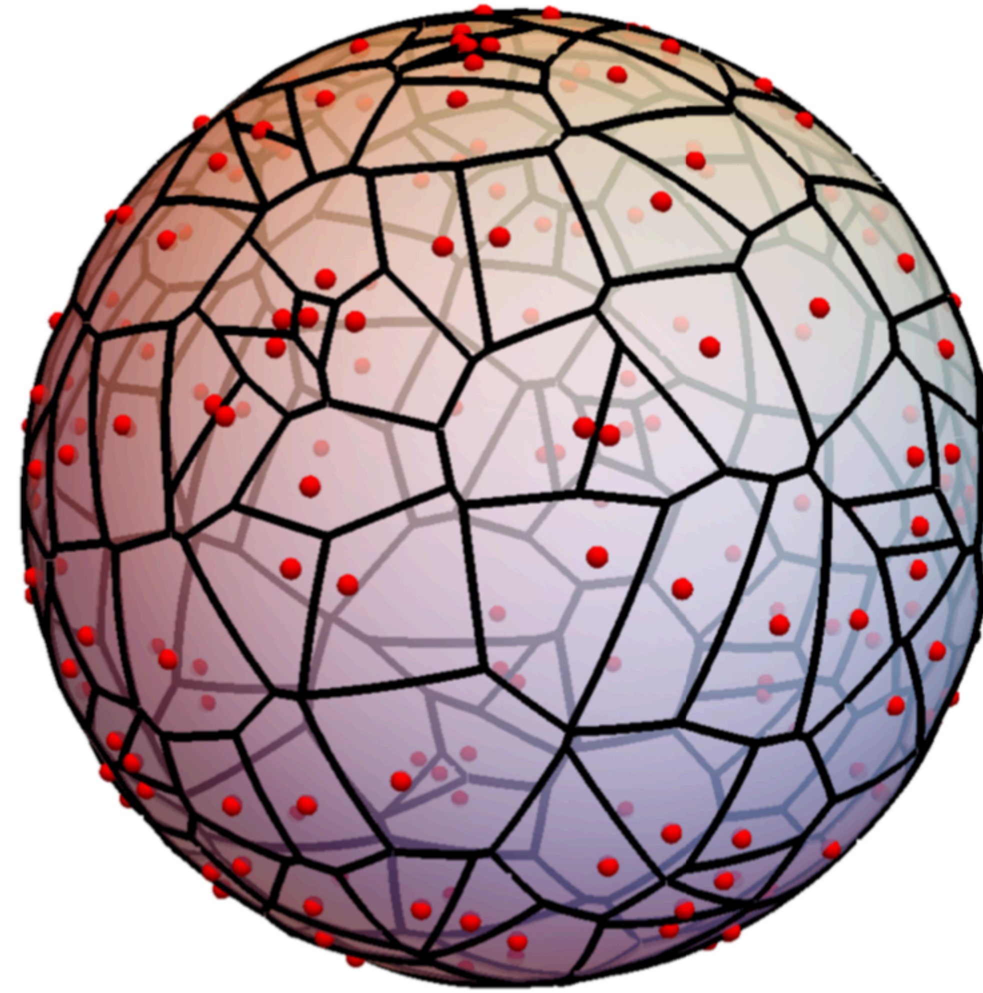

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More precisely: $f_k(V_{n+1}) \overset{d}{=} f_{d-k-1}(\tilde{P}^d_{n,d})$

| | $d=2$ | $d=3$ |
|---|---|---|
| $\mathbb{E}f_0(V_{n+1})$ | $6\left(1-\frac{2}{n+1}\right)$ | $\frac{256}{35\pi}\left(\frac{1}{2\pi}\right)^{n-3}\binom{n}{3}\int_{-\pi/2}^{\pi/2}(\cos x)^8(2x+\sin(2x)+\pi)^{n-3}\,dx$ |
| $\mathbb{E}f_1(V_{n+1})$ | $6\left(1-\frac{2}{n+1}\right)$ | $\frac{3}{2}\mathbb{E}f_0(V_{n+1})$ |
| $\mathbb{E}f_2(V_{n+1})$ | – | $\frac{1}{2}\mathbb{E}f_0(V_{n+1})+2$ |

# Application 3: Typical Voronoi cells in hyperbolic space

Godland, Kabluchko & T. (2022)

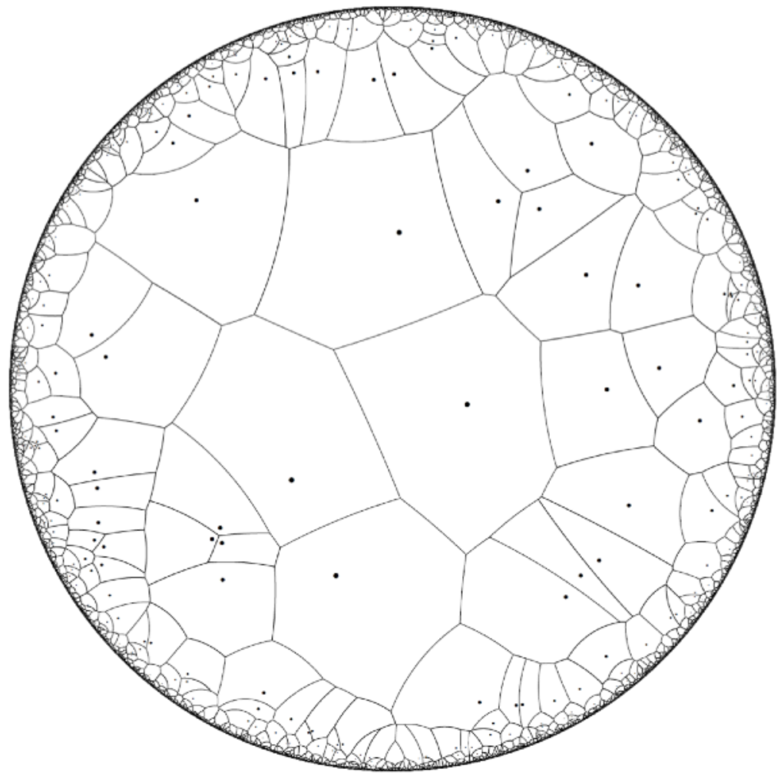
$\eta_t$ stationary Poisson point process in $\mathbb{H}^d$

$C_X = \{z \in \mathbb{H}^d : d_g(z, X) \leq d_g(z, Y) \ \forall Y \in \eta_t\}$

Voronoi cell of $X \in \eta_t$

$\{C_X : X \in \eta_t\}$ Poisson Voronoi tessellation

$V_0$ typical Voronoi cell = cell around an origin



## Application 3: Typical Voronoi cells in hyperbolic space

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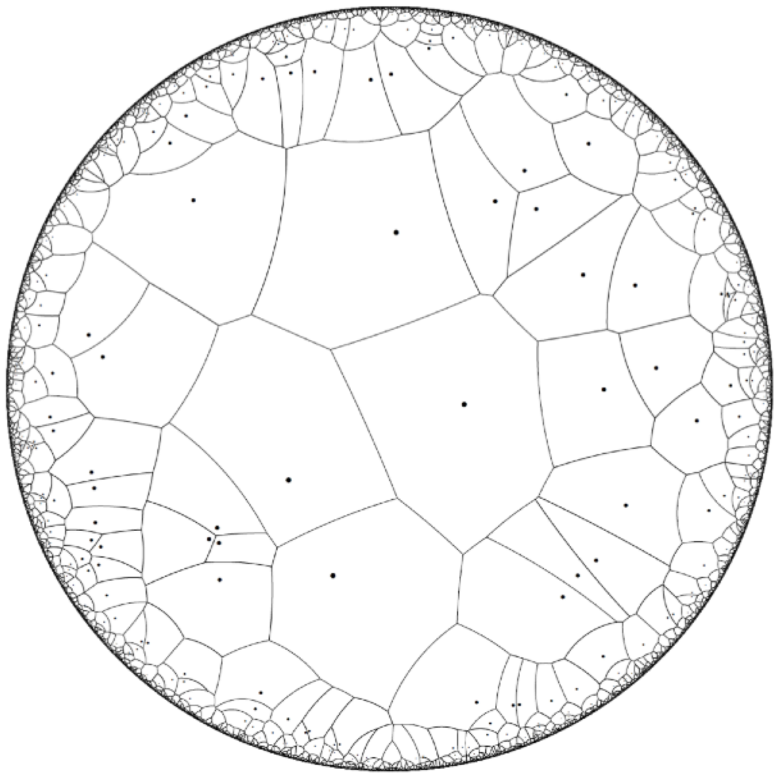

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Theorem: $\mathbb{E}f_k(V_0)$ can be computed explicitly

| | $d=2$ | $d=3$ |
|---|---|---|
| $\mathbb{E}f_0(V_0)$ | $6\left(1+\frac{1}{2\pi t}\right)$ | $\frac{1024\pi^5}{105}\int_0^\infty \sinh(x)^8 e^{-2\pi(\sinh(x)\cosh(x)-x)}\,dx$ |
| $\mathbb{E}f_1(V_0)$ | $6\left(1+\frac{1}{2\pi t}\right)$ | $\frac{3}{2}\mathbb{E}f_0(V_0)$ |
| $\mathbb{E}f_2(V_0)$ | – | $\frac{1}{2}\mathbb{E}f_0(V_0)+2$ |

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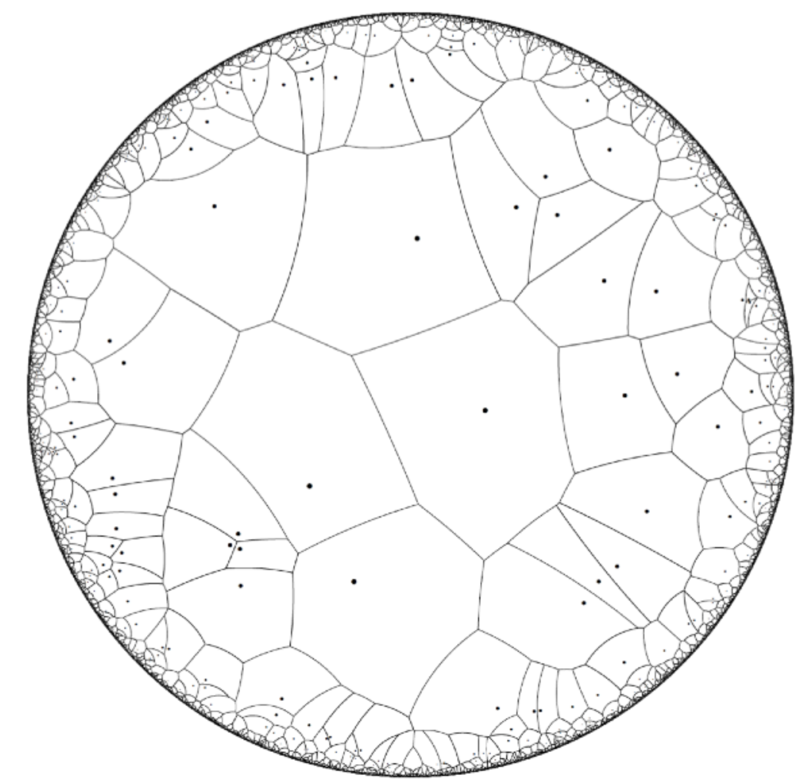
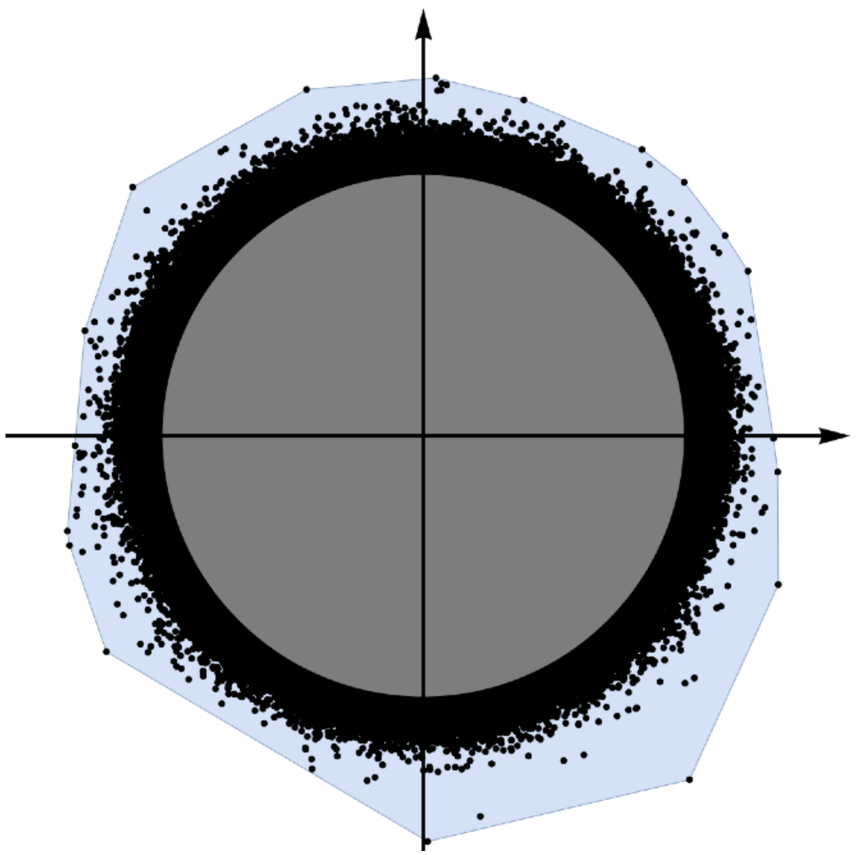


**Theorem:** $\mathbb{E} f_k(V_0)$ can be computed explicitly in terms of a beta-star polytope.



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| $\mathbb{E}f_2(V_0)$ | – | $\frac{1}{2}\mathbb{E}f_0(V_0) + 2$ |

# Thank you!