From Poisson to Quantum Geometry

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Contents

1 Poisson Geometry **4**
1.1 Poisson algebra . . . 4
1.2 Poisson manifolds . . . 6
1.3 The sharp map . . . 11
1.4 The symplectic foliation . . . 14

2 Schouten-Nijenhuis bracket **19**
2.1 Lie-Poisson bracket . . . 19
2.2 Schouten-Nijenhuis bracket . . . 21
2.2.1 Schouten-Nijenhuis bracket computations . . . 24
2.2.2 Lichnerowicz definition of the Schouten-Nijenhuis bracket . . . 25
2.2.3 Jacobi condition and Schouten-Nijenhuis bracket . . . 25
2.2.4 Compatible Poisson tensors . . . 27
2.2.5 Koszul's formula . . . 27
2.3 Poisson homology . . . 28

3 Poisson maps **29**
3.1 Poisson maps . . . 29
3.2 Poisson submanifolds . . . 33
3.3 Coinduced Poisson structures . . . 34
3.4 Completeness . . . 36

4 Poisson cohomology **37**
4.1 Modular class . . . 40
4.2 Computation for Poisson cohomology . . . 42

5 Poisson homology **47**
5.1 Poisson homology and modular class . . . 49

6 Coisotropic submanifolds **51**
6.1 Poisson Morita equivalence . . . 53
6.2 Dirac structures . . . 57

7 Poisson Lie groups **60**
7.1 Poisson Lie groups . . . 60
7.2 Lie bialgebras . . . 63
7.3 Manin triples . . . 65

8 Poisson actions **67**
8.1 Poisson actions 68
8.2 Poisson homogeneous spaces 71
8.3 Dressing actions 79

9 Quantization **83**
9.1 Introduction 83
9.2 Duality 85
9.3 Local, global, special quantizations 87
9.4 Real structures 90
9.5 Dictionary 91
9.6 Quantum subgroups 92
9.7 Quantum homogeneous spaces 92
9.8 Coisotropic creed 96

Bibliography **98**

Chapter 1

Poisson Geometry

1.1 Poisson algebra

Definition 1.1. *A* ***Poisson algebra*** *is an associative algebra A (over a field $\mathbb{K}$) with a linear bracket $\{\cdot,\cdot\}\colon A \otimes A \to A$ such that*

1. $\{f,g\} = -\{f,g\}$ *(antisymetry),*
2. $\{f,gh\} = g\{f,h\} + h\{f,g\}$ *(Leibniz rule),*
3. $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0$ *(Jacobi identity),*

for all $f, g, h \in A$.

Remarks:

- A needs not to be commutative.
- Every associative algebra A can be made into a Poisson algebra by setting $\{f,g\} \equiv 0$.
- When A is unital we get from the assumptions

$$\{f,g\} = \{f, g\cdot 1\} = \{f,g\}\cdot 1 + g\cdot\{f,1\},$$

so $\{f,1\} = 0$ for all $f \in A$.

Exercise 1.2. *Let U be an almost commutative algebra, i.e. filtered associative algebra, $U^0 \subseteq U^1 \subseteq \ldots$, $U^i \cdot U^j \subseteq U^{i+j}$, such that $\mathrm{gr}(U) = \bigoplus_{i=0}^{\infty} U^i/U^j$ is commutative. Let $[x] \in \mathrm{gr}(U)$ be the class of $x \in U^i$, and define*

$$\{[x],[y]\} := [xy - yx] \in \mathrm{gr}(U).$$

Prove that it is a Poisson algebra.

Definition 1.3. ***Poisson morphism*** $(A, \{\}_A) \xrightarrow{\varphi} (B, \{\}_B)$ *is a morphism of algebras such that*

$$\varphi(\{f,g\}_A) = \{\varphi(f), \varphi(g)\}_B, \quad \text{for all} \quad f, g \in A.$$

Exercise 1.4. *Prove that Poisson algebras with Poisson morphism form a category.*

Definition 1.5. *A* ***Poisson subalgebra*** *is a subalgebra closed with respect to $\{\}$.* ***Poisson ideal*** *I is an ideal with respect to the associative product, such that $\{f,i\} \in I$ for all $f \in A$, $i \in I$.*

For Poisson morphism $\varphi\colon A \to B$, $\ker\varphi$ is an ideal in A, $\operatorname{im}\varphi$ is a subalgebra in B, and there is an exact sequence of Poisson algebras

$$0 \to \ker\varphi \to A \to \operatorname{im}\varphi \to 0.$$

Definition 1.6. *Let A be Poisson algebra. An element $f \in A$ is* ***Casimir*** *if $\{f, g\} = 0$ for all $g \in A$.*

Definition 1.7. *Let $X \in \operatorname{End}(A)$. It is called* ***canonical*** *if it is a derivation with respect to both the associative product and the bracket, i.e. for every $f, g \in A$*

1. $X(fg) = (Xf)g + f(Xg)$
2. $X\{f, g\} = \{Xf, g\} + \{f, Xg\}$

The set of all Casimir elements in A will be denoted by $\operatorname{Cas}(A)$, and set of canonical endomorphisms by $\operatorname{Can}(A)$.

Proposition 1.8. *For every $f \in A$, $X_f\colon g \mapsto \{f, g\}$ is canonical.*

Proof. From Leibniz identity:

$$X_f(gh) = \{f, gh\} = \{f, g\}h + g\{f, h\} = (X_f g)h + g(X_f h)$$

From Jacobi identity:

$$X_f(\{g, h\}) = \{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\} = \{X_f g, h\} + \{g, X_f h\}$$

□

Definition 1.9. *Canonical endomorphisms of the form X_f are called* ***hamiltonian*** *and denoted by* $\operatorname{Ham}(A)$*.*

With $\operatorname{Der}(A)$ we will denote the set of derivations of the associative algebra A. We have the following chain of inclusions.

$$\operatorname{Ham}(A) \subseteq \operatorname{Can}(A) \subseteq \operatorname{Der}(A).$$

Let us recall now that $\operatorname{Der}(A)$ is a Lie algebra with respect to the commutator of endomorphisms. $\operatorname{Can}(A)$ is a subalgebra of $\operatorname{Der}(A)$.

Proposition 1.10. $\operatorname{Ham}(A)$ *is an ideal in* $\operatorname{Can}(A)$ *and a subalgebra of* $\operatorname{Der}(A)$*.*

Proof. Let $X \in \operatorname{Can}(A)$, $X_f \in \operatorname{Ham}(A)$. Then

$$\begin{aligned}
[X, X_f](g) &= X(X_f(g)) - X_f(X(g)) \\
&= X\{f, g\} - \{f, X(g)\} \\
&= \{X(f), g\} + \{f, X(g)\} - \{f, X(g)\} \\
&= X_{X(f)}(g),
\end{aligned}$$

so $[X, X_f] = X_{X(f)} \in \operatorname{Ham}(A)$. To prove that $\operatorname{Ham}(A)$ is a subalgebra of $\operatorname{Der}(A)$, one computes

$$\begin{aligned}
([X_f, X_g] - X_{\{f,g\}})h &= X_f(X_g h) - X_g(X_f h) - X_{\{f,g\}}h \\
&= \{f, \{g, h\}\} - \{g, \{f, h\}\} - \{\{f, g\}, h\} \\
&= -\operatorname{Jac}(f, g, h) = 0.
\end{aligned}$$

□

Proposition 1.11. *Let* $(A, \{\cdot,\cdot\}_A)$ *and* $(B, \{\cdot,\cdot\}_B)$ *be Poisson algebras. Then their tensor product* $A \otimes B$ *has a natural structure of Poisson algebra given by*

$$\{a_1 \otimes b_1, a_2 \otimes b_2\} = \{a_1, a_2\}_A \otimes b_1 b_2 + a_1 a_2 \otimes \{b_1, b_2\}_B.$$

The maps $A \to A \otimes B$, $a \mapsto a \otimes 1$, $B \mapsto A \otimes B$, $b \mapsto 1 \otimes B$ *are Poisson morphisms and* $\{a \otimes 1, 1 \otimes b\} = 0$ *for all* $a \in A$, $b \in B$.

Definition 1.12. ***Poisson module*** *structure on a left* A*-module* M *over a Poisson algebra* A *is a linear map*

$$\{\cdot,\cdot\}_M \colon A \otimes M \to M$$

such that

1. $\{\{f,g\}_A, m\}_M = \{f, \{g,m\}_M\}_M - \{g, \{f,m\}_M\}_M$,
2. $\{fg, m\}_M = f \cdot \{g,m\}_M + g \cdot \{f,m\}_M$,
3. $\{f, g \cdot m\}_M = \{f,g\}_A \cdot m + g\{f,m\}_M$

Remark: It is a definition of a flat connection when M is the module of sections of a vector bundle. Indeed, when we denote

$$T \colon M \to \mathrm{Hom}_{\mathbb{K}}(A, M), \quad m \mapsto T_m := \{\cdot, m\}_M$$

then

1. $\Longleftrightarrow T_m(\{f,g\}_A) = \{f, T_m(g)\}_M - \{g, T_m(f)\}_M$ (that is $T_m \in \mathrm{Der}((A, \{\cdot,\cdot\}_A); M)$),
2. $\Longleftrightarrow T_{f \cdot m} = f \cdot T_m(g) + \{f,g\}_A \cdot m = f \cdot T_m(g) + X_f(g) \cdot m$,
3. $\Longleftrightarrow T_m(fg) = fT_m(g) + gT_m(f)$ (that is $T_m \in \mathrm{Der}((A, \cdot); M)$).

One may ask whether this is a reasonable definition of Poisson module. It is, in a sense, the categorical notion of Poisson bimodule as it verifies the so-called square-zero construction which can be summarized as follows: let A be a Poisson algebra and M Poisson A-module; define a Poisson algebra structure on $A \oplus M$ using formulas

$$(f + m) \cdot (f_1 + m_1) := ff_1 + (f \cdot m_1 + f_1 \cdot m),$$
$$\{f + m, f_1 + m_1\} := \{f, f_1\}_A + \{f, m_1\}_M - \{f_1, m\}_M.$$

Proposition 1.13. $A \oplus M$ *is a Poisson algebra if and only if* M *is a Poisson module. Furthermore the projection* $A \oplus M \to A$ *is a map of algebras,* $M^2 = 0$ *and* M *is an ideal.*

1.2 Poisson manifolds

Definition 1.14. *A* ***smooth Poisson manifold*** M *is a smooth manifold together with a Poisson bracket on* $C^\infty(M)$.

An ***affine algebraic Poisson variety*** M *is an affine algebraic variety such that* $A = \mathbb{K}[M]$ *(algebra of regular functions) is a Poisson algebra over* $\mathbb{K}$.

An ***algebraic Poisson variety*** M *is an algebraic variety such that the sheaf of regular functions is a sheaf of Poisson algebras.*

Definition 1.15. *A* ***morphism of Poisson manifolds*** *is a differentiable function* $\varphi\colon M \to N$ *such that* φ^* *is a morphism of Poisson algebras, i.e.*

$$\varphi^*\{f,g\}_M = \{f,g\}_M \circ \varphi = \{f\circ\varphi, g\circ\varphi\}_N = \{\varphi^* f, \varphi^* g\}_N$$

for $f, g \in C^\infty(M)$*.*

The map $f \mapsto X_f$ takes values in $\mathrm{Der}(C^\infty(M)) = \mathfrak{X}^1(M)$. Thus we can write

$$\begin{aligned}\mathrm{Cas}(M) &= \{f \in C^\infty(M) : \{f,g\} = 0 \quad \forall g \in C^\infty(M)\} = \ker(f \mapsto X_f),\\ \mathrm{Ham}(M) &= \text{Hamiltonian vector fields} = \mathrm{im}(f \mapsto X_f).\end{aligned}$$

To put it another way we have the short exact sequence

$$0 \to \mathrm{Cas}(M) \to C^\infty(M) \to \mathrm{Ham}(M) \to 0.$$

The Cartesian product of Poisson manifolds is a Poisson manifold

$$C^\infty(M_1 \times M_2) \supset C^\infty(M_1) \otimes C^\infty(M_2).$$

There is a Poisson structure on $C^\infty(M_1)\otimes C^\infty(M_2)$ and it extends to $C^\infty(M_1 \times M_2)$ by

$$\{f(x_1,x_2), g(x_1,x_2)\} = \{f_{x_2}, g_{x_2}\}_2(x_1) + \{f_{x_1}, g_{x_1}\}_1(x_2), \quad \text{where}$$

$$\begin{aligned} f_{x_1}\colon x_2 &\mapsto f(x_1,x_2), \quad f_{x_1} \in C^\infty(M_2),\\ f_{x_2}\colon x_1 &\mapsto f(x_1,x_2), \quad f_{x_2} \in C^\infty(M_1).\end{aligned}$$

Examples 1.16.

1. Each manifold is a Poisson manifold with trivial bracket $\{\cdot,\cdot\}$.

2. Let (M,ω) be a symplectic manifold i.e. $\omega \in \Omega^2(M)$, $d\omega = 0$, ω nondegenerate ($\omega_x = \sum_{k<j} \omega_{ij}(x) dx^i \wedge dx^j$, where $[\omega_{ij}]$ is an antisymmetric matrix of maximal rank). Define X_f by $\omega(X_f, -) = \langle -df, -\rangle$, that is $i_{X_f}\omega = -df$, $\omega(X_f, Y) = -\langle df, Y\rangle = -Yf$. Now
$$\{f,g\} := -\omega(X_g, X_f) = \omega(X_f, X_g) = -\{g,f\}.$$

   Indeed, $\{\cdot,\cdot\}$ is bilinear:

$$\{f,g\} = i_{X_f} dg, \quad d(g_1+g_2) = dg_1 + dg_2$$

$$\langle X, dg_1 + dg_2\rangle = \langle X, dg_1\rangle + \langle X, dg_2\rangle,$$

$$X_{f+g} = X_f + X_g, \quad \omega(X_{f+g}, -) = \omega(X_f, -) + \omega(X_g, -),$$

   $\{\cdot,\cdot\}$ satisfies Leibniz identity:

$$\begin{aligned}\{f, gh\} &= i_{X_f} d(gh)\\ &= i_{X_f}(g dh + h dg)\\ &= g i_{X_f} dh + h i_{X_f} dg\\ &= g\{f,h\} + h\{f,g\}.\end{aligned}$$

$\{\cdot,\cdot\}$ satisfies Jacobi identity:

$$\begin{aligned}
0 = d\omega(X_f, X_g, X_h) &= X_f\omega(X_g, X_h) - X_g\omega(X_f, X_h) + X_h\omega(X_f, X_g)\\
&\quad - \omega([X_f, X_g], X_h) + \omega([X_f, X_h], X_g) - \omega([X_g, X_h], X_f)\\
&= X_f\{g,h\} - X_g\{f,h\} + X_h\{f,g\}\\
&\quad + [X_f, X_g](h) + [X_g, X_h](f) + [X_h, X_f](g)\\
&= \{f,\{g,h\}\} - \{g,\{f,h\}\} + \{h,\{f,g\}\}\\
&\quad + [X_f, X_g](h) + [X_g, X_h](f) + [X_h, X_f](g)\\
&= \mathrm{Jac}(f,g,h) + X_f(X_g h) - X_g(X_f h) + X_g(X_h f)\\
&\quad - X_h(X_g f) + X_h(X_f g) - X_f(X_h g)\\
&= \mathrm{Jac}(f,g,h) + \{f,\{g,h\}\} - \{g,\{f,h\}\} + \{g,\{h,f\}\}\\
&\quad - \{h,\{g,f\}\} + \{h,\{f,g\}\} - \{f,\{h,g\}\}\\
&= 3\,\mathrm{Jac}(f,g,h).
\end{aligned}$$

We used

$$\begin{aligned}
d\eta(X_1,\dots,X_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}X_i\cdot\eta(X_1,\dots,\widehat{X_i},\dots,X_{k+1})\\
&\quad + \sum_{1\le i<j\le k+1}(-1)^{i+j}\eta([X_i,X_j],X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_{k+1}),
\end{aligned}$$

(with the usual hat notation to denote missing terms) and

$$\begin{aligned}
\omega([X_f, X_g], X_h) &= (i_{X_h}\omega)([X_f, X_g])\\
&= -\langle dh,\,[X_f, X_g]\rangle\\
&= -[X_f, X_g](h).
\end{aligned}$$

Thus every symplectic manifold is a Poisson manifold.

Consider special case of the previous example, $(\mathbb{R}^{2n}, \omega = \sum dp_i \wedge dq_i)$. Every symplectic manifold is locally symplectomorphic to this one (but that does not mean that this is unique symplectic structure on $\mathbb{R}^{2n}$!).

Let $f := f(p_i, q_i)$, $\omega(X_f, Y) = -Yf$. Then for $Y = \partial_{q_i}$ and $Y = \partial_{p_i}$ we have respectively

$$\begin{aligned}
-i_{\partial_{q_i}}\omega &= \omega(-,\partial_{q_i}) = -dp_i\\
-i_{\partial_{p_i}}\omega &= \omega(-,\partial_{p_i}) = -dq_i\\
X_f &= \sum_{i=1}^{n} a_i\partial_{q_i} + b_i\partial_{p_i}.
\end{aligned}$$

Now $\omega(X_f, \partial_{q_i}) = -b_i$, $\omega(X_f, \partial_{p_i}) = -a_i$ and

$$X_f = \sum_{i=1}^{n} -\partial_{p_i} f\partial_{q_i} + \partial_{q_i} f\partial_{p_i},$$

$$\{f,g\} = X_f(g) = \sum_{i=1}^{n} -\partial_{p_i} f\partial_{q_i} g + \partial_{q_i} f\partial_{p_i} g.$$

Exercise 1.17. *Prove by applying definitions that if $M = \mathbb{R}^{2n}$, $\omega = \sum_{i=1}^n dq_i \wedge dp_i$, then $\{f,g\} = \sum_{i=1}^n -\partial_{p_i} f \partial_{q_i} g + \partial_{q_i} f \partial_{p_i} g$.*

Exercise 1.18. *Derive canonical Poisson relations*

$$\begin{aligned}\{q_i, p_j\} &= \delta_{ij},\\ \{q_i, q_j\} &= 0,\\ \{p_i, p_j\} &= 0.\end{aligned}$$

Proposition 1.19. *On every Poisson manifold there is a unique bivector field $\Pi \in \Gamma(\Lambda^2 TM)$ such that*

$$\{f,g\} = \langle \Pi,\, df \wedge dg\rangle.$$

Proof. We need to show that $\{f,g\}(x)$ depends only on $d_x f$ and $d_x g$. Consider f fixed

$$\{f,g\}(x) = (X_f g)(x) = \langle X_f(x),\, d_x g\rangle.$$

Similarly for g fixed

$$\{f,g\}(x) = -\langle X_g(x),\, d_x f\rangle.$$

Furthermore $f \mapsto d_x f$, $C^\infty(M,\mathbb{R}) \to T_x^*M$ is surjective, therefore there exists $\Pi(x)$ bilinear, skewsymmetric on T_x^*M such that

$$\{f,g\}(x) = \Pi(x)(d_x f, d_x g).$$

The map $x \mapsto \Pi(x)$ is a differenital bivector field. Locally in a coordinate chart $\Pi = \sum_{i<j} \Pi_{ij} \partial_{x_i} \wedge \partial_{x_j}$. This means $\{x_i, x_j\} = \Pi_{ij}$. □

Fix on M a coordinate chart $(U; x_1, \dots, x_n)$. Then the bivector Π is locally given by

$$\Pi_U = \sum_{i<j} \Pi_{ij} \partial_{x_i} \wedge \partial_{x_j}$$

where the coefficients Π_{ij} are functions on U explicitely given by $\Pi_{ij} = \{x_i, x_j\}$. Therefore Π is determined once you know brackets of local coordinate functions

$$\{f,g\} = \sum_{i,j=1}^n \{x_i, x_j\} \partial_{x_i} f \partial_{x_j} g.$$

Let $\Pi := \sum_{i<j} \Pi_{ij} \partial_{x_i} \wedge \partial_{x_j}$ be a bivector field, where $\Pi_{ij} = \{x_i, x_j\}$. In many examples a Poisson structure on $\mathbb{R}^{2n}$ will be given simply by lifting brackets of coordinants.

Exercise 1.20. *Prove that the Jacobi identity ($\mathrm{Jac}(x_i, x_j, x_k) = \sum_{\text{cyclic}} \{\{x_i, x_j\}, x_k\} = 0$) is equivalent to*

$$\sum_{k=1,\, i<j<l}^n \frac{\partial \Pi_{ij}}{\partial x_k} \Pi_{kl} + \frac{\partial \Pi_{jl}}{\partial x_k} \Pi_{ki} + \frac{\partial \Pi_{li}}{\partial x_k} \Pi_{kj} = 0. \tag{1.1}$$

Let V be a real n-dimensional vector space. Consider coordinates $x_1, \dots, x_n$. Then $\sum_{i<j} \Pi_{ij} \partial_{x_i} \wedge \partial_{x_j}$ is **Poisson tensor** if and only if (1.1) holds.

Example 1.21. Special cases.

1. $\dim V = 1 \implies \Pi = 0$

2. $\dim V = 2 \implies \Pi = \Pi_{12}\partial_{x_1} \wedge \partial_{x_2}$. In $\mathbb{R}^2$ every bivector defines Poisson bracket $\Pi = f(x,y)\partial_x \wedge \partial_y$, $\{x,y\} = f(x,y)$.

3. $\dim V = 3$ - exercise.

4. Say Π_{ij} are linear functions in $x_1, \ldots, x_n$,

$$\Pi_{ij} = \sum_{k=1}^{n} c_{ij}^k x_k.$$

Therefore $\frac{\partial \Pi_{ij}}{\partial x_k} = c_{ij}^k$,

$$\begin{aligned} 0 &= \sum_{k=1}^{n} c_{ij}^k c_{kl}^h x_h + c_{jl}^k c_{ki}^h x_h + c_{li}^k c_{kj}^h x_h \\ &= \sum_{k=1}^{n} (c_{ij}^k c_{kl}^h + c_{jl}^k c_{ki}^h + c_{li}^k c_{kj}^h) x_h \\ \Leftrightarrow & \sum_{k=1}^{n} (c_{ij}^k c_{kl}^h + c_{jl}^k c_{ki}^h + c_{li}^k c_{kj}^h) = 0, \end{aligned}$$

for all i, j, l, h. Thus c_{ij}^k are structure constants of a Lie algebra. Therefore for any given Lie algebra we have a Poisson structure.

Another way to obtain the same result is to take a Lie algebra $\mathfrak{g}$, $V = \mathfrak{g}^*$ linear functionals on $\mathfrak{g}$ If one knows a Poisson bracket on a basis of $\mathfrak{g}^*$, then knows it on V. Let $X_1, \ldots, X_n$ be basis of $\mathfrak{g}$, $[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k$. Let $\xi_1, \ldots, \xi_n$ be the dual basis of $\mathfrak{g}^*$. Say $\alpha \in \mathfrak{g}^*$, $f, g \in C^\infty(\mathfrak{g}^*)$. Then $d_\alpha f \in (\mathfrak{g}^*)^* = \mathfrak{g}$ and

$$\{f, g\}(\alpha) = \langle \alpha, [d_\alpha f, d_\alpha g] \rangle.$$

For example if $f \simeq X_i$, $g \simeq X_j$, $X_i(\xi_j) = \delta_{ij}$

$$\begin{aligned} \{X_i, X_j\}(\xi_k) &= \langle \xi_k, [X_i, X_j] \rangle \\ &= \langle \xi_k, \sum_{h=1}^{n} c_{ij}^h X_h \rangle \\ &= c_{ij}^k, \\ \{X_i, X_j\} &= c_{ij}^k X_k. \end{aligned}$$

Thus $\Pi = \sum_{k=1}^n c_{ij}^k X_i \wedge X_j$ is a linear Poisson tensor on $\mathfrak{g}^*$. The dual of a Lie algebra has always a canonically defined Poisson tensor.

Example 1.22. Consider

$$M = \mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Then

$$\begin{aligned} \{\alpha, \bar{\alpha}\} &= -i|\beta|^2, \\ \{\beta, \bar{\beta}\} &= 0, \\ \{\alpha, \beta\} &= \tfrac{1}{2} i\alpha\beta \\ \{\alpha, \bar{\beta}\} &= \tfrac{1}{2}\alpha\bar{\beta} \end{aligned}$$

defines uniquely a Poisson bracket on $\mathfrak{su}(2)$. Are you able to find Casimir functions ?

Example 1.23. Let φ be smooth function on $\mathbb{R}^3$. Define

$$\begin{aligned}\{x,y\} &:= \partial_z\varphi\\ \{y,z\} &:= \partial_x\varphi\\ \{z,x\} &:= \partial_y\varphi.\end{aligned}$$

Then for any φ these formulas define a Poisson bracket. In fact

$$\{x,\{y,z\}\}+\{z,\{x,y\}\}+\{y,\{z,x\}\}=$$

$$=\{x,\partial_x\varphi\}+\{z,\partial_z\varphi\}+\{y,\partial_y\varphi\}=$$

$$=\partial_z\varphi(\partial_y\partial_x\varphi)-\partial_y\varphi(\partial_z\partial_x\varphi)-\partial_x\varphi(\partial_y\partial_z\varphi)+\partial_y\varphi(\partial_x\partial_z\varphi)-\partial_z\varphi(\partial_x\partial_y\varphi)+\partial_x\varphi(\partial_z\partial_y\varphi)=0$$

Example 1.24. Let $\mathbb{S}^4=\{(\alpha,\beta,t)\in\mathbb{C}\times\mathbb{C}\times\mathbb{R} : |\alpha|^2+|\beta|^2=t(1-t)\}$. Then

$$\Pi=\alpha\beta\partial_\alpha\wedge\partial_\beta-\alpha\beta^*\partial_\alpha\wedge\partial_{\beta^*}-\alpha^*\beta\partial_{\alpha^*}\wedge\partial_\beta+\alpha^*\beta^*\partial_{\alpha^*}\wedge\partial_{\beta^*}$$

is Poisson tensor. Can you find conditions for f to be a Casimir function ? *Remark*: t is a Casimir function.

Example 1.25. Let V be a real vector space of dimension n, and U, P_j, $j=1,\dots,n-2$ polynomials in variables $x_1,\dots,x_n$. Define

$$J(h_1,\dots,h_n):=\det\left[\frac{\partial}{\partial h_i}x_j\right]$$

Prove that

$$\{f,g\}=UJ(f,g,P_1,\dots,P_{n-2})$$

defines a Poisson bracket.

Example 1.26. On $\mathbb{R}^4$ with coordinates x_0,x_1,x_2,x_3 take real constants J_{12},J_{23},J_{31} and define

$$\begin{aligned}&\{x_i,x_j\}:=2J_{ij}x_0x_k\\ &\{x_0,x_i\}:=-2x_jx_k,\quad \text{where } (i,j,k)=(1,2,3)\quad\text{or cyclic permutation.}\end{aligned}$$

Find the conditions on J_{ij} that implies this is a Poisson bracket. These are called Sklyanin Poisson algebras. Can you find Casimir functions ? *Hint*: two quadratic polynomials.

1.3 The sharp map

Let M be a manifold, and Π a Poisson bivector.

Definition 1.27. *For every Poisson manifold* (M,Π) *we define its* ***sharp map***

$$\#_\Pi\colon T^*M\to TM$$

$$\#_{\Pi,x}(x,\alpha_x):=(x,(i_{\alpha_x}\Pi)(x)),\quad \alpha_x\in T^*_xM.$$

Remark: $(i_{\alpha_x}\Pi)(\beta x)=\langle\Pi,\,\beta_x\wedge\alpha_x\rangle=\Pi_x(\alpha_x,\beta_x)$ for all $\alpha_x,\beta_x\in T^*_xM$.
Properties:

- $\#_\Pi$ is a bundle map on M. It is also called the **anchor** of (M,Π).

- Being a bundle map it induces a map on sections

$$\#_\Pi\colon \Omega^1(M) \to \mathfrak{X}(M), \quad \alpha \mapsto i_\alpha \Pi$$

- in particular on exact 1-forms one easily has $\#_\Pi(df) = X_f$. In fact

$$\langle \#_\Pi(df),\, dg\rangle = \Pi(df, dg) = \{f, g\} = \langle X_f,\, dg\rangle.$$

  Remark then, that a vector field is uniquely determined by its contractions with exact 1-forms (locality of vector fields).

- Local expression

$$\#_\Pi\left(\sum_{i=1}^n a_i dx_i\right) = \sum_{i,j=1}^n \Pi_{ij} a_i \partial_{x_j}$$

$$\#_\Pi(dx_j) = \Pi\left(\sum_{i=1}^n a_i dx_i, dx_j\right) = \sum_{i=1}^n \Pi(a_i dx_i, dx_j) = \sum_{i=1}^n a_i \Pi_{ij}.$$

  If Π_{ij} are smooth, then so is $\#_\Pi$.

- $\operatorname{im} \#_{\Pi,x} = \operatorname{Ham}_x(M)$ - vector subspace of T_xM. This is an easy consequence of $\#_\Pi(df) = X_f$.

Definition 1.28. *The assignment of a vector subspace S_x of T_xM for every $x \in M$ is called a **(generalized) distribution**. A distribution is **differentiable** if for all $x_0 \in M$ and $v_0 \in S_{x_0}$ there exists a neighbourhood U of x_0 and a smooth vector field on U such that $X(y) \in S_y$ for all $y \in U$ and $X(x_0) = v_0$.*

The word generalized refers to the fact that we do not require $\dim S_xM$ to be constant in x. The fact that $\operatorname{im} \#_\Pi$ is locally generated by Hamiltonian vector fields proves that $\operatorname{im} \#_\Pi$ is a differentiable distribution. It will be called the **characteristic distribution** of the Poisson manifold (M, Π).

Exercise 1.29. *On $(\operatorname{im} \#_\Pi)_x$ it is possible to define a natural antisymmetric non-degenerate bilinear product. Let $v, w \in (\operatorname{im} \#_\Pi)_x$ and choose α_x, β_x such that $v = \#_\Pi(\alpha_x)$, $w = \#_\Pi(\beta_x)$,*

$$(v, w) = \langle \Pi_x,\, \alpha_x \wedge \beta_x\rangle.$$

Prove that it is well defined, and its properties.

Definition 1.30. *Let $\rho(x) := \dim \operatorname{im} \#_{\Pi,x}$. We call it the **rank** of the Poisson manifold (at the point x).*

Remarks:

- The reason for the name is that in local coordinates

$$\rho(x) = \operatorname{rank}(\Pi_{ij}(x)) = \operatorname{rank}(\{X_i, X_j\}(x)).$$

- $\rho\colon M \to \mathbb{Z}$; from the differentiability of the distribution it follows that $\rho(x)$ is lower semicontinuos function of x, i.e. it cannot decrease in a neghbourhood of x. Indeed, take $v_1, \ldots, v_r$ - basis of $(\operatorname{im} \#_\Pi)_{x_0}$, $X_1, \ldots, X_r$ corresponding local vector fields, then $X_1(x), \ldots, X_r(x)$ are linearly independent in a neighbourhood of x_0.

Exercise 1.31. *Show that* $\rho(x) \in 2\mathbb{Z}$ *(is always even).*

Definition 1.32. *If* $\rho(x) = k \in \mathbb{Z}$ *for all* $x \in M$ *the Poisson manifold (and also the characteristic distribution) is called a* ***regular***. *If* $x_0 \in M$ *is such that* $\rho(y) = \rho(x_0)$ *for all* y *in a neighbourhood* U_{x_0} *of* x_0, *then* x_0 *is called* ***regular point*** *of* M. *It is called a* ***singular point*** *otherwise (i.e. if for all neighbourhoods* U *of* x_0, *there is* $y \in U$ *such that* $\rho(y) > \rho(x)$*).*

Remark 1.33.

- The set of regular points is open and dense, but not necessarily connected. That it is open is obvious from definition. Let x_0 be a singular point. Take $U \subset U_{x_0}$, there exists $y \in U$ such that $\rho(y) > \rho(x_0)$. We want to prove that y is regular. Say it is not, then there exists $y_1 \in U$ (also neighbourhood of y) such that $\rho(y_1) > \rho(y)$. If y_1 is not regular repeat. Find a seqence $\{y_n\}$ in U such that $\rho(y_{i+1}) > \rho(y_i)$. If it admits a converging subsequence we are OK. We can have this from the fact that M is locally compact therefore we can always choose U to be such that $\overline{U}$ is compact.

- If in M there are singular points then $\operatorname{im} \#_\Pi$ is not a vector subbundle of TM. This is very general (and almost by definition of subbundle): the image of a bundle map is a subbundle if and only if its rank is constant.

Examples 1.34.

1. Let $M = \mathbb{R}^{2n+p}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n, y_1, \ldots, y_p)$. Let $\Pi = \sum_{i=1}^n \partial_{q_i} \wedge \partial_{p_i}$. Then (M, Π) is regular and $\rho(x) = 2n$. Here $\operatorname{im} \#_\Pi$ are tangent spaces to linear subspaces parallel to $y_1 = \ldots = y_p = 0$.

2. Let $\Pi = (x^2 + y^2)\partial_x \wedge \partial_y$ in $M = \mathbb{R}^2$. Then

$$(\operatorname{im} \#_\Pi)_{(0,0)} = \{0\},$$
$$(\operatorname{im} \#_\Pi)_{(x,y)} \simeq \mathbb{R}^2 \quad \text{if } (x,y) \neq (0,0).$$

   $(0,0)$ is singular, $(x,y) \neq (0,0)$ is regular. More generally for $f(x,y)\partial_x \wedge \partial_y$ if $\Gamma_f = \{(x,y) : f(x,y) = 0\}$ then the set of singular points is $\partial\Gamma_f$.

Proposition 1.35. *Let* (M, Π) *be a Poisson manifold. It is the Poisson manifold associated to a symplectic manifold if and only if it is regular, of dimension* $2n$ *and rank* $2n$, *i.e. if and only if* $\#_\Pi$ *is an isomorphism.*

Proof. (sketchy) M symplectic implies $\operatorname{im} \#_\Pi = TM$. In fact locally on $U \subset M$ $\omega|_U = \sum_{i=1}^n dq_i \wedge dp_i$ and the corresponding Poisson bivector is

$$\Pi|_U = \sum_{i=1}^n \partial_{q_i} \wedge \partial_{p_i}.$$

Therefore $\#_\Pi(dq_i) = \partial_{p_i}$, $\#_\Pi(dp_i) = -\partial_{q_i}$ and $\#_\Pi$ is isomorphism.

Let $\#_\Pi$ be an isomorphism (i.e. $\forall\, x \in M$ $\#_{\Pi,x} \colon T_x^*M \to T_xM$ is an isomorphism). Define $\flat_x \colon T_xM \to T_x^*M$, $\flat_x := \#_{\Pi,x}^{-1}$. Define $\Omega_x \in \Lambda^2 T_x^*M$ as $\Omega_x(v,w) = \Pi_x(\flat_x v, \flat_x w)$. Prove that Ω_x is a 2-form such that $\{f,g\} = \Omega(X_f, X_g)$ and therefore $\operatorname{Jac}(f,g,h) = \mathrm{d}\Omega(X_f, X_g, X_h)$. $\square$

1.4 The symplectic foliation

Definition 1.36. *Let S be a distribution on a manifold M. An* ***integral*** *of S is a pair (N, h) of a connected differential manifold N and an immersion $h \colon N \to M$ ($dh_x \colon T_x N \to T_{h(x)}(M)$ injective and h injective) such that*

$$T_x(h(N)) \subset S_x.$$

It is called a ***maximal integral*** *if*

$$T_x(h(N)) = S_x.$$

An ***integral submanifold*** *of S is a connected immersed submanifold N of M such that (N, i_N) is an integral.*

Remark 1.37. Integral manifolds are immersed submanifolds but not necessarily embedded submanifolds. $i \colon N \hookrightarrow M$ is an immersion at x if $\operatorname{rank} i_{*,x} = \dim N$. $i \colon N \hookrightarrow M$ is an embedding if it is an immersion and a homeomorphism from N to $i(N)$ (equipped with the topology of M). In particular integral submanifolds are not necessarily closed.

Definition 1.38. *A distribution is* ***fully integrable*** *if for every $x \in M$ there exists a maximal integral (N, h) of S such that $x \in h(N)$ (maximal at each point).*

Frobenius theorem: A constant rank differentiable distribution is fully integrable if and only if it is involutive, i.e. for all X, Y-sections of S, $[X, Y] \in S$.

For a regular Poisson manifold (M, Π) of rank $2n$, the characteristic distribution is of constant rank, and also involutive, due to $[X_f, X_g] = X_{\{f,g\}}$. Therefore it is fully integrable. Each regular Poisson manifold thus, comes equipped with a regular foliation.

Furthermore on the tangent space to the leaf $(\operatorname{im} \#_\Pi)_x$ it is always possible to define a natural antisymmetric nondegenerate bilinear product.

Computations similar to those of proposition 1.35, allow to prove that if (N, h) is the maximal integral containing x, then there is a symplectic 2-form ω_N on N such that $\omega_N = h^* \omega_\Pi$, where $(\omega_\Pi)_x \in \Lambda^2 (\operatorname{im} \#_\Pi)^*_x$ is determined by the above bilinear product.
We will call this foliation the **symplectic foliation** of M.

Example 1.39. $M = \mathbb{R}^{2n+p}$, $\Pi = \sum_{i=1}^n \partial_{q_i} \wedge \partial_{p_i}$. The leaves are

$$S_{c_1 \ldots c_p} := \{y_1 = c_1, \ldots, y_n = c_n\} \quad \text{- linear subspaces.}$$

On each leaf $\omega = \sum_{i=1}^n dq_i \wedge dp_i$.

Remark: Not every foliation is a symplectic foliation. In fact, first of all leaves need to carry a symplectic structure, a condition which already puts some topological restriction (e.g. you cannot have symplectic structure on $\mathbb{S}^{2n}$ if $n > 1$). Furthermore more delicate obstructions depend on how the symplectic forms vary from leaf to leaf [**?**].
Now we want to generalize this statement to non regular Poisson manifolds.

Theorem 1.40 (Weinstein's splitting theorem). *Let (M, Π) be a Poisson manifold. Let $x_0 \in M$, $\rho_\Pi(x_0) = 2n$. Then there exists a coordinate neighbourhood U of x_0 with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n, y_1, \ldots, y_p)$ ($\dim M = 2n + p = m$) such that*

$$\Pi(x) = \sum_{i=1}^n \partial_{q_i} \wedge \partial_{p_i} + \sum_{i<j=1}^p \varphi_{ij}(x) \partial_{y_i} \wedge \partial_{y_j} \quad \forall x \in U,$$

and such that $\varphi_{ij}(x)$ depends only on coordinates $y_1, \ldots, y_k$ and $\varphi_{ij}(x_0) = 0$.

Proof. Induction on n, $\rho_\Pi(x_0) = 2n$. If $n = 0$ there is nothing to prove. Say it holds for $n-1$. There are $f, g \in C^\infty(M)$ such that

$$\{f, g\}(x_0) \neq 0.$$

Let $p_1 = g$. Then $X_{p_1}(f)(x_0) \neq 0$, so $X_{p_1}(x_0)$ is a nonzero vector field. By the rectifying theorem there exists coordinate neighbourhood centered at x_0 such that $-X_{p_1} = \partial_{q_1}$, hence $\{q_1, p_1\} = -X_{p_1} q_1 = 1$. Remark that X_{p_1} and X_{q_1} are linearly independent (if $X_{q_1} = \lambda X_{p_1}$ then $\{q_1, p_1\} = X_{q_1}(p_1) = \lambda X_{p_1}(p_1) = -\lambda \partial_{q_1} p_1 = 0$). Furthermore $[X_{q_1}, X_{p_1}] = X_{q_1, p_1} = X_{-1} = 0$. Therefore around x_0 these two vector fields span a regular involutive distribution, which is integrable due to Frobenius theorem.

As a consequence there are local coordinates $(y_1, \dots, y_m)$ centered at x_0 such that

$$\begin{aligned} X_{q_1} &= \partial_{y_1} \\ X_{p_1} &= \partial_{y_2} \\ \{q_1, y_i\} &= X_{q_1} y_i = 0 \quad \forall\, i \neq 1 \\ \{p_1, y_i\} &= X_{p_1} y_i = 0 \quad \forall\, i \neq 2. \end{aligned}$$

Lemma 1.41 (Poisson theorem)**.**

$$\begin{aligned} \{p_1, \{y_i, y_j\}\} &= 0 \quad \forall\, i, j \geq 3, \\ \{q_1, \{y_i, y_j\}\} &= 0 \quad \forall\, i, j \geq 3. \end{aligned}$$

Proof. (of lemma) Simply apply Jacobi identity. □

Remark: The reason for giving these equalities the dignity of a separate statement is due to the fact that historically this is the first form in which Jacobi identity was stated.

Now $(q_1, p_1, y_3, \dots, y_m)$ is a new coordinate system, because $(y_1, \dots, y_m)$ is a local coordinate system and the map $\Phi\colon (y_1, \dots, y_m) \mapsto (q_1, p_1, y_3, \dots, y_m)$ has Jacobian

$$\left(\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & * \\ \hline \multicolumn{2}{c|}{0} & I \end{array}\right)$$

We have

$$\Pi = \partial_{q_1} \wedge \partial_{p_1} + \sum_{3 \leq i < j \leq m} \Pi'_{ij}(y_3, \dots, y_m) \partial_{y_i} \wedge \partial_{y_j}.$$

Now apply the induction hypothesis to the right summand which is a Poisson bivector on M of rank $2(n-1)$. □

In the symplectic case this theorem recovers a well-known result:

Corollary 1.42 (Darboux theorem)**.** *Let (M, ω) be a symplectic manifold and $x_0 \in M$. Then there exists a coordinate neighbourhood $(U; q_1, \dots, q_n, p_1, \dots, p_n)$ of x_0 such that*

$$\omega|_U = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

In analogy with this last statement coordinates generated by the splitting theorem are also called Darboux coordinates centered at x_0.

Example 1.43. Let $\mathfrak{g}^*$ be the dual of a Lie algebra $\mathfrak{g}$ and $\Pi = \sum c_{ij}^k X_k \partial_i \wedge \partial_j$. Then these are Darboux coordinates centered at the origin in which there is no symplectic term and all functions φ_{ij} are linear.

Definition 1.44. *Let (M, Π) be a Poisson manifold. A* ***hamiltonian path*** *on M is a smooth curve $\gamma\colon [0,1] \to M$ such that each tangent vector $\dot{\gamma}_t$ at $\gamma(t)$ belongs to $\operatorname{im} \#_{\Pi,\gamma(t)}$ for all $t \in [0,1]$. Let $x, y \in M$. We say that x and y are in* ***hamiltonian relation*** *if there exists a piecewise Hamiltonian curve γ on M connecting x to y.*

Lemma 1.45. *Hamiltonian relation is an equivalence relation.*

Proof. (exercise) Reflexive - trivial; symmetric - change backwards the time parametrization; transitive - concatenation of Hamiltonian paths is an Hamiltonian path. □

Definition 1.46. *Connected components of equivalence classes of this relation are called* ***symplectic leaves*** *of (M, Π).*

Proposition 1.47. *Each symplectic leaf is a maximal integrable submanifold of (M, Π).*

Proof. (sketchy) Let F be a leaf, $x \in F$, $T_xF \subset \operatorname{im} \#_{\Pi,x}$ because all paths on F exiting from x are Hamiltionian paths.

Let $X \in \mathfrak{X}(M)$ such that $X(x) \in \operatorname{im} \#_{\Pi,x}$. Consider the flow of X starting at x. This is a curve $\exp_x(tX)\colon (-\varepsilon, \varepsilon) \to M$ which is an Hamiltonian path. Therefore each point of this curve is in F, so $X(x) \in T_xF$. Thus $T_xF = \operatorname{im} \#_{\Pi,x}$.

Fix a local splitting at $x \in F$.

$$\operatorname{im} \#_{\Pi,x} = \langle \partial_{q_1}, \ldots, \partial_{q_n}, \partial_{p_1}, \ldots, \partial_{p_n} \rangle$$

F is locally given by $y_1 = \ldots = y_p = 0$ therefore F is an immersed submanifold. □

Proposition 1.48. *Let (M, Π) be a Poisson manifold. On each symplectic leaf F there is a well defined symplectic structure such that the inclusion map $i\colon F \to M$ is a Poisson map.*

Proof. Let $x \in F$ and F_x has splitting coordinate neighbourhood $(U; q_1, \ldots, q_n, p_1, \ldots, p_n, y_1, \ldots, y_p)$. Therefore $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ are local coordinates for F around x. Define

$$\omega_x = \sum_{i=1}^{n} d_x q_i \wedge d_x p_i, \quad x \in U.$$

This defines a symplectic structure on F. To verify that $i\colon F \to M$ is Poisson, it is enough to check it on brackets of coordinate functions. □

Examples 1.49.

1. Each symplectic manifold is a symplectic leaf of itself.
2. $(\mathbb{R}^2, f(x,y)\partial_x \wedge \partial_y)$. Each point of $\Gamma_f = \{(x,y) : f(x,y) = 0\}$ is a 0-dimensional symplectic leaf. Each connected component of $\mathbb{R}^2 \setminus \Gamma_f$ is a 2-dimensional symplectic leaf.
3. $M = \mathfrak{g}^*$. By the Leibniz rule

$$\operatorname{im} \#_{\Pi,x} = \operatorname{span}\{X_f(x) : f \quad \text{linear} \ \}$$

Let $f, g \in C^\infty(\mathfrak{g}^*)$ be linear i.e. $f, g \in \mathfrak{g}$. Fix $\alpha \in \mathfrak{g}^*$

$$\{f, g\}(\alpha) = \alpha([f, g]) = \langle \alpha, \operatorname{ad}_f g\rangle = -\langle \operatorname{ad}_f^* \alpha, g\rangle$$

$X_f(\alpha) = (-ad_f^*)(\alpha)$, where ad_f^* is an infinitesimal coadjoint action i.e. fundamental vector field of the adjoint action. Symplectic leaves are coadjoint orbits. Therefore each coadjoint orbit in $\mathfrak{g}^*$ carries a naturally defined symplectic form, called the Kirillov-Kostant-Souriau form (KKS).

Proposition 1.50. *Let (M, Π) be a Poisson manifold. Casimir functions are constant along the leaves (therefore leaves are contained in connected components of level sets of Casimirs).*

Proof. Let F be a leaf, f a Casimir function. We want to prove that $f|_F$ is constant. This is equivalent to $Xf = 0$ for all $X \in TF$ (vector fields tangent to F, locally). But locally $\mathfrak{X}(F) = \operatorname{im}\#_\Pi = \operatorname{Ham}(M)$ and

$$X_g f = \{g, f\} = 0 \quad \forall\, g \in C^\infty(M).$$

□

Proposition 1.51. *Let (M, Π) be a Poisson manifold, $\dim M = m$. Let $x_0 \in M$ be such that $\operatorname{rank}_\Pi(x_0) = \rho_\Pi(x_0) = m$. Then the symplectic leaf through x_0 is open in M.*

Proof. Around x_0 rank is constant and equal $\dim M$. Therefore $(\operatorname{im}\#_\Pi)_{x_0} = T_x M$ and due to lower semicontinuity the same holds for any y in an open neighbourhood U of x_0. Thus every path on M exiting x_0 is locally Hamiltonian, hence the thesis. □

Proposition 1.52. *Let (M, Π) be a Poisson manifold. Let $f_1, \dots, f_p$ be Casimir functions on M. Let*

$$\Gamma_{i,c} := \{x \in M : f_i(x) = c\}$$

If $\Gamma_{1,c} \cap \dots \cap \Gamma_{p,c}$ has $\dim = \operatorname{rank}\Pi_0$ *and is smooth, then its connected components are symplectic leaves.*

Remark: But $\Gamma_{1,c} \cap \dots \cap \Gamma_{p,c}$ may have $\dim \neq \operatorname{rank}\Pi_0$.

Let us consider a general polynomial Poisson bracket on $\mathbb{R}^n$, i.e. a Poisson bracket such that $\{x_i, x_j\} = P_{ij}(x_1, \dots, x_n)$. A function $f \in C^\infty(\mathbb{R}^n)$ is a Casimir function if and only if $\{x_i, f\} = 0$ for any $i = 1, \dots, n$. This can be rewritten as $\sum_j P_{ij}\partial_{x_j} f = 0$. Therefore f has to be a smooth solution of a system of linear first order PDE's. If, as in this case, we are considering a linear Poisson structure the system has constant coefficients (which are the structural constant of the Lie algebra) and its solutions can be explicitly determined.

Example 1.53. Consider $\mathfrak{su}(2)^* \cong \mathbb{R}^3$. Its Lie-Poisson bivector is

$$\Pi(x_1, x_2, x_3) = x_1\partial_{x_2} \wedge \partial_{x_3} + x_2\partial_{x_3} \wedge \partial_{x_1} + x_3\partial_{x_1} \wedge \partial_{x_2}.$$

Find the symplectic foliation.

$$\begin{aligned}
\#_\Pi(dx_1) &= -x_2\partial_{x_3} + x_3\partial_{x_2},\\
\#_\Pi(dx_2) &= x_1\partial_{x_3} - x_3\partial_{x_1},\\
\#_\Pi(dx_3) &= -x_1\partial_{x_2} + x_2\partial_{x_1}.
\end{aligned}$$

Thus

$$\Pi_{ij}(x_1, x_2, x_3) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

Compute rank $\Pi_{ij}(x_1, x_2, x_3)$.
If $(x_1, x_2, x_3) = (0, 0, 0)$ then $\rho_\Pi((0, 0, 0)) = 0$, and if $(x_1, x_2, x_3) \neq (0, 0, 0)$ then $\rho_\Pi((x_1, x_2, x_3)) = 2$ (always 2×2 minor $\neq 0$). Therefore we have everywhere rank 2, except at origin, which is an isolated 0-dimensional symplectic leaf.
Remark that $x_1^2 + x_2^2 + x_3^2$ is a Casimir function

$$\begin{aligned} \{x_1, -\} &= 2x_2\{x_1, x_2\} + 2x_3\{x_1, x_3\} = 2x_2x_3 - 2x_3x_2 = 0, \\ \{x_1, -\} &= 2x_1\{x_2, x_1\} + 2x_3\{x_2, x_3\} = 2x_1x_3 - 2x_3x_1 = 0, \\ \{x_1, -\} &= 2x_1\{x_3, x_1\} + 2x_2\{x_3, x_2\} = 2x_1x_2 - 2x_2x_1 = 0. \end{aligned}$$

Thus symplectic leaves are contained in spheres $x_1^2 + x_2^2 + x_3^2 = r^2$. Each leaf is a connected open 2-manifold in $\mathbb{S}^2$, so each leaf is homeomorphic to $\mathbb{S}^2$.

It is easily checked that the corresponding symplectic structure is the unique SU(2) invariant volume form on the sphere of radius r.

Example 1.54. (Natsume-Olsen Poisson sphere)

$$\mathbb{S}^2 \subset \mathbb{C} \times \mathbb{R}, \quad \zeta\bar{\zeta} + z^2 = 1$$

The following Poisson brackets on $\mathbb{S}^2$ were introduced in [62]

$$\begin{aligned} \{\zeta,\, z\} &= i(1 - z^2)\zeta, \\ \{\bar{\zeta},\, z\} &= -i(1 - z^2)\bar{\zeta}, \\ \{\zeta,\, \bar{\zeta}\} &= -2i(1 - z^2)z. \end{aligned}$$

Therefore

$$\Pi = (1 - z^2)[i\zeta\partial_\zeta \wedge \partial_z - i\bar{\zeta}\partial_{\bar{\zeta}} \wedge \partial_z - 2iz\partial_z \wedge \partial_{\bar{\zeta}}] = (1 - z^2)\Pi_0.$$

where Π_0 is the standard rotation invariant symplectic (i.e. volume) form on $\mathbb{S}^2$. The sharp map $\#_\Pi$ is given by:

$$\begin{aligned} dz &\mapsto (z^2 - 1)i(\zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}}), \\ dz &\mapsto (1 - z^2)i(\zeta\partial_z - 2z\partial_{\bar{\zeta}}), \\ dz &\mapsto (1 - z^2)i(-\bar{\zeta}\partial_z + 2z\partial_\zeta). \end{aligned}$$

so that in the obvious basis it is represented by the matrix:

$$\begin{pmatrix} 0 & (1 - z^2)i\zeta & -i(1 - z^2)\bar{\zeta} \\ (z^2 - 1)i\zeta & 0 & 2i(1 - z^2)z \\ -i(z^2 - 1)\bar{\zeta} & 2i(z^2 - 1)z & 0 \end{pmatrix}$$

Such matrix has rank $= 0$ if $\zeta = \bar{\zeta} = 0$, that is if $1 - z^2 = 0$. This happens in two points

$$\begin{aligned} \zeta = \bar{\zeta} = 0, &\quad z = -1 \text{ south pole}, \\ \zeta = \bar{\zeta} = 0, &\quad z = 1 \text{ north pole}. \end{aligned}$$

Chapter 2

Schouten-Nijenhuis bracket

2.1 Lie-Poisson bracket

Let (M, Π) be a Poisson manifold.

Theorem 2.1. *There exists a unique* $\mathbb{R}$*-linear skewsymmetric bracket* $[-,-]_\Pi \colon \Omega^1 M \times \Omega^1 M \to \Omega^1 M$ *such that*

1. $[df, dg] = d\{f, g\}$ *for all* $f, g \in C^\infty(M)$,
2. $[\alpha, f\beta] = f[\alpha, \beta] + (\#_\Pi(\alpha) f)\beta$ *for all* $\alpha, \beta \in \Omega^1 M,\ f \in C^\infty(M)$.

It is given by

$$[\alpha, \beta] = L_{\#_\Pi(\alpha)}\beta - L_{\#_\Pi(\beta)}\alpha - d(\Pi(\alpha, \beta)).$$

Furthermore $[-,-]_\Pi$ *is a Lie bracket and* $\#_\Pi \colon \Omega^1 M \to \mathfrak{X}(M)$ *is a Lie algebra homomorphism:*

$$[\#_\Pi(\alpha), \#_\Pi(\beta)] = \#_\Pi([\alpha, \beta]_\Pi).$$

Proof. Such $[-,-]_\Pi$ should be local i.e. if $\beta_1|_U = \beta_2|_U$ in a neighbourhood U of x_0, then $[\alpha, \beta_1]_\Pi(x_0) = [\alpha, \beta_2]_\Pi(x_0)$. Take a compact neighbourhood $x_o \in V_{x_0} \subset U$ and $f \in C^\infty(M)$ such that $f = 1$ on V_{x_0}, $f|_{M \setminus U} = 0$. Then

$$[\alpha, f\beta_1](x_0) = \underbrace{f(x_0)}_{=1}[\alpha, \beta_1]_\Pi(x_0) + (\underbrace{\#_\Pi(\alpha) f}_{=0})(x_0)\beta_1(x_0)$$

$$[\alpha, f\beta_1](x_0) = \underbrace{f(x_0)}_{=1}[\alpha, \beta_2]_\Pi(x_0).$$

Applying twice property 2. one has the following:

Lemma 2.2.

$$[h\alpha, f\beta]_\Pi = (hf)[\alpha, \beta] + h(\#_\Pi(\alpha) f)\beta - f(\#_\Pi(\beta) h)\alpha.$$

Take $\alpha = \sum \alpha_i dx_i$, $\beta = \sum \beta_i dx_i$, $\Pi = \sum \Pi_{ij}\partial_{x_i} \wedge \partial_{x_j}$. Then

$$\begin{aligned}
[\alpha, \beta]_\Pi &= \sum_{i,j} [\alpha_i dx_i, \beta_j dx_j]_\Pi \\
&= \sum_{i,j} \alpha_i\beta_j [dx_i, dx_j]_\Pi + \alpha_i(\#_\Pi(dx_i)\beta_j)dx_j - \beta_j(\#_\Pi(dx_j)\alpha_i)dx_i \\
&= \sum_{i,j} \left(\alpha_i\beta_j d\{x_i, x_j\} + \sum_k (\alpha_i \Pi_{ik}\partial_k\beta_j dx_j - \beta_j\Pi_{jk}\partial_k\alpha_i dx_i) \right) \\
&= d\left(\sum_{i,j} \Pi_{ij}\alpha_i\beta_j \right) - \sum_{i,j}\Pi_{ij}\beta_j d\alpha_i - \sum_{i,j}\Pi_{ij}\alpha_i d\beta_j \\
&= d\langle \Pi,\, \alpha\wedge\beta\rangle + i_{\#_\Pi(\alpha)}d\beta - i_{\#_\Pi(\beta)}d\alpha.
\end{aligned}$$

This does not depend on the choice of local coordinates, so we have the existence of $[-,-]_\Pi$.

Recall Cartan's magic formula:

$$L_X = di_X + i_X d$$

Using it we get:

$$\begin{aligned}
[df, dg]_\Pi &= L_{\#_\Pi(df)}dg - L_{\#_\Pi(dg)}df - d\Pi\{df, dg\} \\
&= L_{X_f}dg - L_{X_g}df - d\{f, g\} \\
&= d\underbrace{i_{X_f}dg}_{X_f(g)} - d\underbrace{i_{X_g}df}_{X_g(f)} - d\{f, g\} \\
&= d\{f, g\} + d\{f, g\} - d\{f, g\} \\
&= d\{f, g\}.
\end{aligned}$$

Furthermore

$$\begin{aligned}
[\alpha, f\beta]_\Pi &= d\langle \Pi,\, \alpha\wedge f\beta\rangle + i_{\#_\Pi(\alpha)}d(f\beta) - i_{\#_\Pi(f\beta)}d\alpha \\
&= d(f\langle \Pi,\, \alpha\wedge\beta\rangle) + i_{\#_\Pi(\alpha)}d(f\beta).
\end{aligned}$$

The Jacobi identity is proved locally using *adf*, *bdg*, *cdh*. □

Definition 2.3. *Let M be a manifold, $E \to M$ vector bundle. Then E is called a* ***Lie algebroid*** *if there exists a bilinear bracket*

$$[-,-]\colon \Gamma(E)\times\Gamma(E) \to \Gamma(E)$$

and a bundle map, called the ***anchor****, $\rho\colon E \to TM$ ($\rho\colon \Gamma(E) \to \mathfrak{X}(M)$) such that*

1. *$(\Gamma(E), [-,-])$ is a Lie algebra,*
2. *ρ is a Lie algebra homomorphism,*
3. *$[v, fw] = f[v, w] + (\rho(v)f)w$ for all $v, w \in \Gamma(E)$, $f \in C^\infty(M)$.*

Remarks:

- Given any Lie algebroid, the image of the anchor is always a generalized integrable distribution; its maximal integrable submanifolds are called **orbits** of the Lie algebroid.

- The tangent bundle TM to a manifold is always a Lie algebroid with the trivial anchor map $\rho = \mathrm{id}$.
- Theorem 2.1 proves that for any Poisson manifold M, its cotangent bundle T^*M is a Lie algebroid with anchor the sharp map. The orbits of this algebroid are the symplectic leaves of M

2.2 Schouten-Nijenhuis bracket

Let M be a smooth manifold of dimension M. Recall

$$\begin{aligned} \Omega^p(M) &= \Gamma(\Lambda^p T^*M) \text{ differential } p\text{-forms, } p \geq 1, \\ \mathfrak{X}^p(M) &= \Gamma(\Lambda^p TM) \ p\text{-multivector fields, } p \geq 1, \\ \Omega^0(M) = \mathfrak{X}^0(M) &= C^\infty(M), \\ \Omega^\bullet(M) &= \bigoplus_p \Omega^p(M), \\ \mathfrak{X}^\bullet(M) &= \bigoplus_p \mathfrak{X}^p(M) \text{ graded vector spaces.} \end{aligned}$$

External product gives both spaces a structure of graded, associative algebra, $\mathbb{Z}_2$-commutative (supercommutative) i.e.

$$P \wedge Q = (-1)^{\deg Q \deg P} Q \wedge P.$$

The natural duality pairing between T_xM and T_x^*M extends to a natural pairing between $\Omega^\bullet(M)$ and $\mathfrak{X}^\bullet(M)$ as follows

$$\langle \alpha,\, X\rangle_x := \langle \underbrace{\alpha(x)}_{\in T_x^*M},\, \underbrace{X(x)}_{\in T_xM}\rangle \quad \text{for } \alpha \in \Omega^1(M),\ X \in \mathfrak{X}^1(M),$$

$$\langle \omega,\, P\rangle = \begin{cases} 0 & p \neq q, \\ \det(\langle \alpha_i,\, X_j\rangle) & p = q \quad \text{for } \omega = \alpha_1 \wedge \cdots \wedge \alpha_q,\ P = X_1 \wedge \cdots \wedge X_p, \\ & \alpha_i \in \Omega^1(M),\ X_j \in \mathfrak{X}^1(M). \end{cases}$$

Remark 2.4. $\langle \omega,\, P\rangle(x)$ depends only on $\omega(x)$, $P(x)$. Locally every q-form (resp. p-vector field) is decomposable. Therefore formula above defines a $C^\infty(M)$-bilinear pairing on the whole space M.

Inner product for $P \in \mathfrak{X}^\bullet(M)$, $\omega \in \Omega^\bullet(M)$:

$$\langle i_P\omega,\, Q\rangle = \langle \omega,\, P \wedge Q\rangle \quad \forall\, Q \in \mathfrak{X}^\bullet(M).$$

It is the left transpose of external product.

$\mathfrak{X}^1(M)$ is a Lie algebra with bracket $[X, Y] = XY - YX$ of vector fields. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$.

Proposition 2.5. *There is a unique bracket on $\Lambda^\bullet\mathfrak{g}$ which extends the Lie bracket on $\mathfrak{g}$ and such that if $A \in \Lambda^a\mathfrak{g}$, $B \in \Lambda^b\mathfrak{g}$, $C \in \Lambda^c\mathfrak{g}$ then:*

1. $[A, B] = -(-1)^{(a-1)(b-1)}[B, A]$;
2. $[A, B \wedge C] = [A, B] \wedge C + (-1)^{(a-1)b} B \wedge [A, C]$;

3. $(-1)^{(a-1)(c-1)}[A,[B,C]] + (-1)^{(b-1)(c-1)}[B,[C,A]] + (-1)^{(c-1)(b-1)}[C,[A,B]] = 0$*;*

4. The bracket of an element in $\Lambda^\bullet$ *with an element in* $\Lambda^0\mathfrak{g} = \mathbb{K}$ *is 0.*

Remark: It is not correct to say that $\Lambda^\bullet\mathfrak{g}$ is a graded Lie algebra. In fact in a graded Lie algebra the 0-component should be a Lie subalgebra, therefore the 0-component should be $\mathfrak{g}$. $\Lambda^{\bullet+1}\mathfrak{g}$ is a graded Lie algebra.

Proof. Start from 2 to prove a formula for

$$[A, B_1 \wedge \cdots \wedge B_n] = \sum_{i=1}^{n} (-1)^* B_1 \wedge \cdots \wedge [A, B_i] \wedge \cdots \wedge B_n$$

Then using 1 prove the formula for $[B_1 \wedge \cdots \wedge B_n, A]$ and having that prove the formula for $[B_1 \wedge \cdots \wedge B_n, A_1 \wedge \cdots \wedge A_n]$. Extend by $\mathbb{K}$-linearity to sum of decomposables. Verify Jacobi identity. □

Proposition 2.6. *Let* M *be a manifold. Then there exists a unique* $\mathbb{R}$*-bilinear bracket* $[-,-]\colon \mathfrak{X}^\bullet(M) \times \mathfrak{X}^\bullet(M) \to \mathfrak{X}^\bullet(M)$ *such that*

1. $[-,-]$ *is of degree -1;*

2. For all $X \in \mathfrak{X}^1(M)$ *and* $Q \in \mathfrak{X}^\bullet(M)$

$$[X, Q] = L_X Q.$$

In particular the bracket coincides with the usual Lie bracket of vector fields on $\mathfrak{X}^1(M)$*;*

3. For all $P \in \mathfrak{X}^p(M)$ *and* $Q \in \mathfrak{X}^q(M)$

$$[P, Q] = -(-1)^{(p-1)(q-1)}[Q, P]$$

4. For all $P \in \mathfrak{X}^p(M)$*,* $Q \in \mathfrak{X}^q(M)$*,* $R \in \mathfrak{X}^\bullet(M)$

$$[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{(p-1)q} Q \wedge [P, R].$$

Such bracket will be called the ***Schouten-Nijenhuis bracket*** *of multivector fields. It will furthermore satisfy:*

5. For all $P \in \mathfrak{X}^p(M)$*,* $Q \in \mathfrak{X}^q(M)$*,* $R \in \mathfrak{X}^r(M)$

$$(-1)^{(p-1)(r-1)}[P,[Q,R]] + (-1)^{(q-1)(p-1)}[Q,[R,P]] + (-1)^{(r-1)(q-1)}[R,[P,Q]] = 0.$$

Proof. The first step is to proof that such bracket has to be a local operation, i.e. for all U open in M, $[P,Q]|_U$ depends only on $P|_U$, $Q|_U$. The proof of this fact is similar to the analogous proof in theorem 2.1. Due to antisymmetry it is enough to show that if $Q_1|_U = Q_2|_U$ then $[P, Q_1](x_0) = [P, Q_2](x_0)$ for a neighbourhood U of x_0. Take $f = 0$ outside U, $f = 1$ in a compact neighbourhood of x_0 contained in U. Then $fQ_1 = fQ_2$ on M. Applying property 4. with $Q = f \in \mathfrak{X}^0 M$ we get

$$[P, fR] = [P, f] \wedge R - f[P, R] = (L_P f)R - f[P, R].$$

Now show that

$$[P, fQ_1](x_0) = [P, Q_1](x_0)$$
$$[P, fQ_2](x_0) = [P, Q_2](x_0)$$

Locality allows us to work in a coordinate chart. P and Q can therefore be taken as finite sums of exterior products of vector fields. Remark that from property 4. we get

$$[X, Q_1 \wedge \cdots \wedge Q_n] = \sum_{i=1}^{n} (-1)^* Q_1 \wedge \cdots \wedge [X, Q_i] \wedge \cdots \wedge Q_n,$$

$$[P_1 \wedge \cdots \wedge P_n, Q_1 \wedge \cdots \wedge Q_m] =$$
$$= \sum_{i<j} (-1)^* [P_i, Q_j] \wedge P_1 \wedge \cdots \wedge \widehat{P_i} \wedge \cdots \wedge P_n \wedge Q_1 \wedge \cdots \wedge \widehat{Q_j} \wedge \cdots \wedge Q_m.$$

Now

$$[P_1 \wedge \cdots \wedge P_n, fQ_1 \wedge \cdots \wedge Q_m] =$$
$$= \underbrace{[P_1 \wedge \cdots \wedge P_n, f]}_{=(-1)^n [f, P_1 \wedge \cdots \wedge P_n]} \wedge Q_1 \wedge \cdots \wedge Q_m + (-1)^m f [P_1 \wedge \cdots \wedge P_n, Q_1 \wedge \cdots \wedge Q_m]$$

and

$$(-1)^n [f, P_1 \wedge \cdots \wedge P_n] = (-1)^n \sum_{i=1}^{n} L_{P_i}(f) P_1 \wedge \cdots \wedge \widehat{P_i} \wedge \cdots \wedge P_n.$$

These fixes all values and thus proves unicity. Finally one has to prove the independence of local coordinates - on the overlapping coordinate domains you have the same result. Other properties are proved by direct (lengthy) computation. □

Definition 2.7. *A* ***Gerstenhaber algebra*** *is a triple* $(\mathfrak{A}, \wedge, [-,-])$ *such that*

1. *a is a* $\mathbb{N}$*-graded vector space,* $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \ldots$*;*
2. $\wedge$ *is an associative, supercommutative multiplication of degree 0 (*$\mathfrak{A}_i \wedge \mathfrak{A}_j \subset \mathfrak{A}_{i+j}$*);*
3. $[-,-]$ *is a super Lie algebra structure of degree* (-1) *(*$[\mathfrak{A}_i, \mathfrak{A}_j] \subset \mathfrak{A}_{i+j-1}$*) satisfying*

$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{(|a|-1)|b|} b \wedge [a, c].$$

Examples 2.8.

- Multivector fields on a manifold M are a Gerstenhaber algebra with respect to Schouten-Nijenhuis bracket.
- Differential forms on Poisson manifold are a Gerstenhaber algebra.
- Similarly from any Lie algebroid E there is a natural construction of Gerstenhaber algebra on $\Gamma(\Lambda^\bullet E)$ generalizing the costruction of the Schouten-Nijenhuis bracket (just remark that the proof of proposition 2.6 uses exactly the fact that TM is a Lie algebroid with anchor $\rho = \mathrm{id}$).
- From any Lie algebra $\mathfrak{g}$ you can construct a Gerstenhaber algebra $\Lambda^\bullet \mathfrak{g}$.

- Hochschild cohomology has a Gerstenhaber algebra structure (coefficients in the given algebra). Hochschild-Kostant-Rosenberg map

$$\phi_{HKR}\colon \mathrm{HH}^\bullet_{\mathrm{cont}}(C^\infty(M)) \to \mathfrak{X}^\bullet(M)$$

  fails to be a Gerstenhaber algebra morphism. This is what leads to L_∞-algebra structures and Kontsevich formality.

Let $(\mathfrak{A}, \wedge, [-,-])$ be a Gerstenhaber algebra. An operator $D\colon \mathfrak{A}^\bullet \to \mathfrak{A}^{\bullet-1}$ is said to generate the Gerstenhaber algebra if for all $a \in \mathfrak{A}^i$, $b \in \mathfrak{A}$

$$[a,b] = (-1)^i (D(a\wedge b) - Da \wedge b - (-1)^i a \wedge Db).$$

If $D^2 = 0$ we say that our Gerstenhaber algebra is **exact** or **Batalin-Vilkovisky algebra**.

We will show that the Gerstenhaber algebra of differential forms on a Poisson manifold is a Batalin-Vilkovisky algebra. Its generating operator will be called **Poisson** (or **canonical** or **Brylinski**) **differential**.

2.2.1 Schouten-Nijenhuis bracket computations

Fix a system of local coordinates and consider two vector fields

$$X = \sum_i a_i \partial_{x_i}, \quad Y = \sum_i b_i \partial_{x_i}, \quad X, Y \in \mathfrak{X}(M), \quad x_1, \dots, x_n \text{ coordinates}$$

$$[X,Y] = \sum_i a_i \left(\sum_j \frac{\partial b_j}{\partial x_i} \partial_{x_j} \right) - \sum_i b_i \left(\sum_j \frac{\partial a_j}{\partial x_i} \partial_{x_j} \right).$$

Let $\zeta_i = \partial_{x_i}$ and consider it as an odd formal variable

$$\zeta_i \zeta_j = -\zeta_j \zeta_i \quad (\partial_{x_i} \wedge \partial_{x_j} = -\partial_{x_j} \wedge \partial_{x_i}).$$

Then

$$X := \sum_i a_i \zeta_i, \quad Y := \sum_i b_i \zeta_i$$

$$\begin{aligned} [X,Y] &= \sum_i \left(\frac{\partial X}{\partial \zeta_i} \frac{\partial Y}{\partial x_i} - \frac{\partial Y}{\partial \zeta_i} \frac{\partial X}{\partial x_i} \right) \\ &= \left(\sum_i \partial_{\zeta_i} \wedge \partial_{x_i} \right) (X \otimes Y). \end{aligned}$$

Extend this idea to multivector fields

$$P \in \mathfrak{X}^p(M), \quad P = \sum_{i_1 < \ldots < i_p} \partial_{x_{i_1}} \wedge \ldots \wedge \partial_{x_{i_p}} = \sum_{i_1 < \ldots < i_p} P_{i_1 \ldots i_p} \zeta_{i_1} \ldots \zeta_{i_p}.$$

Fix the following differentiation rule

$$\begin{aligned} &\partial_{\zeta_{i_p}} (\zeta_{i_1} \ldots \zeta_{i_p}) = \zeta_{i_1} \ldots \zeta_{i_{p-1}} \\ &\partial_{\zeta_{i_k}} (\zeta_{i_1} \ldots \zeta_{i_p}) = (-1)^{p-k} \zeta_{i_1} \ldots \widehat{\zeta_{i_k}} \ldots \zeta_{i_{p-1}}. \end{aligned}$$

Then we claim that

$$[P,Q]_{SN} = \sum_i \partial_{\zeta_i} P \, \partial_{x_i} Q - (-1)^{(p-1)(q-1)} \partial_{\zeta_i} Q \, \partial_{x_i} P.$$

2.2.2 Lichnerowicz definition of the Schouten-Nijenhuis bracket

Lichnerowicz defined the Schouten-Nijenhuis bracket implicitly as follows.

Proposition 2.9. *For all* $P \in \mathfrak{X}^p M$*,* $Q \in \mathfrak{X}^q M$*,* $\omega \in \Omega^{p+q-1} M$

$$\langle \omega,\, [P, Q] \rangle = (-1)^{(p-1)(q-1)} \langle d(i_Q \omega),\, P \rangle - \langle d(i_P \omega),\, Q \rangle + (-1)^p \langle d\omega,\, P \wedge Q \rangle \tag{2.1}$$

With respect to our explicit construction this formula has the advantage of being well adapted and easy to use in "global type" computations.

Look at what happens, for example, when $X, Y \in \mathfrak{X}^1(M)$, $\omega \in \Omega^1 M$

$$\langle \omega,\, [X, Y] \rangle = \langle d(i_Y \omega),\, X \rangle - \langle d(i_X \omega),\, Y \rangle - \langle d\omega,\, X \wedge Y \rangle$$

which you can rewrite as

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

i.e. the formula for the differential of 1-form.

In our approach this formula needs a proof. It suffices to show that the bracket defined by 2.1 has the same algebraic properties as the Schouten-Nijenhuis bracket. Unicity then implies the claim.

2.2.3 Jacobi condition and Schouten-Nijenhuis bracket

Let Π be a bivector on M, so $[\Pi, \Pi] \in \mathfrak{X}^3 M$. Let ω be a 3-form on M. Then $\langle \omega,\, [\Pi, \Pi] \rangle$ is a function and

$$\langle \omega,\, [\Pi, \Pi] \rangle = -\langle d(i_\Pi \omega),\, \Pi \rangle - \langle d(i_\Pi \omega),\, \Pi \rangle + \langle d\omega,\, \Pi \wedge \Pi \rangle.$$

Let $\omega = df \wedge dg \wedge dh$. Put $\{f, g\} = \langle df \wedge dg,\, \Pi \rangle$. Remark that

$$\langle i_\Pi \omega,\, X \rangle = \langle df \wedge dg \wedge dh,\, \Pi \wedge X \rangle$$

for all $X \in \mathfrak{X}^1 M$. Since $d(df \wedge dg \wedge dh) = 0$ we have

$$\langle \omega,\, [\Pi, \Pi] \rangle = -2 \langle d(i_\Pi \omega),\, \Pi \rangle.$$

Lemma 2.10.

$$\langle df \wedge dg \wedge dh,\, [\Pi, \Pi] \rangle = \pm 2\, \mathrm{Jac}(f, g, h)$$

Proof.

$$\begin{aligned}
\langle df \wedge dg \wedge dh,\, [\Pi, \Pi] \rangle &= -2 \langle d(i_\Pi (df \wedge dg \wedge dh)),\, \Pi \rangle = \\
&= -2 \langle d(\{g,\, h\} df - \{f,\, h\} dg + \{f,\, g\} dh),\, \Pi \rangle = \\
&= -2 \langle d\{g, h\} \wedge df - d\{f,\, h\} \wedge dg + d\{f,\, g\} \wedge dh,\, \Pi \rangle = \\
&= -2 (\{\{g,\, h\},\, f\} - \{\{f,\, h\},\, g\} + \{\{f,\, g\},\, h\}) = \\
&= 2\, \mathrm{Jac}(f, g, h).
\end{aligned}$$

□

Corollary 2.11. *A bivector* $\Pi \in \mathfrak{X}^2 M$ *is Poisson if and only if* $[\Pi, \Pi] = 0$.

As an application of corollary (2.11) consider a Lie algebra $\mathfrak{g}$, manifold M, $\xi\colon \mathfrak{g} \to \mathfrak{X}(M)$ an infinitesimal map (a Lie algebra homomorphism). Then ξ extends uniquely to a degree 0 map

$$\wedge\xi\colon \Lambda^\bullet\mathfrak{g} \to \mathfrak{X}^\bullet(M)$$
$$x_1 \wedge \cdots \wedge x_n \mapsto \xi(x_1) \wedge \cdots \wedge \xi(x_n)$$

which preserves the graded brackets

$$\wedge\xi[\alpha,\beta] = [\wedge\xi(\alpha), \wedge\xi(\beta)]_{SN}$$

This is a simple consequence of the fact that both brackets are determined by their values in degree 0 and 1.

Another easy consequence of characterization $[\Pi,\Pi] = 0$ is that if $\dim M = 2$, then any bivector $\Pi \in \mathfrak{X}^2 M$ is Poisson. Indeed $[\Pi,\Pi] \in \mathfrak{X}^3 M = 0$.

Let now G be a connected Lie group such that $\mathrm{Lie}(G) = \mathfrak{g}$ (not necessarily simply connected). Let us denote the translation operators by

$$L_g\colon G \to G, \quad h \mapsto gh,$$
$$R_g\colon G \to G, \quad h \mapsto hg.$$

We will use the following notations for the tangent maps $T_eG \to T_gG$.

$$L_{g,*}\colon T_hG \to T_{gh}G \qquad R_{g,*}\colon T_hG \to T_{hg}G$$
$$L^\wedge_{g,*}\colon \Lambda^\bullet T_eG \to \Lambda^\bullet T_gG \qquad R^\wedge_{g,*}\colon \Lambda^\bullet T_eG \to \Lambda^\bullet T_gG$$

Let now $\alpha \in \Lambda^\bullet\mathfrak{g}$. We will denote with α^L (resp. α^R) the left (resp. right) invariant multivector field on G whose value at $e \in G$ (identity of G) is α i.e.

$$\alpha^L(g) := L^\wedge_{g,*}\alpha, \quad (\text{resp. } \alpha^R(g) := R^\wedge_{g,*}\alpha)$$

In the same way we consider α^R to be the right invariant multivector field on G whose value at e is α.

Proposition 2.12. *For any $\gamma \in \Lambda^2\mathfrak{g}$ the following are equivalent*

1. *γ^L is a left invariant Poisson structure.*
2. *γ^R is a right invariant Poisson structure.*
3. *$[\gamma,\gamma] = 0$ (bracket in $\Lambda^\bullet\mathfrak{g}$)*

Proof. $L_{X,g}\colon \mathfrak{g} \to \mathfrak{X}(G)$ is an infinitesimal action.

$$[\gamma^L,\gamma^L]_{SN}(g) = [L^\wedge_{*,g}\gamma, L^\wedge_{*,g}\gamma](g) = L^\wedge_{*,g}[\gamma,\gamma],$$

so the left hand side is zero if and only if the right hand side is zero. $[\gamma^L,\gamma^L]_{SN} = 0$ is the condition for being Poisson. The computation for right invariant multivectors is exactly the same. □

The condition $[\gamma,\gamma] = 0$ is called **classical Yang-Baxter equation**. The above proposition can be stated as:

Corollary 2.13. *There is a one to one correspondence between left (resp. right) invariant Poisson structures on a Lie group G and solutions of classical Yang-Baxter equation on* $\mathrm{Lie}(G)$.

2.2.4 Compatible Poisson tensors

Say that Π_1, Π_2 are Poisson bivectors on M.
Question: is $\Pi_1 + \Pi_2$ a Poisson bivector on M ?
If this is the case we will say that they are **compatible Poisson tensors**.

Proposition 2.14. *Π_1 and Π_2 are compatible if and only if $[\Pi_1, \Pi_2] = 0$.*

In this case $a\Pi_1 + b\Pi_2$ is Poisson for all $a, b \in \mathbb{R}$ and $\{a\Pi_1 + b\Pi_2 : a, b \in \mathbb{R}\}$ is called a **Poisson pencil**.

Proof.

$$[\Pi_1 + \Pi_2, \Pi_1 + \Pi_2] = \underbrace{[\Pi_1, \Pi_1]}_{=0} + \underbrace{[\Pi_2, \Pi_2]}_{=0} + 2[\Pi_1, \Pi_2],$$

so $[\Pi_1 + \Pi_2, \Pi_1 + \Pi_2] = 0$ if and only if $[\Pi_1, \Pi_2] = 0$.
Furthermore

$$[a\Pi_1 + b\Pi_2, a\Pi_1 + b\Pi_2] = 2ab[\Pi_1, \Pi_2].$$

□

2.2.5 Koszul's formula

Theorem 2.15 (Koszul formula). *Let $P \in \mathfrak{X}^p M$, $Q \in \mathfrak{X}^q M$. Then*

$$i_{[P,Q]} = (-1)^{(p-1)(q-1)} i_P d i_Q - i_Q d i_P + (-1)^p i_{P \wedge Q} d + (-1)^q d i_{P \wedge Q} \tag{2.2}$$

Remark: Koszul formula implies Lichnerowicz formula (2.1) after contracting with $(p + q - 1)$-form.

Proof. (of Koszul formula) By induction using Leibniz rule. Say first you want to prove it for $\deg P = 0$, $\deg Q$ whatever.

$$[P, Q] = L_Q f \omega \quad P = f, \quad i_P \omega = f \omega$$

$$i_{L_Q f} = f d i_q - i_Q df + \underbrace{i_{fQ} d + (-1)^q d i_{fQ}}_{L_{fQ}}$$

$$L_{fQ} \omega = f L_Q \omega$$

$$i_{L_Q f} \omega = \langle L_Q f, \omega \rangle.$$

□

This formula can be also memorized as

$$i_{[P,Q]} = [[i_P, d], i_Q]$$

but with graded commutators on the right !

2.3 Poisson homology

Definition 2.16. ***Canonical*** *(or* ***Brylinski****)* ***operator***

$$\partial_\Pi := i_\Pi d - d i_\Pi \colon \Omega^k M \to \Omega^{k+1} M$$

Proposition 2.17. *The following identities are verified*

1. $d\partial_\Pi + \partial_\Pi d = 0$.

2. $\partial_\Pi i_\Pi - i_\Pi \partial_\Pi = 0$.

3. $\partial_\Pi^2 = 0$.

Proof.

1. $d\partial_\Pi = d i_\Pi d = -\partial_\Pi d$.

2. Apply Koszul's formula to see that

$$0 = i_{[\Pi,\Pi]} = [[i_\Pi, d], i_\Pi] = [\partial_\Pi, i_\Pi].$$

3. $\partial_\Pi i_\Pi = i_\Pi \partial_\Pi$ as a consequence of Koszul formula. Thus

$$2 i_\Pi d i_\Pi = i_\Pi^2 d - d i_\Pi^2.$$

Apply d on the left

$$2 d i_\Pi d i_\Pi = d i_\Pi^2 d.$$

Apply d on the right

$$2 i_\Pi d i_\Pi d = d i_\Pi^2 d.$$

Therefore $d i_\Pi d i_\Pi = -i_\Pi d i_\Pi d$ and

$$\begin{aligned}\partial_\Pi^2 &= (i_\Pi d - d i_\Pi)(i_\Pi d - d i_\Pi)\\ &= i_\Pi d i_\Pi d - d i_\Pi i_\Pi d + d i_\Pi d i_\Pi\\ &= 2 i_\Pi d i_\Pi d - d i_\Pi i_\Pi d = 0.\end{aligned}$$

□

Definition 2.18. *The homology of the complex* $(\Omega^\bullet, \partial_\Pi)$ *is called* ***Poisson*** *(or* ***canonical****)* ***homology*** *and it is denoted by* $\mathrm{H}_k^\Pi(M)$*.*

Chapter 3

Poisson maps

3.1 Poisson maps

Recall that if (M_1, Π_1), (M_2, Π_2) are Poisson manifolds then $\varphi\colon M_1 \to M_2$ is a Poisson map if $\varphi^*\colon C^\infty(M_2) \to C^\infty(M_1)$ verifies

$$\varphi^*\{f, g\}_{M_2} = \{\varphi^* f, \varphi^* g\}_{M_1}$$

for all $f, g \in C^\infty(M_2)$. Here $\varphi^*\colon f \mapsto f \circ \varphi$.

Recall also from differential geometry that having a map $\varphi\colon M \to N$, you can pull-back forms, but in general you cannot push-forward vector fields.

Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Then the two vector fields X and Y are said to be **φ-related** if

$$\varphi_{*,x}(X_x) = Y_{\varphi(x)}$$

for all $x \in M$, $\varphi_{*,x}\colon T_x M \to T_{\varphi(x)} N$. This relation does not define a map. In fact you can have more than one vector field on N related to a fixed vector field on M. (Think of $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi(x, y) = (x, 0)$. Then saying that Y is φ-related to X says something only about values of Y on the line $(x, 0)$.) You can also have none. (In the example as before if $X_{(x,0)}$ and $X_{(x,1)}$ have different projections on $\operatorname{im} \varphi_{*,x}$).

If φ is a diffeomorphism then $\varphi_*\colon \mathfrak{X}(M) \to \mathfrak{X}(N)$. Remark that you can define φ-relation on multivectors simply by considering $\varphi^\wedge_{*,x}$.

Proposition 3.1. *Let (M_1, Π_1), (M_2, Π_2) be Poisson manifolds and $\varphi\colon M_1 \to M_2$ smooth map. The following are equivalent.*

1. *φ is Poisson map.*

2. *$X_{\varphi^* f} \in \operatorname{Ham}(M_1)$ and $X_f \in \operatorname{Ham}(M_2)$ are φ-related for all $f \in C^\infty(M_2)$ i.e.*

$$\varphi_{*,x}(X_{\varphi^* f}(x)) = X_f(\varphi(x))$$

for all $x \in M$.

3. *Let ${}^t\varphi_{*,x}\colon T^*_{\varphi(x)} M_2 \to T^*_x M_1$. Then*

$$\#_{\Pi_2, \varphi(x)} = \varphi_{*,x} \circ \#_{\Pi_1, x} \circ^t \varphi_{*,x}$$

4. Π_1, Π_2 *are* φ*-related i.e.*

$$\langle \Pi_{2,\varphi(x)},\, \alpha \wedge \beta \rangle = \langle \Pi_{1,x},\, {}^t\varphi_{*,x}\alpha \wedge^t \varphi_{*,x}\beta \rangle$$

for all $\alpha, \beta \in T^*_{\varphi(x)} M_2$ *and all* $x \in M_1$.

Proof. (3) $\iff$ (4) by definitions.
(1) $\iff$ (4) using

$$\{f,\, g\}(x) = \langle \Pi_x,\, d_x f \wedge d_x g \rangle$$

and the fact that for all $\alpha \in T^*_{\varphi(x)} M_2$ there exists $f \in C^\infty(M_2)$ such that $d_{\varphi(x)} f = \alpha$.
(1) $\iff$ (2) using

$$\{f,\, g\} = X_f g = \langle X_f,\, dg \rangle.$$

□

Remark 3.2. From property (3):

$$\rho_{\Pi_1}(x) \geq \rho_{\Pi_2}(\varphi(x))$$

because $(\operatorname{im} \#_{\Pi_2})_{\varphi(x)} \subseteq \varphi_{*,x}(\operatorname{im} \#_{\Pi_1,x})$. This fact has remarkable, though easy consequences.

- Let $x_0 \in M_1$ be a 0-dimensional symplectic leaf. Then $\varphi(x_0)$ is a 0-dimensional symplectic leaf. Thus for example there is no Poisson map $\varphi\colon \mathfrak{g}^* \to M$ if M is symplectic and $\mathfrak{g}$ is a Lie algebra.

- Let $\varphi\colon M_1 \to M_2$ be a Poisson immersion, i.e. a Poisson map such that $\varphi_{*,x}$ is injective. Then $\operatorname{rank}_{\Pi_1}(x) = \operatorname{rank}_{\Pi_2}(x)$. This in particular holds if φ is a Poisson (local) diffeomorphism (even more so for Poisson automorphism).

- Let $\varphi\colon M_1 \to M_2$ be a Poisson map between symplectic manifolds. Then

$$\underbrace{\rho_{\Pi_1}(x)}_{\dim M_1} \geq \underbrace{\rho_{\Pi_2}(\varphi(x))}_{\dim M_2}$$

  and φ has to be a submersion i.e. $\varphi_{*,x}$ surjective, because $\operatorname{im} \varphi_{*,x}$ is forced to be $T_{\varphi(x)} M_2$ for all $x \in M_1$. So the only Poisson maps between symplectic manifolds are submersions.

This shows that being a Poisson map between symplectic manifolds is very different from being a symplectic map (which means $\varphi\colon M_1 \to M_2$, $\varphi^*\omega_2 = \omega_1$). This difference is made explicit by the following two examples.

Example 3.3.

$$\begin{aligned} i\colon \mathbb{R}^2 &\to \mathbb{R}^4, \\ (q_1, p_1) &\mapsto (q_1, p_1, 0, 0), \\ \omega_1 &= dq_1 \wedge dp_1, \\ \omega_2 &= dq_1 \wedge dp_1 + dq_2 \wedge dp_2. \end{aligned}$$

Then i is a symplectic map but it is not a Poisson map:

$$\underbrace{\{q_2,\, p_2\} \circ i}_{=1} \neq \underbrace{\{q_2 \circ i,\, p_2 \circ i\}}_{=0}.$$

Example 3.4.

$$\begin{aligned} \psi\colon \mathbb{R}^4 &\to \mathbb{R}^2, \\ (q_1, p_1, q_2, p_2) &\mapsto (q_1, p_1), \\ \omega_1 &= dq_1 \wedge dp_1 + dq_2 \wedge dp_2, \\ \omega_2 &= dq_1 \wedge dp_1. \end{aligned}$$

Then ψ is a Poisson map:

$$\underbrace{\{q_2,\, p_2\} \circ \psi}_{=1} = \underbrace{\{q_2 \circ \psi,\, p_2 \circ \psi\}}_{=1}$$

but it is not symplectic:

$$\psi^*(dq_1 \wedge dp_1) = dq_1 \wedge dp_1 \neq dq_1 \wedge dp_1 + dq_2 \wedge dp_2.$$

This difference between morphisms in the Poisson and symplectic categories implies, obviously, that related concepts such as subobjects and quotients have different behaviours. We will see later an example of this issue when referring to submanifolds.

Proposition 3.5. *Let (M_i, Π_i), $i = 1, 2, 3$ be Poisson manifolds. Let $\varphi\colon M_1 \to M_2$ and $\psi\colon M_2 \to M_3$ be smooth maps.*

1. *If φ and ψ are Poisson, then $\psi \circ \varphi$ is Poisson.*
2. *If φ and $\psi \circ \varphi$ are Poisson, and φ is surjective, then ψ is Poisson.*
3. *If φ is Poisson and a diffeomorphism, then φ^{-1} is Poisson.*

Proof.

1. Obvious.
2. Take $y \in M_2$, $y = \varphi(x)$.

$$\begin{aligned} \#_{\Pi_3, \psi(y)} &= (\psi \circ \varphi)_{*,x} \circ \#_{\Pi_{1,x}} \circ^t (\psi \circ \varphi)_{*,x} \\ &= \psi_{*,y} \circ \varphi_{*,x} \circ \#_{\Pi_{1,x}} \circ^t \varphi_{*,x} \circ^t \psi_{*,y} \\ &= \psi_{*,y} \circ \#_{\Pi_{2,x}} \circ^t \psi_{*,y}. \end{aligned}$$

3. Follows immediately from (2).

□

Examples 3.6.

1. Let $\phi\colon \mathfrak{h} \to \mathfrak{g}$ be a Lie algebra morphism. Prove that $\phi^*\colon \mathfrak{g}^* \to \mathfrak{h}^*$ is a Poisson map. Is the converse true ?
2. Any Poisson map from M to a connected symplectic manifold S, $\varphi\colon M \to S$ is a submersion.

Proof.

$$\varphi(T_xM) \subseteq T_xS$$

It is a submersion if and only if equality holds. Say there is no equality.

$$\underbrace{\varphi_{*,x}(\Pi_M(x))}_{=\Pi_S(x)} \subseteq \varphi_{*,x}\Lambda^2 T_xM.$$

But then choose $\xi \in T_x^*S$ such that $\xi \in \varphi_{*,x}(T_xM)^\perp$, $\xi \neq 0$. Then $\langle \xi, \Pi_S(x)\rangle = 0$ contradicting nondegeneracy of Π_S. □

3. Any Poisson map $\varphi\colon M \to N$ such that M is symplectic is called a **symplectic realization** of (N, Π_N). It can be proven that any Poisson manifold admits a surjective symplectic realization. Note that surjectivity of φ implies that functions in $C^\infty(N)$ are faithfully represented as vector fields on M by $f \mapsto X_{\varphi^* f}$.

4. Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$. Consider T^*G with the standard symplectic structure. Let $L\colon T^*G \to \mathfrak{g}^*$ be defined by $(g, p) \mapsto (L_g)^* p$ for $L_g^*\colon T_g^*G \to T_e^*G = \mathfrak{g}^*$. Then L is always symplectic realization.

5. Any Poisson map $\mu\colon M \to \mathfrak{g}^*$ is called a **moment map**. When there is such a map, then M carries an infinitesimal $\mathfrak{g}$-action by infinitesimal Poisson automorphisms of which this is the moment map. This ia consequence of the fact that the composition

$$\mathfrak{g} \hookrightarrow C^\infty(\mathfrak{g}^*) \xrightarrow{\mu^*} C^\infty(M) \to \mathfrak{X}_{\mathrm{Ham}}(M)$$

is a Lie algebra homomorphism.

Definition 3.7. *Let (M, Π) be a Poisson manifold. A* ***Poisson vector field*** *$X \in \mathfrak{X}(M)$ (or* ***infinitesimal Poisson automorphism****) is a vector field such that its flow φ induces for all $t \in \mathbb{R}$ a local Poisson morphism $\varphi_t\colon M \to M$.*

Proposition 3.8. *Let (M, Π) be a Poisson manifold, and $X \in \mathfrak{X}(M)$. The following are equivalent:*

1. *X is a Poisson vector field*

2. *X is a canonical derivation i.e.*

$$X\{f, g\} = \{Xf, g\} + \{f, Xg\}$$

3. *$L_X\Pi = 0$.*

Proof. We have

$$\begin{aligned} L_X(\Pi(df, dg)) &= (L_X\Pi)(df, dg) + \Pi(L_X df, dg) + \Pi(df, L_X dg) \\ &= (L_X\Pi)(df, dg) + \Pi(dL_X f, dg) + \Pi(df, dL_X g), \end{aligned}$$

because $L_X d = dL_X$. Now rewriting this with brackets we get

$$X\{f, g\} - \{Xf, g\} - \{f, Xg\} = (L_X\Pi)(df, dg).$$

so 2) $\iff$ 3).

Let $X \in \mathfrak{X}(M)$, and let φ_t be its flow. Let $f, g \in C^\infty(M, \mathbb{R})$.

$$\underbrace{\frac{d}{dt}\varphi_{-t}^*\{\varphi_t^* f, \varphi_t^* g\}|_{t=t_0}}_{\frac{d}{dt}\{f\circ\varphi_t, g\circ\varphi_t\}\circ\varphi_{-t}} = -\varphi_{-t_0}^*(X\{\varphi_{t_0}^* f, \varphi_{t_0}^* g\}) + \varphi_{-t_0}^*(\{X\varphi_{t_0}^* f, \varphi_{t_0}^* g\}) + \varphi_{-t_0}^*(\{\varphi_{t_0}^* f, X\varphi_{t_0}^* g\})$$

Now 1) $\iff$ left hand side is 0, and 2) $\iff$ right hand side is 0. □

Remark 3.9.

1. From $L_{[X,Y]} = [L_X, L_Y]$ it immediately follows that the bracket of Poisson vector fields is a Poisson vector field.

2. Let X be a Poisson vector field. Then the rank $\rho_\Pi(x)$ is constant along the flow of X, but the flow of X need not be contained in a leaf.

3. Hamiltonian vector field is Poisson vector field, but the opposite is false. For example on $(M, \Pi = 0)$ every vector field is Poisson ($L_X 0 = 0$), but only the 0 vector field is hamiltonian. For $(\mathbb{R}^{2n}, \text{std})$ we have $\text{Ham}(M) = \text{Poiss}(M) = \mathfrak{X}(M)$.

Definition 3.10. *Let $d_\Pi \colon \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M)$, $\Pi \mapsto [\Pi, P]$. Then d_Π is called the **Lichnerowicz coboundary**.*

Remark 3.11. If Π is Poisson, then $d_\Pi^2 = 0$. In fact $[\Pi, [\Pi, P]] = \frac{1}{2}[[\Pi, \Pi], P]$ from the Jacobi identity of the Schouten bracket.

Definition 3.12. *The cohomology of the complex $(\mathfrak{X}^\bullet(M), d_\Pi)$ is called **Poisson (Lichnerowicz) cohomology** of (M, Π) and is denoted by $\text{H}_\Pi^k(M)$.*

We have $\text{H}_\Pi^0(M) = \text{Cas}(M)$, $[\Pi, f] = X_f$, and $\text{H}_\Pi^1(M) = \text{Poiss}(M)/\,\text{Ham}(M)$.

3.2 Poisson submanifolds

Recall that for Poisson manifolds (M_1, Π_1), (M_2, Π_2) $\varphi \colon M_1 \to M_2$ is a Poisson map if and only if $\varphi_{*,x}^{\wedge 2}(\Pi_1(x)) = \Pi_2(\varphi(x))$ for all $x \in M_1$.

Recall that a submanifold of M is a pair (N, i) where N is a manifold and $i \colon N \hookrightarrow M$ is an injective immersion.

Definition 3.13. *Let (M, Π_M) be a Poisson manifold. Then (N, i) is is a **Poisson submanifold** if N has a Poisson structure Π_N such that i is a Poisson map.*

Remark 3.14. If i is an immersion, then $i_{*,x}^{\wedge 2}$ is injective at every x and

$$i_{*,x}^{\wedge 2}(\Pi_N(x)) = \Pi_M(i(x))$$

uniquely determines Π_N to be Poisson diffeomorphic to the restriction of Π_M to $i(N)$. The symplectic leaves of a Poisson manifold are a natural example of Poisson submanifolds.

Proposition 3.15. *Every open subset U of (M, Π_M) is an open Poisson submanifold. A closed submanifold N of (M, Π_M) is Poisson if and only if it is a union of symplectic leaves.*

Proof. From Π being Poisson we have $\Pi|_U$ is Poisson for all open $U \subseteq M$.

Let (N, i) be a closed submanifold. The question is whether Π_N is a Poisson bivector on N. This is true if and only if Π_N is tangent to N at any of its points, which locally, around x, means exactly that the leaf through N is contained in N. Now apply the usual open-closed argument. □

Example 3.16. When M is a symplectic manifold the only Poisson submanifolds are open subsets. This is in contrast with what happens for symplectic submanifolds (think again at the case $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$). To relax this rigidity the notion of Poisson–Dirac submanifold of a Poisson manifold was recently introduced (see [22, 80]).

An interesting way to construct a Poisson manifold with prescribed Poisson submanifolds is that of gluing symplectic structures on given symplectic leaves. The following theorem ([72], page 26, gives a characterization for such construction. Let us remark that in general using topological constructions in the differential geometrical setting of Poisson manifold is, at the same time, an interesting and difficult procedure, related to what is called flexibility of the geometrical structure. A construction of suspension of Poisson structures on spheres was realized in [7]. For other constructions and some general consideration see [48, 23].

Proposition 3.17. *Let M be a differentiable manifold and let $\mathcal{F}$ be a generalized foliation on M such that every leaf $F \in \mathcal{F}$ is endowed with a symplectic structure ω_F. For any $f \in C^\infty(M)$ define $X_f(x) = \#_{\omega_F^{-1}}(d_x f)$. If all X_f's are differentiable vector fields then there exists a unique Poisson structure on M having symplectic (F, ω_F) as symplectic leaves.*

Example 3.18. Let $D = D(0,1)$ be the disc and let Π_1, Π_2 be Poisson structures on D going to zero at the boundary. Then there exists a unique Poisson structure Π on $\mathbb{S}^2$ such that

$$i_+ \colon D \to \mathbb{S}^2$$

sending D to the upper hemisphere is a Poisson submanifold (D, Π_1),

$$i_- \colon D \to \mathbb{S}^2$$

sending D to the lower hemisphere is a Poisson submanifold (D, Π_2),

$$i_0 \colon \mathbb{S}^1 \to \mathbb{S}^2$$

sending $\mathbb{S}^1$ to the equator is a Poisson submanifold $(\mathbb{S}^1, 0)$.

Proof. There exists a uniquely determined bivector Π on $\mathbb{S}^2$ with such properties. The question is whether this bivector is Poisson and smooth. Being $[\Pi, \Pi] = 0$ a local condition it certainly holds true at any point of the lower and upper hemisphere. Then if Π is smooth, being $[\Pi, \Pi]$ smooth, by continuity $[\Pi, \Pi] = 0$ on all of $\mathbb{S}^2$. So smoothness is the only real issue here. But the smoothness is true if by "going to zero at the border" we mean that it can be smoothly extended over the border. Then with a partition of unity argument you can prove smooth gluing. □

3.3 Coinduced Poisson structures

Let $\varphi \colon M_1 \to M_2$ be a surjective map. Then if we want it to be Poisson, then Π_2 is uniquely determined by Π_1.

Definition 3.19. *A surjective mapping from a Poisson manifold can be Poisson for at most one Poisson structure on M_2. If this is the case we will say that the Poisson structure on M_2 is **coinduced** via φ from that on M_1.*

Proposition 3.20. *Let (M_1, Π_1) be a Poisson manifold. If $\varphi \colon (M_1, \Pi_1) \to M_2$ is a surjective differenitable map, then M_2 has a coinduced Poisson structure if and only if*

$$\{\varphi^* f,\, \varphi^* g\}_{M_1}$$

is constant along the fibers of φ for all $f, g \in C^\infty(M)$.

Proof. If $\{\varphi^* f,\, \varphi^* g\}_{M_1}$ is constant then define

$$\{f,\, g\}_{M_2}(\varphi(x)) := \{\varphi^* f, \varphi^* g\}_{M_1}(x).$$

This is well defined, i.e. it does not depend on x but only on $\varphi(x)$ (and it is defined everywhere, because φ is surjective). That it is Poisson is an easy consequence of $\{-,\, -\}_{M_1}$ being Poisson (remark that here you are using again surjectivity of φ).

Conversely, let Π_2 exists. Then

$$\{\varphi^* f,\, \varphi^* g\}_{M_1}(\varphi^{-1}(y)) = \{f,\, g\}(y)$$

and the right hand side does not depend on ϕ, hence the left hand side is constant along fibers. $\square$

Example 3.21. Consider $\mathbb{S}^2 \xrightarrow{\varphi} \mathbb{R}P^2$ the projection being given by identification of antipodal points $(x, -x) \mapsto [x]$. There exists a coinduced Poisson bivector on $\mathbb{R}P^2$ if and only if for any given pair of functions on $\mathbb{R}P^2$

$$\{\varphi^* f,\, \varphi^* g\}_{D^2}(x) = \{\varphi^* f,\, \varphi^* g\}(-x).$$

So if we identify $C^\infty(\mathbb{R}P^2) \hookrightarrow C^\infty(\mathbb{S}^2)^{\mathbb{Z}_2}$, the previous equality states that maps $\widehat{f}, \widehat{g}$ belonging to $C^\infty(\mathbb{S}^2)^{\mathbb{Z}_2}$ have to satisfy

$$\langle \Pi(x),\, d_x\widehat{f} \wedge d_x\widehat{g}\rangle = \{\widehat{f},\, \widehat{g}\}(x) = \{\widehat{f},\, \widehat{g}\}(-x) = \langle \Pi(-x),\, d_x\widehat{f} \wedge d_x\widehat{g}\rangle.$$

for all $x \in \mathbb{S}^2$. Choosing functions giving you a basis of the cotangent space this implies

$$\Pi(x) = \Pi(-x).$$

In particular if Π on $\mathbb{S}^2$ is constructed by gluing this implies $\Pi_1(x) = \Pi_2(-x)$ where now $x \in D$. This does not say that for any surjective map $\varphi\colon \mathbb{S}^2 \to \mathbb{R}P^2$ you have the same condition.

Proposition 3.22. *Let (M_1, Π_1) be a Poisson manifold. Let $\varphi\colon M_1 \to M_2$ be a surjective submersion with connected fibers. If*

$$\ker \varphi_{*,x} \subseteq \#_{\Pi,x}(M_1)$$

is locally spanned by hamiltonian vector fields, then M_2 has coinduced Poisson structure.

Proof. Take $f, g \in C^\infty(M_2)$. We want to prove that $\{\varphi^* f,\, \varphi^* g\}_{M_1}$ is constant along the fibers. Because φ is a submersion fibers are submanifolds. Since $\ker \varphi_{*,x} \subseteq \#_{\Pi,x}(M_1)$ it is enough to prove that for all $\lambda \in C^\infty(M_1)$ if $\lambda \in \ker \varphi_*$ then $X_\lambda(\{\varphi^* f,\, \varphi^* g\}_{M_1}) = 0$ (because $\ker \varphi_*$ is the tangent space to the fibers). But this follows from Jacobi identity. In fact

$$X_\lambda\{\varphi^* f,\, \varphi^* g\}_{M_1} = \{\varphi^* f,\, X_\lambda(\varphi^* g)\}_{M_1} + \{X_\lambda(\varphi^* f),\, \varphi^* g\}_{M_1}.$$

But $\varphi^* f$ and $\varphi^* g$ are constant along the fibers (by definition) and therefore

$$X_\lambda(\varphi^* g) = X_\lambda(\varphi^* f) = 0,$$

from which

$$X_\lambda(\{\varphi^* f,\, \varphi^* g\}_{M_1}) = 0$$

hence thesis. $\square$

3.4 Completeness

Let $\varphi\colon M \to N$ be a Poisson map and F a leaf in M. One could ask whether φ brings symplectic leaves of M into symplectic leaves of N. This is easily seen not to be the case. Let us take $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$, $\varphi(x, y) = x$ is Poisson with respect to the standard Poisson structure in $\mathbb{R}^2$ and zero structure in $\mathbb{R}$. But $\varphi(\mathbb{R}^2)$ is a union of leaves. From this example one could guess that in general $\phi(F)$ is a union of leaves. Even this turns out to be wrong, though for a subtler reason. Consider $U \subseteq \mathbb{R}^{2n}$ open set and $i\colon U \to \mathbb{R}^{2n}$ with the standard Poisson bivector Π on $\mathbb{R}^{2n}$ and $\Pi|_U$ on U. The image of the leaf U is not a whole leaf but just an open set in the leaf. Why is it so?

Consider now $\varphi(F)$ and take $\varphi(x) \in S$, where S is a leaf through $\varphi(x)$ in N. Take $y \in S$ and a piecewise Hamiltonian curve from y to $\varphi(x)$. We would like to lift this curve from N to M. Say the first Hamiltonian piece is the flow of X_h. Even if X_h is complete $X_{\varphi^* h}$ is not necessarily complete.

Definition 3.23. *A* ***complete Poisson map*** *is a Poisson map $\varphi\colon M \to N$ such that X_h complete implies $X_{\varphi^* h}$ complete.*

Then we immediately have

Proposition 3.24. *Let (M_1, Π_1) and (M_2, Π_2) be Poisson manifolds and $\varphi\colon M_1 \to M_2$ a complete Poisson map. Take F to be a leaf of M_1. Then $\varphi(F)$ is a union of symplectic leaves in M_2.*

Remark 3.25.

- Let M_1 be compact. Then any Poisson map $\varphi\colon M_1 \to M_2$ is complete.
- Let $\varphi\colon M_1 \to M_2$ be a proper Poisson map. Then it is complete.

Remark that also when we consider algebraic smooth Poisson varieties and alegbraic maps between them, properness, in the algebraic sense, implies completeness. This is often used when dealing with algebraic Poisson groups.

Chapter 4

Poisson cohomology

Let us recall the definition of Poisson cohomology. Let (M,Π) be a Poisson manifold. Consider the cochain complex $(\mathfrak{X}^k(M), d_\Pi)$, where

$$d_\Pi \colon \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M), \quad P \mapsto [\Pi, P], \tag{4.1}$$

where $[-,-]$ is the Schouten bracket. Then $d_\Pi^2 = 0$ as a consequence of the graded Jacobi identity together with $[\Pi,\Pi] = 0$. Remark that the Poisson tensor itself always defines a 2–cocycle and, thus, a Poisson cohomology class. When $[\Pi] = 0$ the Poisson manifold is said to be **exact**. We would like now to give a different, more explicit expression for this coboundary operator.

Proposition 4.1. *In the above hypothesis, for all $P \in \mathfrak{X}^k(M)$ and for all $\alpha_i \in \Omega^k(M)$, $i = 0, \dots, k$*

$$(d_\Pi P)(\alpha_0, \dots, \alpha_k) = \sum_{i=0}^{k} (-1)^{i+1} \#_\Pi(\alpha_i) P(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_k) + \tag{4.2}$$

$$\sum_{0 \le i < j \le k} (-1)^{i+j-1} P([\alpha_i, \alpha_j], \alpha_0, \dots, \widehat{\alpha_i}, \dots, \widehat{\alpha_j}, \dots, \alpha_k).$$

Proof. Let us first remark that the formula is true for $k = 0, 1$

$$\begin{aligned} k &= 0 \qquad (d_\Pi f)(\alpha) = \#_\Pi(\alpha) f, \\ k &= 1 \qquad (d_\Pi X)(df, dg) = X\{f,\, g\} + \{g,\, Xf\} - \{f,\, Xg\}. \end{aligned}$$

Let now P be a decomposable k-vector and prove (4.2) by induction on k. Due to the graded Leibniz identity for the Schouten bracket

$$d_\Pi P = [\Pi, P_1 \wedge \cdots \wedge P_k] = [\Pi, P] \wedge (P_2 \wedge \cdots \wedge P_k) + (-1)^1 P_1 \wedge [\Pi, P_2 \wedge \cdots \wedge P_k].$$

Therefore

$$\begin{aligned} (d_\Pi P)(\alpha_0, \dots, \alpha_k) &= [\Pi, P] \wedge (P_2 \wedge \cdots \wedge P_k)(\alpha_0, \dots, \alpha_k) - P_1 \wedge [\Pi, P_2 \wedge \cdots \wedge P_k](\alpha_0, \dots, \alpha_k) \\ &= \sum_{0 \le i < j \le k} [\Pi, P_1](\alpha_i, \alpha_j) P_2 \wedge \cdots \wedge P_k(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \widehat{\alpha_j}, \dots, \alpha_k) \\ &\quad - \sum_{i=0}^{k} P_1(\alpha_i) d_\Pi (P_2 \wedge \cdots \wedge P_k)(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_k). \end{aligned}$$

We have thus proven our claim on all decomposable k–vector fields. Due to locality of the Schouten–Nijenhuis bracket (together with the fact that locally any k-vector field is decomposable) tha claim holds true for all k–vector fields. □

Remark 4.2. From this expicit expression it would be tempting to say that the Poisson cohomology is some sort of Lie algebra cohomology, and precisely the Lie algebra cohomology of $(C^\infty(M), \{-,-\})$. This is not precise, because we do not have an identification between cochains (which are linear maps $\Lambda^k C^\infty(M) \to C^\infty(M)$) with multivectors. Multivectors are exactly those cochains which are differentiable in each argument. From this remark one can construct a homomorphism

$$j_* \colon \mathrm{H}^k_\Pi(M) \to \mathrm{H}^k_{\mathrm{Lie}}(C^\infty(M), \{-,-\}).$$

Computations of the cohomology on the right hand side are even harder than those for Poisson cohomology. This is one of the reasons why such cohomology is seldom considered.

Remark 4.3. Let f be a Casimir function for (M, Π). Let $P \in \mathfrak{X}^k(M)$. Then $d_\Pi(fP) = [\Pi, f] \wedge P + f[\Pi, P] = f[\Pi, P] = f d_\Pi P$. Hence we can define a product $f \cdot [P] = [fP]$. So there is a structure of $\mathrm{H}^0_\Pi(M) = \mathrm{Cas}(M)$-module on each $\mathrm{H}^k_\Pi(M)$.

Proposition 4.4. *The external product of multivector fields induces an associative and super commutative product in Poisson cohomology.*

$$\wedge \colon \mathrm{H}^k_\Pi(M) \times \mathrm{H}^l_\Pi(M) \to \mathrm{H}^{k+l}_\Pi(M).$$

this product will be called the ***Poisson product****.*

Proof.

$$[\Pi, P \wedge Q] = [\Pi, P] \wedge Q + (-1)^{p-1} P \wedge [\Pi, Q]$$

Therefore if $[\Pi, P] = [\Pi, Q] = 0$ also $[\Pi, P \wedge Q] = 0$, hence the product of two cocycles is a cocycle

$$[-,-] \colon Z^k_\Pi \times Z^q_\Pi \to Z^{p+q-1}_\Pi.$$

This product descends to cohomology. Define $[P] \wedge [Q] := [P \wedge Q]$. This is well defided, because

$$\begin{aligned}
[P + [\Pi, R]] \wedge [Q] &= [(P + [\Pi, R]) \wedge Q] \\
&= [P \wedge Q] + [[\Pi, R] \wedge Q] \\
&= [P \wedge Q] + (-1)^{p-2}[P, R \wedge \underbrace{[\Pi, Q]}_{=0}] + [P, [\Pi, R \wedge Q]].
\end{aligned}$$

The algebraic properties are a trivial consequence of analogous properties of $\wedge$. □

Remark 4.5. In a similar way it is easy to verify that also the Schouten bracket descends to cohomology via

$$[[P], [Q]] := [[P, Q]].$$

Here the key property is connected to the Jacobi identity for $[[\Pi, \Pi], Q]$.

Remark 4.6. H^k_Π is not functorial. In fact given a Poisson map $\varphi \colon M_1 \to M_2$ you do not have a corresponding map on chains $\varphi_* \colon \mathfrak{X}^k(M_1) \to \mathfrak{X}^k(M_2)$, where as we remarked already, only the weaker notion of φ_*-relatedness survive.

Theorem 4.7. *Let (M, Π) be a Poisson manifold. The sharp map intertwines the Poisson and de Rham cochain complexes, i.e.*

$$\#_\Pi \colon \Omega^k(M) \to \mathfrak{X}^k(M), \quad \#_\Pi \circ d = d_\Pi \circ \#_\Pi,$$

and therefore induces a homomorphism

$$\#_\Pi \colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k_\Pi(M).$$

This morphism is an algebra morphism with respect to cup products. Furthermore, if M is symplectic then $\#_\Pi$ is an isomorphism.

Proof. Here $\#_\Pi$ is extended to k-forms as

$$(\#_\Pi\omega)(\alpha_1, \ldots, \alpha_k) = \omega(\#_\Pi(\alpha_1), \ldots, \#_\Pi(\alpha_k)).$$

Let $\omega \in \Omega^k(M)$ and $\alpha_0, \ldots, \alpha_k \in \Omega^1(M)$. Then

$$\begin{aligned}
(d_\Pi(\#_\Pi(\omega)))(\alpha_0, \ldots, \alpha_k) &= \sum_{i=0}^{k}(-1)^{i+1}\#_\Pi(\alpha_i)(\#_\Pi(\omega))(\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_k) \\
&\quad + \sum_{i<j}(-1)^{i+j-1}\#_\Pi(\omega)([\alpha_i, \alpha_j], \alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_k) \\
&= \sum_{i=0}^{k}(-1)^{i+1}\#_\Pi(\alpha_i)\omega(\#_\Pi(\alpha_0), \ldots, \widehat{\#_\Pi(\alpha_i)}, \ldots, \#_\Pi(\alpha_k)) \\
&\quad + \sum_{i<j}(-1)^{i+j-1}\omega([\#_\Pi(\alpha_i), \#_\Pi(\alpha_j)], \#_\Pi(\alpha_0), \ldots, \widehat{\#_\Pi(\alpha_i)}, \\
&\quad \ldots, \widehat{\#_\Pi(\alpha_j)}, \ldots, \#_\Pi(\alpha_k)) \\
&= d\omega(\#_\Pi(\alpha_0), \ldots, \#_\Pi(\alpha_k)) \\
&= \#_\Pi(d\omega).
\end{aligned}$$

The fact that the sharp map respects cup product is obvious from definitions already at the chain level

$$\#_\Pi(\omega_1 \wedge \omega_2) = \#_\Pi(\omega_1) \wedge \#_\Pi(\omega_2).$$

Lastly if M is symplectic, $\#_\Pi$ is invertible at the chain level and therefore it remains such on cohomology. □

Proposition 4.8. *Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{g}^*$ the dual vector space with the Lie-Poisson bracket. Then*

$$\mathrm{H}^k_\Pi(\mathfrak{g}^*) \cong \mathrm{H}^k_L(\mathfrak{g}) \otimes \mathrm{Cas}(\mathfrak{g}^*),$$

where on the left H^k_L is the Lie algebra cohomology of $\mathfrak{g}$.

Remark 4.9. To complete the list of basic examples consider that if $(M, 0)$ is considered as a Poisson manifold then $\mathrm{H}^k_\Pi(M) = \mathfrak{X}^k(M)$. Therefore the Poisson cohomology has a huge variety of behaviours and is in general likely to be infinite dimensional over $\mathbb{R}$. We will see in examples that even the weaker property of being finitely generated as $H^0_\pi(M)$–modules is not always satisfied by Poisson cohomology groups.

Theorem 4.10 (Mayer-Vietoris sequence for Poisson cohomology)**.** *Let (M, Π) be a Poisson manifold. Let U and V be open subsets of M, considered as Poisson manifolds under restriction of the bivector $(U, \Pi|_U)$, $(V, \Pi|_V)$. Then there is a long exact sequence*

$$\ldots \to \mathrm{H}^{k-1}_{\Pi}(U \cap V) \xrightarrow{\partial} \mathrm{H}^{k}_{\Pi}(U \cup V) \to \mathrm{H}^{k}_{\Pi}(U) \oplus \mathrm{H}^{k}_{\Pi}(V) \to \mathrm{H}^{k}_{\Pi}(U \cap V) \xrightarrow{\partial} \mathrm{H}^{k+1}_{\Pi}(U \cup V) \to \ldots$$

Proof. Here one basically recalls how the proof of Mayer-Vietoris theorem goes on forms.

Given two open sets U and V, let $P \in \mathfrak{X}^k(U)$, $Q \in \mathfrak{X}^k(V)$, $R \in \mathfrak{X}^k(U \cup V)$. Then $R \mapsto R|_U$, $R \mapsto R|_V$ are maps to $\mathfrak{X}^k(U)$ and $\mathfrak{X}^k(V)$ respectively. Being the Shouten bracket a local operator one has $[\Pi|_U, R|_U] = [\Pi, R]|_U$ and $[\Pi|_V, R|_V] = [\Pi, R]|_V$. Therefore restriction induces a map on chains, trivially injective. Now start from P and Q. We want a vector field on $U \cap V$. We can of course consider $P - Q|_{U \cap V}$. If $[\Pi|_U, P] = 0 = [\Pi|_V, Q]$ then $[\Pi, P - Q|_{U \cap V}] = 0$. So again we have a cochain map. This map is surjective. Indeed, given a vector field S on $U \cap V$ we may extend, by the usual trick of smoothing function, to P on U and Q on V such that. $P - Q = S$ on $U \cap V$. Therefore we have a short exact sequence of cochain complexes. This induces as usual a long exact sequence in cohomology. Given $S \in \mathfrak{X}^k(U \cap V)$, $[\Pi, S] = 0$ consider (P, Q) as before such that $P - Q = S$. Being $[\Pi, (P, Q)] = ([\Pi|_U, P], [\Pi|_V, Q])$ we have $[\Pi|_U, P] - [\Pi|_V, Q] = [\Pi|_{U \cap V}, P - Q] = 0$. Therefore there exists $T \in \mathfrak{X}(U \cup V)$ such that $P = T|_U$, $Q = T|_V$. Define $\partial[S] := [T]$. The usual arguments, based on the snake lemma, prove the theorem. □

4.1 Modular class

Let (M, Π) be a Poisson manifold. Let us assume, for simplicity that M is orientable. Let Ω be a volume form on Ω. Consider for any $f \in C^\infty(M)$, $L_{X_f}\Omega \in \Omega^n M$. There exists a function $\phi_\Omega(f)$ such that $L_{X_f}\Omega = \phi_\Omega(f)\Omega$.

Fact 4.11. *ϕ_Ω is a vector field.*

Proof. We have to prove $\phi_\Omega(fg) = \phi_\Omega(f)g + f\phi_\Omega(g)$. But $X_{fg} = gX_f + fX_g$ (from $\{fg, h\} = g\{f, h\} + f\{g, h\}$). Therefore

$$L_{X_{fg}}\Omega = L_{gX_f + fX_g}\Omega = gL_{X_f}\Omega + X_f(g)\Omega + fL_{X_g}\Omega + X_g(f)\Omega = g\phi_\Omega(f)\Omega + f\phi_\Omega(g)\Omega$$

hence thesis. □

Definition 4.12. *ϕ_Ω is called the **modular vector field** of (M, Π) with respect to Ω.*

Fact 4.13. *The modular vector field is an infinitesimal Poisson field.*

Proof. We have seen that this is equivalent to $\phi_\Omega \in \mathrm{Der}(C^\infty(M), \{-, -\})$. Now

$$\begin{aligned}
L_{X_{\{f,\,g\}}} &= L_{[X_f, X_g]}\Omega \\
&= [L_{X_f}, L_{X_g}]\Omega \\
&= L_{X_f}(\phi_\Omega(g)\Omega) - L_{X_g}(\phi_\Omega(f)\Omega) \\
&= \phi_\Omega(g)L_{X_f}\Omega + \{f,\, \phi_\Omega(g)\}\Omega - \phi_\Omega(f)L_{X_g}\Omega - \{g,\, \phi_\Omega(f)\}\Omega \\
&= \phi_\Omega(g)\phi_\Omega(f)\Omega + \{f,\, \phi_\Omega(g)\}\Omega + \phi_\Omega(f)\phi_\Omega(g)\Omega + \{\phi_\Omega(f),\, g\}\Omega,
\end{aligned}$$

so

$$\phi_\Omega(\{f,\, g\}) = \{\phi_\Omega(f),\, g\} + \{f,\, \phi_\Omega(g)\}.$$

□

Fact 4.14. $L_{\phi_\Omega}\Omega = 0$.

Proof.

$$L_{\phi_\Omega}\Omega = di_{\phi_\Omega}\Omega + i_{\phi_\Omega}d\Omega = di_{\phi_\Omega}\Omega = d(di_\Pi\Omega),$$

because $di_\Pi\Omega = i_{\phi_\Omega}\Omega$. □

Take another volume form $\Omega' = a\Omega$. Then

$$L_{X_f}\Omega' = \phi_{\Omega'}(f)\Omega' = \phi_{\Omega'}(f)a\Omega$$

$$L_{X_f}(a\Omega) = aL_{X_f}\Omega + X_f(a)\Omega = a\phi_\Omega(f) + X_f(a)$$

Furtermore

$$a\phi_{\Omega'}(f) = a\phi_\Omega(f) + X_f(a),$$

$$\phi_{\Omega'}(f) = \phi_\Omega(f) + \frac{1}{a}X_f(a),$$

and

$$\frac{1}{a}X_f(a) = \frac{1}{a}\{f,\, a\} = \{f,\, \log|a|\} = -\{\log|a|,\, f\}$$

Hence the modular vector fields with respect to different volume forms differ for a hamiltonian vector field.

$$\phi_{\Omega'} = \phi_\Omega + X_{-\log|a|}.$$

Definition 4.15. *The vector field ϕ_Ω defines a class $[\phi_\Omega] \in \mathrm{H}^1_\Pi(M)$. This class is independent of Ω, and is called the **(Poisson) modular class**.*

Definition 4.16. *Let (M, Π) be a Poisson manifold such that $[\phi_\Omega] = 0$. Then (M, Π) is called **unimodular**.*

Exercise 4.17. *On $(\mathbb{R}^2, f(x, y)dx \wedge dy)$ compute the modular class.*

Examples 4.18.

1. Let (M, ω) be a compact symplectic manifold, $\Pi = \omega^{-1}$. Then the modular class is 0. In fact the volume form $\frac{\omega^n}{n!}$ is invariant under all Hamiltonian vector fields.

2. Let $(M, \Pi) = (\mathfrak{g}^*, \Pi_{\text{lin}})$. Then M is unimodular if and only if $\mathfrak{g}$ is unimodular as Lie algebra, i.e. $\mathrm{tr}(\mathrm{ad}_X) = 0$ for all $X \in \mathfrak{g}$.

3. Let (M, Π) be a regular Poisson structure. It can be proved that there exists an injective map

$$\mathrm{H}^1(M) \hookrightarrow \mathrm{H}^1_\Pi(M)$$

sending Reeb class $[Reeb]$ to $[\phi_\Omega]$. The Reeb class is an obstruction to the existence of a volume form of the usual bundle invariant for vector fields tangent to leaves [1].

Let (M, Π) be compact unimodular Poisson manifold. Then there exists Ω such that $L_{\phi_\Omega}\Omega = 0$. Then

$$\begin{aligned}\int_M \{f,\, g\}\Omega &= \int_M (L_{x_f} g)\Omega \\ &= \int_M L_{X_f}(g\Omega) - \int_M g L_{X_f}\Omega \\ &= \underbrace{\int_M d(i_{X_f} g\Omega)}_{=0 \text{ by Stokes theorem}} + \underbrace{i_{X_f} d(g\Omega)}_{=0} - \int_M g L_{X_f}\Omega \\ &= -\int_M g\phi_\Omega(f)\Omega.\end{aligned}$$

This is called also **infinitesimal KMS condition**. Being (M, Π) unimodular, we can choose a volume form Ω such that $\phi_\Omega \equiv 0$, so $\int_M \{f,\, g\}\Omega = 0$, i.e.

$$\int_M \Omega\colon C^\infty(M) \to \mathbb{R}$$

is a **Poisson trace**.

4.2 Computation for Poisson cohomology

Let us consider the quadratic Poisson structure on $\mathbb{R}^2$

$$\Pi_0(x, y) = (x^2 + y^2)\partial_x \wedge \partial_y.$$

We want to prove the following

Proposition 4.19 (Ginzburg)**.** *The Poisson cohomology of* $(\mathbb{R}^2, \Pi_0)$ *is given by*

$$\begin{aligned}&\mathrm{H}^0_{\Pi_0}(\mathbb{R}^2) = \mathbb{R}, \\ &\mathrm{H}^1_{\Pi_0}(\mathbb{R}^2) = \mathbb{R}\langle x\partial_x + y\partial_y, y\partial_x - x\partial_y\rangle, \\ &\mathrm{H}^2_{\Pi_0}(\mathbb{R}^2) = \mathbb{R}\langle \partial_x \wedge \partial_y, \Pi_0\rangle.\end{aligned}$$

Proof. To make computations easier let us identify $\mathbb{R}^2 \simeq \mathbb{C}$, $z = x + iy$, $\bar{z} = z - iy$, $\partial_z = \partial_z - i\partial_y$, $\partial_{\bar{z}} = \partial_x + i\partial_y$, $\Pi_0 = -2iz\bar{z}\partial_z \wedge \partial_{\bar{z}}$. We will omit the factor $(-2i)$ from now on.

Let us start by considering vector fields having coefficients which are formal power series in z and $\bar{z}$ (real coefficients): $\mathfrak{X}^k_f(\mathbb{R}^2)$. Let us denote with V_n the space of homogeneous polynomials in z and $\bar{z}$ of degree n, $V_n = \langle z^n, z^{n-1}\bar{z}, \ldots, \bar{z}^n\rangle$, $\dim V_n = n + 1$.

$$\begin{aligned}\mathfrak{X}^0_f(\mathbb{R}^2) &= \text{ formal power series in } z \text{ and } \bar{z} \ = \prod_{i=1}^{\infty} V_i, \\ \mathfrak{X}^1_f(\mathbb{R}^2) &= \prod_{i=1}^{\infty} V_i\partial_z \oplus \prod_{i=1}^{\infty} V_i\partial_{\bar{z}} = \prod_{i=1}^{\infty}(V_i\partial_z \oplus V_i\partial_{\bar{z}}), \\ \mathfrak{X}^2_f(\mathbb{R}^2) &= \prod_{i=1}^{\infty} V_i\partial_z \wedge \partial_{\bar{z}}.\end{aligned}$$

Now we compute $[\Pi, f]$ on $f \in C^\infty(\mathbb{R}^2)$:

$$[\Pi, f] = z\bar{z}((\partial_z(f))\partial_{\bar{z}} - (\partial_{\bar{z}}(f))\partial_z), \tag{4.3}$$

and $[\Pi, X]$ with $X \in \mathfrak{X}^1(\mathbb{R}^2)$, $X = f(z,\bar{z})\partial_z + g(z,\bar{z})\partial_{\bar{z}}$:

$$[\Pi, X] = z\bar{z}(\partial_z(f) + \partial_{\bar{z}}(g)) - \bar{z}f - zg. \tag{4.4}$$

From this formulae it is evident that

$$d_\Pi \colon V_{i-1} \to V_i\partial_z \oplus V_i\partial_{\bar{z}}$$

$$d_\Pi \colon V_i\partial_z \oplus V_i\partial_{\bar{z}} \to V_{i+1}\partial_z \wedge \partial_{\bar{z}}.$$

Therefore the complex on formal vector fields splits into a direc sum complexes

$$0 \to V_{i-1} \to V_i^{\oplus 2} \to V_{i+1} \to 0$$

If we denote with $\varphi_i := d_\pi|_{V_i} \colon V_i \to V_{i+1}^{\oplus 2}$, and $\psi_i := d_\Pi|_{V_i^{\oplus 2}} \colon V_i^{\oplus 2} \to V_{i+1}$, this means that

$$\mathrm{H}^0_\Pi = \bigoplus_{i\in\mathbb{N}} \ker\varphi_i$$
$$\mathrm{H}^1_\Pi = \bigoplus_{i\in\mathbb{N}} \ker\psi_i / \operatorname{im}\varphi_i$$
$$\mathrm{H}^2_\Pi = \bigoplus_{i\in\mathbb{N}} \operatorname{im}\psi_i$$

Let us first consider the case $i \geq 2$.

$$0 \to V_{i-1} \to V_i^{\oplus 2} \to V_{i+1} \to 0$$

The cohomology contributions of these complexes are

$$\ker\varphi_{i-1} \hookrightarrow \mathrm{H}^0_\Pi, \quad \ker\psi_i/\operatorname{im}\varphi_{i-1} \hookrightarrow \mathrm{H}^1_\Pi, \quad V_{i+1}/\operatorname{im}\psi_i \hookrightarrow \mathrm{H}^2_\Pi\,.$$

Observe that $\dim V_{i-1} = i$, $\dim V_i^{\oplus 2} = 2(i+1)$, $\dim V_{i+1} = i+2$. Now

1. φ_{i-1} is injective. In fact from 4.3

   $$\varphi_{m+l}(z^m\bar{z}^l) = mz^m\bar{z}^{l+1}\partial_{\bar{z}} - lz^{m+1}\bar{z}^l\partial_z.$$

   Therefore $\ker\varphi_{i-1} = \{0\}$, $\dim\operatorname{im}\varphi_{i-1} = i$.

2. ψ_i is surjective. In fact from 4.4

   $$\psi_{m+l}(z^m\bar{z}^l\partial_z) = (m-1)z^m\bar{z}^{l+1}\partial_z \wedge \partial_{\bar{z}},$$

   $$\psi_{m+l}(z^m\bar{z}^l\partial_{\bar{z}}) = (l-1)z^{m+1}\bar{z}^l\partial_z \wedge \partial_{\bar{z}}.$$

   Therefore $V_{i+1}/\operatorname{im}\psi_i = 0 \hookrightarrow \mathrm{H}^2_\Pi$.

3. lastly $\dim\ker\psi_i = \dim V_i^{\oplus 2} - \dim\operatorname{im}\psi_i = 2(i+2) - (i+2) = i+2$ which implies $\ker\psi_i/\operatorname{im}\varphi_{i-1} = 0 \hookrightarrow \mathrm{H}^1_\Pi$.

Thus no contributions to cohomology comes from $i \geq 2$. Let us look what happens when $i = 0, 1$.

$$\begin{array}{ccccc} V_0 & & V_0 \oplus V_0 & & V_0 \\ & \overset{\varphi_0}{\searrow} & & \overset{\psi_0}{\searrow} & \\ & & V_1 \oplus V_1 & & V_1 \\ & & & \overset{\psi_1}{\searrow} & \\ & & & & V_2 \end{array}$$

Again an easy and explicit computation shows that

$$\psi_0 \colon \begin{cases} \partial_z \mapsto \bar{z}\partial_z \wedge \partial_{\bar{z}} \\ \partial_{\bar{z}} \mapsto z\partial_z \wedge \partial_{\bar{z}}, \end{cases}$$

$$\varphi_0 = 0,$$

$$\psi_1 \colon \begin{cases} z\partial_z \mapsto 0 \\ \bar{z}\partial_z \mapsto -\bar{z}^2\partial_z \wedge \partial_{\bar{z}} \\ z\partial_{\bar{z}} \mapsto -z^2\partial_z \wedge \partial_{\bar{z}} \\ \bar{z}\partial_{\bar{z}} \mapsto 0. \end{cases}$$

Now

$$\ker \varphi_0 = V_0 \simeq \mathbb{R} \hookrightarrow \mathrm{H}^0_\Pi$$

$$\ker \psi_0 \oplus \ker \psi_1 / \operatorname{im} \varphi_0 = \ker \psi_1 \hookrightarrow \mathrm{H}^1_\Pi, \quad \ker \psi_1 = \langle z\partial_z, \bar{z}\partial_{\bar{z}} \rangle$$

$$V_0 \oplus V_1 / \operatorname{im} \psi_0 \oplus V_2 / \operatorname{im} \psi_1 = V_0 \oplus \langle z\bar{z}\partial_z \wedge \partial_{\bar{z}} \rangle \hookrightarrow \mathrm{H}^2_\Pi .$$

Moving to real coordinates

$$\begin{aligned} V_0 &= \langle \partial_x \wedge \partial_y \rangle \\ \langle z\bar{z}\partial_z \wedge \partial_{\bar{z}} \rangle &= \langle \Pi \rangle \\ z\partial_z &= x\partial_x + y\partial_y \\ \bar{z}\partial_{\bar{z}} &= y\partial_x - x\partial_y. \end{aligned}$$

Now we need to prove that only this formal vector fields contribute to the Poisson smooth cohomology. Define flat functions to be those $f \in C^\infty(\mathbb{R}^2)$ such that all their derivatives at the origin are 0 and $f(0) = 0$. Similarly flat multivector fields are those with flat coefficients. Then we have a short exact sequence of complexes

$$0 \to \mathfrak{X}^\bullet_{flat}(\mathbb{R}^2) \to \mathfrak{X}^\bullet(M) \to \mathfrak{X}^\bullet_{formal}(M) \to 0$$

Exactness is a consequence of Borel's theorem.
If we prove that $\mathrm{H}^*_{flat,\Pi_0}(\mathbb{R}^2) = 0$ we are done.

Now consider $\#_\Pi \colon \Omega^*_{flat} \to \mathfrak{X}^*_{flat}$ (here as usual by $\#_\Pi$ we denote the extension to all $\Omega^* M$; as we have seen $\#_\Pi(f) = X_f$). We claim that $\#_\Pi$ is an isomorphism. Let us prove it on 1-forms

$$\#_\Pi(f dx + g dy) = (x^2 + y^2)(f\partial_y - g\partial_x).$$

Now tha point is that if f is flat, then $(x^2 + y^2)f$ is also flat, but also the other way around, i.e.

$$(x^2 + y^2)f = \bar{f}$$

has always a flat solution in f, i.e.

$$\frac{\bar{f}}{x^2+y^2}$$

is a well defined flat function.

The key points are that Π has polynomial coefficients and has isolated singular points (this can be weakened). $\square$

Example 4.20. Let SU(2) have Poisson structure we already mentioned. The adjoint action of SU(2) on $\mathfrak{su}(2) \simeq \mathbb{R}^3$ is then action by rotations. The isotropy subgroup of $(1,0,0)$ is $\left\{\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} : |\alpha| = 1\right\}$. The orbit of $(1,0,0)$ is $\mathbb{S}^2$. The map

$$\phi\colon \mathrm{SU}(2) \to \mathrm{SU}(2)/\,\mathrm{U}(1) \simeq \mathbb{S}^2$$

is given by the formula

$$\phi\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = (\underbrace{|\alpha|^2 - |\beta|^2}_{x_1}, \underbrace{-i(\alpha\beta - \bar{\alpha}\bar{\beta})}_{x_2}, \underbrace{-(\alpha\beta + \bar{\alpha}\bar{\beta})}_{x_3})$$

Check that indeed $x_1^2 + x_2^2 + x_3^2 = 1$.

We claim that ϕ coinduces a Poisson structure on $\mathbb{S}^2$. Using the explicit expression for p one can explicitly compute

$$\begin{aligned}
\{x_1,\, x_2\} &= (1-x_1)x_3 \\
\{x_2,\, x_3\} &= (1-x_1)x_1 \\
\{x_3,\, x_1\} &= (1-x_1)x_2 \\
\Pi_0 &= (1-x_1)[x_3\partial_{x_1} \wedge \partial_{x_2} + x_1\partial_{x_2} \wedge \partial_{x_3} + x_2\partial_{x_3} \wedge \partial_{x_1}] \\
&= (1-x_1)\Pi,
\end{aligned}$$

Symplectic foliation consists of two 0-leaves - the north and south pole, and complement which is a 2-leaf.

Example 4.21. Take the stereographic projection from the south pole, i.e.

$$\begin{aligned}
\mathbb{R}^2 &\to \mathbb{S}^2 \setminus \{N\} \\
(x,y) &\mapsto \left(\frac{x_2}{1+x_1}, \frac{x_3}{1+x_1}\right)
\end{aligned}$$

Then Π_0 on $\mathbb{R}^2$ becomes $(x^2+y^2)\partial_x \wedge \partial_y$.

Of course if you take the stereographic projection from the north pole you get

$$\mathbb{R}^2 \to \mathbb{S}^2 \setminus \{S\}$$

with the symplectic structure (it can be proved that it is the standard one).

We wanto to use the Mayer-Vietoris exact sequence to compute the Poisson cohomology of $(\mathbb{S}^2, \Pi_0)$

$U = \mathbb{S}^2 \setminus \{N\}$	$V = \mathbb{S}^2 \setminus \{N\}$	$U \cap V = \mathbb{S}^2 \setminus \{N, S\}$
$(x^2+y^2)\partial_x \wedge \partial_y$	symplectic	symplectic
$\mathrm{H}^0_\Pi(U) = \mathbb{R}$	$\mathrm{H}^0_\Pi(V) = \mathbb{R}$	$\mathrm{H}^0_\Pi(U \cap V) = \mathbb{R}$
$\mathrm{H}^1_\Pi(U) = \mathbb{R}^2$	$\mathrm{H}^1_\Pi(V) = 0$	$\mathrm{H}^1_\Pi(U \cap V) = \mathbb{R}$
$\mathrm{H}^2_\Pi(U) = \mathbb{R}^2$	$\mathrm{H}^2_\Pi(V) = 0$	$\mathrm{H}^2_\Pi(U \cap V) = 0$

The sequence is

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to$$
$$\to \mathrm{H}^1_\Pi(\mathbb{S}^2) \to \mathbb{R}^2 \oplus 0 \to \mathbb{R} \to$$
$$\to \mathrm{H}^2_\Pi(\mathbb{S}^2) \to \mathbb{R}^2 \oplus 0 \to 0$$

The first row is exact (a Casimir function is constant on each of U and V).

$$0 \to \mathrm{H}^1_\Pi(\mathbb{S}^2) \to \underbrace{\mathbb{R}^2 \xrightarrow{\lambda} \mathbb{R}}_{x\partial_x + y\partial_y \mapsto y\partial_x - x\partial_y} \xrightarrow{\mu} \mathrm{H}^2_\Pi(\mathbb{S}^2) \to \mathbb{R}^2 \to 0$$

Because λ is surjecitve $\dim\ker\lambda = 0$ and $\dim\operatorname{im}\mu \leq 1$, so $\mu = 0$ and $\mathrm{H}^2_\Pi(\mathbb{S}^2) \simeq \mathbb{R}^2$. If you know that $\mathrm{H}^1_\Pi(\mathbb{S}^2)$ is nontrivial then $\mathrm{H}^1_\Pi(\mathbb{S}^2) \simeq \mathbb{R}$.

Chapter 5

Poisson homology

Recall that

$$\partial_\Pi = i_\Pi d - d i_\Pi \colon \Omega^k M \to \Omega^{k-1} M.$$

We showed that $\partial_\Pi^2 = 0$ and defined Poisson homology as the homology of the complex $(\Omega^\bullet, \partial_\Pi)$.

Proposition 5.1. *The Poisson homology is explicitely computed (in local coordinates) by*

$$\partial_\Pi(f_0 df_1 \dots df_k) = \sum_{1 \le i \le k} (-1)^{i+1} \{f_0,\, f_i\} df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_k$$

$$+ \sum_{1 \le i < j \le k} (-1)^{i+j} f_0 d\{f_i,\, f_j\} df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_k$$

Proof.

$$\begin{aligned}
\partial_\Pi(f_0 df_1 \dots df_k) &= i_\Pi(df_0 \wedge df_1 \dots df_k) \\
&\quad - d[\sum_{1 \le i < j \le k} (-1)^{i+j} f_0 d\{f_i,\, f_j\} df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_k] \\
&= \sum_{1 \le i < j \le k} (-1)^{i+j+1} \{f_i,\, f_j\} df_0 \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_k \\
&\quad + \sum_{1 \le i < j \le k} (-1)^{i+j} \{f_i,\, f_j\} df_0 \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_k \\
&\quad + \sum_{1 \le i < j \le k} (-1)^{i+j} f_0 d\{f_i,\, f_j\} df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_k.
\end{aligned}$$

□

Remark 5.2. One could use this formulas a definition for ∂_Π. This is correct, but requires also checking that the formula does not depend on local choices and this is quite difficult.

Note that $\partial_\pi(f_0 df_1) = \{f_0,\, f_1\}$ and therefore The 0–th Poisson homology is just given by: $C^\infty(M)/\{C^\infty(M),\, C^\infty(M)\}$. Thus it can be considered as the dual space to Poisson traces. This apparently easy definition does not mean that, even in very explicit examples, such invariant can be easily computed.

Theorem 5.3 (Brylinski)**.** *If M is symplectic manifold then*

$$\mathrm{H}_k^\Pi(M) \simeq \mathrm{H}_{DR}^{m-k}(M; \mathbb{R}) \simeq \mathrm{H}_\Pi^{m-k}(M).$$

Proof. (sketch) Given ω symplectic form, take the volume form $\frac{\omega^m}{m!} = \Omega_0$. You can use it to define a "Hodge-like" *-operator

$$*\colon \Omega^k(M) \to \Omega^{2m-k}(M),$$

implicitly as

$$(\beta \wedge *\alpha) = \underbrace{\Pi^{\wedge k}(\alpha, \beta)}_{\in C^\infty(M)} \Omega.$$

This operator verifies the following:

1. $** = \mathrm{id}$,
2. $\beta \wedge (*\alpha) = (-1)^k \alpha \wedge (*\beta)$,
3. on $\Omega^k(M)$, $\partial_\Pi = (-1)^{k+1} * d *$.

Therefore it intertwines d with * and therefore induces an isomorphism in homology. □

Remark 5.4. This map is similar to Poincare duality. In fact one could recover the same result through the existing duality between Poisson homology and cohomology.

Poisson homology is functorial. Given a Poisson map $\varphi\colon M_1 \to M_2$ there is a map $\varphi^*\colon \mathrm{H}_k^\Pi(M_2, \Pi_2) \to \mathrm{H}_k^\Pi(M_1, \Pi_1)$. In particular for any leaf S of M

$$\begin{array}{ccc}
\mathrm{H}_k^\Pi(S) & \xrightarrow{\varphi^*} & \mathrm{H}_k^\Pi(M) \\
\simeq \downarrow & & \nearrow \\
\mathrm{H}_\Pi^{n-k}(S) & & \varphi * \\
\simeq \downarrow & \nearrow & \\
\mathrm{H}_{\mathrm{DR}}^{n-k}(S) & &
\end{array}$$

Again deciding whether this map is injective or surjective is a difficult problem.

In the canonical double (mixed) complex you have $d\partial_\Pi + \partial_\Pi d = 0$

$$\begin{array}{ccccc}
\downarrow & & \downarrow & & \downarrow \\
\Omega^2(M) & \xleftarrow{d} & \Omega^1(M) & \xleftarrow{d} & \Omega^0(M) \\
\downarrow \partial_\Pi & & \downarrow \partial_\Pi & & \\
\Omega^1(M) & \xleftarrow{d} & \Omega^0(M) & & \\
\downarrow \partial_\Pi & & & & \\
\Omega^0(M) & & & &
\end{array}$$

Starting from this you can define cyclic (negative, periodic) Poisson homology and a long exact sequence of Connes-type.

Example 5.5. Consider on $\mathbb{R}^3$ the Poisson bracket

$$\{x_2,\, x_3\} = 2px_2x_3 - qx_1^2 = g_1,$$
$$\{x_1,\, x_3\} = 2px_1x_3 - qx_2^2 = g_2,$$
$$\{x_1,\, x_2\} = 2px_1x_2 - qx_3^2 = g_3.$$

Check that

$$\phi = \frac{q}{3}(x_1^3 + x_2^3 + x_3^3) - 2px_1x_2x_3$$

is a Casimir element. Can you prove that there are no other functionally independent Casimirs?

Let

$$\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}).$$

Verify that

$$\nabla\phi = (g_1, g_2, g_3),$$

and that

$$\nabla \times (g_1, g_2, g_3) = 0.$$

(here we are denoting $\nabla\times$ to be the curl as in usal vector calculus). Then, again by direct computation you can verify that

$$\partial_\Pi(x_1dx_1 + x_2dx_2 + x_3dx_3) = \nabla(x_1, x_2, x_3) \cdot \nabla\phi,$$

$$\partial_\Pi(x_1dx_2 \wedge dx_3 + x_2dx_3 \wedge dx_1 + x_3dx_1 \wedge dx_2) = \nabla(x_1, x_2, x_3)d\phi - d[(x_1, x_2, x_3) \cdot \nabla\phi],$$

$$\partial_\Pi(f dx_1 \wedge dx_2 \wedge dx_3) = -df \wedge d\phi.$$

These formulas are basically all one needs to thoroughly compute in an explicit manner the Poisson homology groups, as explained in [77].

The result of computation of Poisson homology is that $\mathrm{H}_*^\Pi(\mathbb{R}^3)$ is a free $\mathbb{R}[\phi]$-module of rank 8, 8, 1, 1. $\mathrm{H}_2^\Pi(\mathbb{R}^3)$ is generated by $x_1dx_2dx_3$, $x_2dx_3dx_1$, $x_3dx_1dx_2$. $\mathrm{H}_3^\Pi(\mathbb{R}^3)$ is generated by $dx_1dx_2dx_3$.

5.1 Poisson homology and modular class

Say Ω is a volume form on M.

$$\partial_\Pi\Omega = i_\Pi d\Omega - di_\Pi\Omega = -i_{\phi_\Omega}\Omega.$$

If M is unimodular Poisson then there exists $\Omega \in \Omega^n(M)$ such that $\phi_\Omega = 0$, so $\partial_\Pi\Omega = 0$ and thus $[\Omega] \neq 0 \in \mathrm{H}_n^\Pi(M)$.

For this reason in quantization you can regard Connes axiom of having "quantum" homological dimension equal classical dimension as a condition of unimodularity of the underlying Poisson manifold.

Let us now consider the Poisson structure of example (5.5). We want to compute its modular form starting from the standard volume form $\Omega = dx_1 \wedge dx_2 \wedge dx_3$. This means we want, for any $f \in C^\infty(M)$

$$L_{X_f}\Omega = \phi(f)\Omega$$

Then, explicitely

$$\begin{aligned}L_{X_{x_1}}\Omega &= di_{g_3\partial_2 - g_2\partial_3}\Omega \\ &= d\left(g_3 i_{\partial_2}\Omega - g_2 i_{\partial_3}\Omega\right) \\ &= d\left(-g_3 dx_1 \wedge dx_3 - g_2 dx_1 \wedge dx_2\right) \\ &= \left(\partial_2 g_3 - \partial_3 g_2\right)\Omega\end{aligned}$$

And similar computations show that

$$\phi_\Omega(x_i) = \det\begin{pmatrix} \partial_j & \partial_k \\ g_j & g_k \end{pmatrix}$$

with $(i,j,k) = (1,2,3)$ or cyclic permutations. Therefore $\phi_\Omega = \nabla \times g$ (up to now we've never used the explicit form of g). Lastly, as remarked, g is defined in such a way that $\nabla \times g = 0$ and therefore such Poisson structure is unimodular. It is worth remarking that van den Bergh in its paper was commenting that this condition is exactly what makes computations of Poisson homology accesible through explicit formulas (unimodularity was at that time not recognized as an easily accesible, though very relevant, invariant of Poisson manifolds).

Chapter 6

Coisotropic submanifolds

Let (M, Π) be a Poisson manifold, C a submanifold of M and N^*C its conormal bundle defined as:

$$N^*C = \{\alpha \in T^*M : \langle \alpha, v \rangle = 0 \quad \forall\, v \in TC\}.$$

Definition 6.1. *C is called* ***coisotropic sumbanifold*** *of M if*

$$\#_\Pi(N^*C) \subseteq TC$$

Remark 6.2. On symplectic manifolds, for a submanifold N of M you consider TN and

$$TN^{\perp\omega} = \{w \in TM : \omega(v, w) = 0 \quad \forall\, v \in TN\}.$$

Then you have

$$\begin{aligned} TN &\subseteq TN^{\perp\omega} \quad \textbf{isotropic}, \\ TN &= TN^{\perp\omega} \quad \textbf{Lagrangian}, \\ TN &\supseteq TN^{\perp\omega} \quad \textbf{coisotropic}. \end{aligned}$$

Exercise 6.3. *Prove that if (M, Π) is the Poisson manifold associated to a symplectic manifold then a submanifold verifies*

$$\#_\Pi(N^*C) \subseteq TC \quad \textit{iff} \ (TC)^{\perp\omega} \subseteq TC.$$

Proposition 6.4. *The following are equivalent*

1. *C is coisotropic in (M, Π).*
2. *For all $f, g \in C^\infty(M)$ such that $f|_C, g|_C = 0$, $\{f, g\}|_C = 0$.*
3. *For all $f \in C^\infty(M)$ such that $f|_C = 0$, $X_f|_C$ is tangent to C.*

Proof. The point here is that if $I = \{f \in C^\infty(M) | f|_C = 0\}$ then

$$\{d_x f : f \in I\} = N^*_x C,$$

$$\langle d_x f, v_x \rangle = v(f)(x).$$

The fact that we get all conormal vectors as differentials of functions in I follows from local equalities for C of the form $x^1 = \ldots = x^p = 0$ in a coordinate neighbourhood $(U; x^1, \ldots, x^n)$ $(p \leq n)$ adapted to C.
Then we have easily (3) $\implies$ (1) $\implies$ (2) $\implies$ (3). $\square$

Remark 6.5.

- If C is Poisson submanifold then I is a Poisson ideal.
- If C is coisotropic then I is a Poisson subalgebra.

Exercise 6.6. *Let $\mathfrak{h}$ be a Lie subalgebra in $\mathfrak{g}$. Prove that $\mathfrak{h}^\perp$ is a coisotropic submanifold in $\mathfrak{g}^*$.*

Theorem 6.7. $\varphi\colon (M_1, \Pi_1) \to (M_2, \Pi_2)$ *is a Poisson map if and only if*

$$\Gamma_\varphi := \{(x, \varphi(x)) : x \in M_1\}$$

is a coisotropic submanifold of $M_1 \times \overline{M_2}$.

The notation:

$$M_1 \times \overline{M_2} = (M_1 \times M_2, \Pi_1 \oplus (-\Pi_2))$$

with product Poisson structure.

Proof. We have

$$T_{(x,\varphi(x))}\Gamma_\varphi = \{(v, \varphi_{*,x}v) : v \in T_xM_1\}$$
$$N^*\Gamma_\varphi = \{(-\varphi^*\lambda, \lambda) : \lambda \in T^*_{\varphi(x)}M_2\}$$

Then

$$\#_\Pi(N^*\Gamma_\varphi) \subseteq T\Gamma_\varphi$$

is equivalent to

$$\varphi_*(\#_{\Pi_1}(-\varphi^*\lambda)) = -\#_{\Pi_2}\lambda, \quad \forall\, \lambda \in T^*_{\varphi(x)}M_2,$$

which is one of the conditions equivalent to being Poisson. □

Definition 6.8. *Let C be a coisotropic submanifold of (M, Π) and let $I := \{f \in C^\infty(M) : f|_C = 0\}$. Define*

$$N(I) := \{g \in C^\infty(M) : \{g,\, I\} \subseteq I\}.$$

Proposition 6.9. *$N(I)$ is a Poisson subalgebra of $C^\infty(M)$, I is a Poisson ideal of $N(I)$ and therefore $N(I)/I$ is a Poisson algebra.*

Proof. From the Jacobi identity we get the first part:

$$\{\{g_1,\, g_2\},\, f\} = -\{\{g_2,\, f\},\, g_1\} + \{\{g_1,\, f\},\, g_2\}$$

Furthermore

$$N(I)/I = C^\infty(\underline{C}) = \{f \in C^\infty(C) : Xf = 0 \quad \forall\, X \in \Gamma(\#_\Pi N^*C)\} \subseteq \text{Poisson manifold}$$

□

Proposition 6.10. *A submanifold C is coisotropic if and only if $f|_C = 0$ and $g|_C = 0$ implies $\{f,\, g\}|_C = 0$.*

Remark 6.11. Is it true that C is a coisotropic submanifold of M if and only if $C \cap F$ is a coisotropic submanifold of any leaf F of M? To show this is not true take for example $\mathbb{R}^3$, $\Pi = \partial_x \wedge \partial_y$. Symplectic foliation is given by planes parallel to $\{z = 0\}$, $F_h = \{z = h\}$. The standard embedding $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$, $x^2 + y^2 + z^2 = 1$ gives a coisotropic submanifold. This can be checked directly proving that functions which are zero on $\mathbb{S}^2$ form a Poisson subalgebra or through the following:.

Exercise 6.12. *Show that every codimension 1 locally closed submanifold is coisotropic.*

Now let us look at intersections:

$$\mathbb{S}^2 \cap F_h = \begin{cases} \emptyset & h \notin [-1,1] \\ * & h \in \{-1,1\} - \text{ is never coisotropic in the leaf to which it belongs} \\ \mathbb{S}^1 & h \in (-1,1) \end{cases}$$

Therefore a submanifold maybe coisotropic without its intersections being coisotropic in the leaves. From this example it is also quite evident the reason for it: the submanifold and the leaves may intersect not transversally. In fact adding suitable transversality conditions it is possible to relate coisotropy to coisotropy in the leaves (see for example [72]).

6.1 Poisson Morita equivalence

Take (M, Π) to be Poisson, (S, ω) symplectic,

$$\begin{aligned} \#_\Pi \colon T^*M &\to TM \\ \#_{\omega^{-1}} \colon T^*S &\to TS \\ \flat_\omega \colon TS &\to T^*S \end{aligned}$$

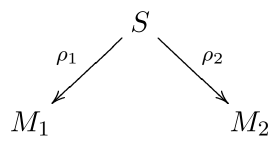
Say we have $\rho\colon S \to M$ surjective submersion,

$$\rho_{*,p} \colon T_pS \to T_{\rho(p)}M.$$

For $p \in S$, $x = \rho(p)$, $\rho^{-1}(x)$ is a closed submanifold. We have

$$\begin{aligned} \ker(\rho_{*,p}) &= \{v \in T_pS : \rho_{*,p}(v) = 0\} = T_p\rho^{-1}(x), \\ N_p^*\rho^{-1}(x) &= \{\alpha \in T_p^*S : \langle \alpha, v\rangle = 0 \quad \forall\, v \in T_p\rho^{-1}(x)\} \\ &= \{\alpha \in T_p^*S : \alpha = d_pf,\ f \in \rho^*(C^\infty(M))\}, \\ (\ker(\rho_{*,p}))^{\perp\omega} &= \{w \in T_pS : \omega(v,w) = 0 \quad \forall\, v \in T_p\rho^{-1}(x)\} \\ &= \{w \in T_pS : \langle \flat_\omega(w), v\rangle = 0 \quad \forall\, v \in T_p\rho^{-1}(x)\} \\ &= \{w \in T_pS : \flat_\omega(w) \in N_p^*\rho^{-1}(x)\} \\ &= \#_{\omega^{-1}}(N_p^*\rho^{-1}(x)) \\ &= \{X^\omega_{\rho^*(f)} : f \in \rho^*(C^\infty(M))\}. \end{aligned}$$

Definition 6.13. *Two Poisson manifolds (M_1, Π_1) and (M_2, Π_2) form a* ***dual pair*** *if there exists a symplectic manifold (S, ω) and two Poisson submersions (i.e. symplectic realizations)*



such that the fibers are symplectic orthogonal, i.e. for any $p \in S$, $\rho_1(p) = x$, $\rho_2(p) = y$

$$T_p\rho_1^{-1}(x) = (T_p\rho_2^{-1}(y))^{\perp\omega}.$$

The pair is called ***full*** *if ρ_1, ρ_2 are surjective.*

Remark 6.14. We have seen that in some sense a symplectic realization of M_1 is a notion like "one sided module over M_1". The dual pair is thus a notion of bimodule.

Our task now is to unravel this definition.

Proposition 6.15. *Let (S, ω) with $\rho_i \colon (S, \omega) \to (M, \Pi_i)$, $i = 1, 2$ be a full dual pair. Then*

$$\{\rho_1^*(f),\, \rho_2^*(g)\}_S = 0 \quad \forall\, f \in C^\infty(M_1),\ g \in C^\infty(M_2). \tag{6.1}$$

Condition 6.1 is equivalent to symplectic orthogonality of tangent spaces if fibers are connected.

Proof.

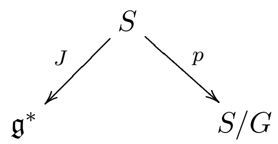
$$\begin{aligned} \ker((\rho_1)_{*,p}) = T_p\rho_1^{-1}(x) = (T_p\rho_2^{-1}(y))^{\perp\omega} = (\ker((\rho_2))_{*,p})^{\perp\omega} \\ = \{X^\omega_{\rho_2^*(g)}(p) : g \in \rho_2^*(C^\infty(M_2))\}. \end{aligned}$$

Take $f \in C^\infty(M_1)$

$$\{\rho_1^*(f),\, \rho_2^*(g)\}(p) = -X^\omega_{\rho_2^*(g)}(\rho_1^*(f))(p) = 0$$

because $-X^\omega_{\rho_2^*(g)} \in T\rho^{-1}(x)$ and $\rho_1^*(f)$ is constant along $\rho^{-1}(x)$.
The argument can be reversed provided fibers are connected. □

Example 6.16. Let S be a symplectic manifold, $J \colon S \to \mathfrak{g}^*$ constant rank Poisson map. (Moment map, Hamiltonian action of G on S). Assume that J is a surjective submersion and that G-action on S is regular, S/G is a manifold. Then there exists a coinduced Poisson structure on S/G and



form a full dual pair.

If regularity is missing one can ask if $p^*(C^\infty(S/G))$ is the Poisson commutant of $J^*(C^\infty(\mathfrak{g}^*))$ (admissible functions),

$$\{C^\infty(S)^G,\, J^*(C^\infty(\mathfrak{g}^*))\} = 0.$$

One can also ask such questions only after restriction to an open subset U of $\mathfrak{g}^*$.

Proposition 6.17. *Let (M_i, Π_i), $i = 1, 2$, be Poisson manifolds. Let (S, ω) be a symplectic manifold. Let $\rho_i \colon S \to M_i$, $i = 1, 2$, form a full dual pair with connected fibers. Then there is a 1-1 correspondence between symplectic leaves of M_1 and M_2, inducing homeomorphism on leaf spaces.*

Proof. The basic idea is the following. Take a leaf F_1 in M_1. Consider $\rho_2(\rho_1^{-1}(F_1))$, which is a leaf in M_2. The correspondence $\Phi \colon F_1 \mapsto \rho_2(\rho_1^{-1}(F_1))$ is bijective and Φ is a homeomorphism.

The details are as follows. Fix $x \in M_1$ and let F_1 be a leaf through x

$$T_xF_1 = \operatorname{im} \#_{\Pi_1, x}.$$

Consider $\rho_1^{-1}(F_1)$ and take $p \in \rho_1^{-1}(x)$. Prove that

$$(\rho_2)_*(T_p\rho^{-1}(F_1)) = \operatorname{im} \#_{\Pi_2, \rho_2(p)}.$$

Indeed

$$\begin{aligned} T_{\rho_1(p)}F_1 &= \{X_f^{\Pi_1}(\rho_1(p)) : f \in C^\infty(M_1)\} \\ &= \{(\rho_1)_{*,p}X_f^\omega(p) : f \in \rho_1^*(C^\infty(M_2))\} \\ &= (\rho_1)_{*,p}(\#_{\omega^{-1}}N^* \ker((\rho_1)_{*,p})). \end{aligned}$$

Take $\mathcal{D}_1$, $\mathcal{D}_2$ be the distributions spanned by Hamiltonian vector fields of pull-backs.

$$\begin{aligned} \mathcal{D}_1 &= \{X_{\rho_1^* f}^\omega : f \in \rho_1^*(C^\infty(M_1))\} \\ \mathcal{D}_2 &= \{X_{\rho_2^* f}^\omega : f \in \rho_2^*(C^\infty(M_2))\} \end{aligned}$$

Take as usual $p \in S$, $\rho_1(p) = x$, $\rho_2(p) = y$.

$\mathcal{D}_1$ and $\mathcal{D}_2$ are integrable disributions. In fact we know also the maximal integral submanifolds - fibers. Surjectivity grants that each point belongs to a fiber. Connectedness grants that the fibers are submanifolds.

Consider the distribution $\mathcal{D}_1 + \mathcal{D}_2$. We claim that it is also integrable. Let F_1 be a symplectic leaf in M_1. Show that $\rho_1^{-1}(F_1)$ is a connected integral sumanifold of $\mathcal{D}_1 + \mathcal{D}_2$.

$$\begin{aligned} T_p(\rho^{-1}(F_1)) &= (\ker \rho_{1,*})_p + \{v \in T_pS : (\rho_{1,*})_p v \in T_{\rho_1(p)}F_1\} \\ &= (\ker \rho_{2,*})_p^{\perp\omega} + \{X_{\rho_1^* f}^\omega(p) : f \in C^\infty(M_1)\} \\ &= \{X_{\rho_2^* g}^\omega(p) : g \in C^\infty(M_2)\} + \{X_{\rho_1^* f}^\omega(p) : f \in C^\infty(M_1)\} \\ &= \mathcal{D}_1 + \mathcal{D}_2. \end{aligned}$$

Now also $\rho_2^{-1}(F_2)$, F_2 symplectic leaf of M_2 are integral submanifolds of $\mathcal{D}_2 + \mathcal{D}_2$.

Let $\mathcal{L}$ be the set of integral submanifolds of $\mathcal{D}_1 + \mathcal{D}_2$. Then $\mathcal{L}$ is in one to one correspondence with the set of leaves of M_1 and the set of leaves of M_2. The bijection on the sets of leaves is given by

$$F_1 \mapsto \rho_2(\rho_1^{-1}(F_1))$$

It remains to show that it is homeomorphism of topological spaces. □

Lemma 6.18. *Let $f\colon S^{(n)} \to M^{(m)}$ be a submersion, $n \geq m$, F is $(m-k)$-dimensional submanifold of M, $x \in S$, $f(x) = y \in F$. Then we can find local coordinates*

$$\begin{aligned} &(U, \phi) \text{ around } x \text{ in } S, \\ &(V, \psi) \text{ around } y \text{ in } M, \end{aligned}$$

such that for all $w \in f^{-1}(V) \cap U$

$$\psi_i(f(w)) = \phi_i(w), \quad i = 1, \ldots, m$$

and

$$f(w) \in F \cap V \iff \psi_1 = \ldots = \psi_k = 0.$$

Therefore $f^{-1}(F)$ is an $(n-k)$-dimensional submanifold of S given by $\phi_1 = \ldots = \phi_k = 0$.

Definition 6.19 ([78])**.** *Two Poisson manifolds are called **Poisson Morita equivalent** if there exists a full dual pair (S, ω), ρ_1, ρ_2 between (M_1, Π_1) and $(M_2, -\Pi_2)$ such that*

1. *ρ_1, ρ_2 are complete,*

2. fibers of ρ_1, ρ_2 are connected, simply connected.

$$\begin{array}{ccc} & S & \\ {\scriptstyle \rho_1}\swarrow & & \searrow{\scriptstyle \rho_2} \\ M_1 & & M_2 \end{array}$$

Remark 6.20. Despite its name Poisson Morita equivalence is not an equivalence relation as it fails to be reflexive. In such cases it is natural to single out the subclass of objects on which a relation indeed defines an equivalence:

Definition 6.21. *Poisson manifolds Poisson Morita equivalent to themeselves are called **integrable**.*

Reason for the name is that the associated Lie algebroid can be integrated to a Lie grupoid.

Proposition 6.22. *Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. They are Poisson Morita equivalent if and only if they have isomorphic fundamental groups.*

In particular any connected and simply connected symplectic manifold is Poisson-Morita equivalent to a point.

Proof. Let

$$\begin{array}{ccc} & S & \\ {\scriptstyle \rho_1}\swarrow & & \searrow{\scriptstyle \rho_2} \\ M_1 & & M_2 \end{array}$$

be the Poisson-morita equivalence.

Look at the long exact sequence in homotopy

$$\begin{array}{ccccccc} 0 = \pi_1(\text{fiber}_1) & \longrightarrow & \pi_1(S) & \longrightarrow & \pi_1(M_1) & \longrightarrow & \pi_0(\text{fiber}_1) = 0 \\ & \nearrow & & \searrow & & & \\ 0 = \pi_1(\text{fiber}_2) & & & & \pi_1(M_2) & \longrightarrow & \pi_0(\text{fiber}_2) \end{array}$$

Conversely: say $\pi_1(M_1) \simeq \pi_1(M_2) \simeq G$. Let $\widetilde{M_j}$ be the universal cover of M_j, $j = 1, 2$. Both are principal G-bundles over M_j. The product $\widetilde{M_1} \times \overline{\widetilde{M_2}}$ has symplectic structure given by $(\omega_1, -\omega_2)$.

$$\begin{array}{ccc} & \widetilde{M_1} \times \overline{\widetilde{M_2}} & \\ {\scriptstyle \rho_1}\swarrow & & \searrow{\scriptstyle \rho_2} \\ M_1 & & M_2 \end{array}$$

□

Example 6.23. Let (S, ω) be a connected, simply connected symplectic manifold and let M be a connected manifold with the zero Poisson structure. Then M is Poisson-Morita equivalent to S.

$$\begin{array}{ccc} & S \times T^*M & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ S & & M \end{array}$$

where p_1 denotes the projection of $S \times T^*M$ on its first component, while p_2 is the projection on the second component composed with the cotangent bundle projection.

Proposition 6.24 (Lu-Ginzburg)**.** *Poisson-Morita equivalent manifolds have isomorphic first Poisson cohomology* $\mathrm{H}^1_\Pi(-)$*, but can have non-isomorphic* $\mathrm{H}^k_\Pi(-)$*.*

Remark 6.25. With some more work one can prove that the induced map between set of leaves is in fact a heomeomorphism of topological spaces.

Remark 6.26. The first Poisson cohomology and modular class are Poisson-Morita invariants.

6.2 Dirac structures

Definition 6.27. *Let M be a smooth manifold. A **Dirac structure** on M is a subbundle $L \subset TM \oplus T^*M$ which gives pointwise linear Dirac structures in $T_xM \oplus T^*_xM$ and such that its sections are closed under the **Courant bracket***

$$[(X, \alpha), (Y, \beta)] = ([X, Y], L_X\beta - L_Y\alpha + \tfrac{1}{2}d(\alpha(Y) - \beta(X))) \tag{6.2}$$

Remark 6.28. The Courant bracket is not a Lie bracket. However it turns out to be a Lie bracket on sections of a Dirac bundle.

Proposition 6.29. *Let $\Pi \in \Gamma(\Lambda^2 TM)$ be a bivector on M. Then* $\mathrm{graph}(\Pi)$ *defines a subbundle of $TM \oplus T^*M$ which is pointwiese a linear Dirac structure;* $\Gamma(\Pi)$ *is a Dirac structure if and only if* Π *is Poisson.*

Remark 6.30. Not every Dirac structure comes from a Poisson bivector.

Proof. For any $\Pi \in \Gamma(\Lambda^2 TM)$ define

$$\Gamma_\Pi = \{(\#_\Pi(\alpha), \alpha) : \alpha \in \Omega^1(M)\}$$

This is pointwise Dirac.

$$\begin{aligned}
[(\#_\Pi(\alpha), \alpha), (\#_\Pi(\beta), \beta)] &= ([\#_\Pi(\alpha), \#_\Pi(\beta)], L_{\#_\Pi(\alpha)}\beta - L_{\#_\Pi(\beta)}\alpha + \tfrac{1}{2}d(\alpha(\#_\Pi(\beta)) - \beta(\#_\Pi(\alpha)))) \\
&= ([\#_\Pi(\alpha), \#_\Pi(\beta)], [\alpha, \beta]_\Pi)
\end{aligned}$$

Now the point is that $\#_\Pi$ is a Lie algebra map if and only if $[\Pi, \Pi]_{\mathrm{SN}} = 0$. ☐

Proposition 6.31. *Let B be a skewsymmetric bilinear form on V, $B \in \Lambda^2 V^*$. Then for any linear Dirac structure L*

$$\mathcal{C}_B(L) := \{(v, \mu + Bv) : (v, \mu) \in L\}$$

is a linear Dirac structure.

Proof. Dimension is obviously unchanged. Therefore it suffices to show isotropy

$$\begin{aligned}
((v, \mu + Bv), (w, \eta + Bw)) &= \tfrac{1}{2}((\mu + Bv)(w) + (\eta + Bw)(v)) \\
&= \tfrac{1}{2}(\mu(w) + \eta(v)) + \tfrac{1}{2}(\underbrace{B(v, w) + B(w, v)}_{=0}) \\
&= ((v, \mu), (w, \eta)) = 0.
\end{aligned}$$

☐

Proposition 6.32. *Let $\Pi \in \Lambda^2 V$ and let Γ_Π be the linear Dirac structure corresponding to the graph of Π. Let $B \in \Lambda^2 V^*$. Then there exists $\Pi' \in \Lambda^2 V$ such that*

$$\mathcal{C}_B(\Gamma_\Pi) = \Gamma_{\Pi'} \Longleftrightarrow (\mathrm{id} + \flat_B \circ \#_\Pi) \text{ is invertible.}$$

Here

$$\flat_B \colon V \to V^*,\ \flat_B(v) = B(v, -),$$

$$\#_\Pi \colon V^* \to V,\ \#_\Pi(\xi) = \Pi(\xi, -)$$

with the identifcation $V \simeq V^{**}$.

Proof.

$$\mathcal{C}_B(\Gamma_\Pi) = \Gamma_{\Pi'} \Longleftrightarrow \mathcal{C}_B(\Gamma_\Pi) \cap V = \{0\}$$

Now $\mathrm{id}\, \flat_B \circ \#_\Pi \colon V^* \to V^*$ is invertible if and only if it is injective, therefore

$$\alpha + B(\#_\Pi(\alpha)) = 0,\ \alpha \neq 0 \Longleftrightarrow \mathrm{id} + \flat_B \circ \#_\Pi \text{ is injective.}$$

□

Proposition 6.33. *Let L be a Dirac structure on M and let $B \in \Omega^2(M)$. Then*

$$\mathcal{C}_B(L) \text{ is Dirac } \Longleftrightarrow dB = 0$$

Proof. As we have already seen $\mathcal{C}_B(L)$ is pointwise a linear Dirac structure. We have to show what happens if we require in $\mathcal{C}_B(L)$ closeness with respect to the Courant bracket.

$$\begin{aligned}
[(X, \eta + B(X)), (Y, \xi + B(Y))] &= ([X, Y], L_X(\mu + B(Y)) - L_Y(\omega + B(X)) \\
&\quad + \tfrac{1}{2} d((\omega + B(X))(Y) - (\mu + B(Y))(X))) \\
&= ([X, Y], L_X\mu - L_Y\omega + \tfrac{1}{2} d(\omega(Y) - \mu(X)) \\
&\quad + L_X B(Y) - L_Y B(X) + d(B(X, Y)))
\end{aligned}$$

Lemma 6.34.

$$L_X B(Y) - L_Y B(X) + d(B(X, Y)) = (dB)(X, Y) - B([X, Y])$$

Proof.

$$L_X = di_X + i_X d,\ L_Y = di_Y + i_Y d$$

$$L_X B(Y) = d(B(Y, X)) + i_X(dB(Y))$$

$$L_Y B(X) = d(B(X, Y)) + i_Y(dB(X))$$

$$(i_X d(B(Y)))(Z) = \langle d(B(Y)),\ X \wedge Z\rangle = ZB(X, Y) - XB(Y, Z) - B(Y, [X, Z])$$

Use formula for

$$(dB)(X, Y, Z) = XB(Y, Z) - YB(X, Z) + ZB(X, Y) - B([X, Y], Z) + B([X, Z], Y) - B([Y, Z], X).$$

□

□

Definition 6.35. *Two Poisson bivectors* Π_1, Π_2 *on the manifold* M *are said to be* ***gauge equivalent*** *if there exists a closed 2-form* B *such that*

$$\mathcal{C}_B(\Gamma_{\Pi_1}) = \Gamma_{\Pi_2}$$

(i.e. if the corresponding Dirac structures are equivalent)

Two Poisson manifolds (M_1, Π_1) *and* (M_2, Π_2) *are said to be* ***gauge equivalent up to diffeomorphism*** *if there exists a Poisson diffeomorphism*

$$\varphi\colon (M_1, \Pi_1) \to (M_2, \Pi_0)$$

such that Π_0 *and* Π_2 *are gauge equivalent.*

Remark 6.36. Two symplectic structures on a given manifold are gauge equivalent. Two symplectic manifolds are gauge equivalent up to diffeomorphism if and only if they are symplectomorphic.

Chapter 7

Poisson Lie groups

7.1 Poisson Lie groups

Recall the two presentations of a Poisson manifold:

1. $\{-,-\}\colon C^\infty(M)\otimes C^\infty(M)\to C^\infty(M)$ such that
   - $\{-,-\}$ is a Lie bracket (antisymmetric + Jacobi identity)
   - $\{f, gh\} = \{f, g\}h + g\{f, h\}$ (Leibniz rule)
2. (M,Π), $\Pi\in\Gamma(\Lambda^2 TM)$ such that $[\Pi,\Pi]=0$

connected by the equality

$$\{f, g\}(x) = \langle \Pi(x),\, d_x f\otimes d_x g\rangle$$

Recall that a smooth map $\phi\colon M\to N$ between Poisson manifolds is a map that preserves Poisson brackets

$$\{f_1, f_2\}_M\circ\phi = \{f_1\circ\phi,\, f_2\circ\phi\}_N$$

or equivalently

$$\phi_{*,x}^{\otimes 2}\Pi_M(x) = \Pi_N(\phi(x))$$

Recall also that if M, N are Poisson manifolds the structure of product Poisson manifold on $M\times N$ is the one given by

$$\{f_1, f_2\}_{M\times N}(x,y) = \{f_1(-,y),\, f_2(-,y)\}_M(x) + \{f_1(x,-),\, f_2(x,-)\}_N(y)$$

or equivalently

$$\Pi_{M\times N} = \Pi_M\oplus\Pi_N\in\Gamma(\Lambda^2 T(M\times N)) = \Gamma(\Lambda^2 TM\oplus\Lambda^2 TN)$$

Proposition 7.1. *Let G be a Lie group, Π Poisson tensor on G. Then the following are equivalent:*

1. *The product $m\colon G\times G\to G$ is a Poisson map*
2. $$\Pi(g_1g_2) = L_{g_1,*}\Pi(g_2) + R_{g_2,*}\Pi(g_1),$$
   where
   $$L_g\colon G\to G,\quad h\mapsto gh,\qquad R_g\colon G\to G,\quad h\mapsto hg$$
   and $L_{g,*}$, $R_{g,*}$ are derivatives.

Proof. Let $m\colon G \times G \to G$ be a Poisson map, that is

$$\{f_1,\, f_2\}(m(g_1, g_2)) = \{f_1 \circ m,\, f_2 \circ m\}_{G\times G}(g_1, g_2)$$

i.e.

$$\{f_1,\, f_2\}(g_1 g_2) = \{f_1 \circ L_{g_1},\, f_2 \circ L_{g_1}\}(g_2) + \{f_1 \circ R_{g_2},\, f_2 \circ R_{g_2}\}(g_1)$$

or equivalently

$$\langle \Pi(g_1 g_2),\, d_{g_1 g_2} f_1 \otimes d_{g_1 g_2} f_2 \rangle =$$
$$\langle \Pi(g_2),\, d_{g_2}(f_1 \circ L_{g_1}) \otimes d_{g_2}(f_2 \circ L_{g_1}) \rangle + \langle \Pi(g_1),\, d_{g_1}(f_1 \circ R_{g_2}) \otimes d_{g_1}(f_2 \circ R_{g_2}) \rangle$$

Now use

$$d_g(f \circ L_h) = L_{h,*} d_g f, \quad d_g(f \circ R_h) = R_{h,*} d_g f$$

to obtain

$$\langle \Pi(g_1 g_2),\, d_{g_1 g_2} f_1 \otimes \Pi_{g_1 g_2} f_2 \rangle = \langle \Pi(g_2),\, L^{\otimes 2}_{g_1,*}(d_{g_2} f_1 \otimes d_{g_2} f_2) \rangle + \langle \Pi(g_1),\, R^{\otimes 2}_{g_2,*}(d_{g_1} f_1 \otimes d_{g_1} f_2) \rangle$$
$$= \langle L^{\otimes 2}_{g_1,*} \Pi(g_2),\, d_{g_2} f_1 \otimes d_{g_2} f_2 \rangle + \langle R^{\otimes 2}_{g_2,*} \Pi(g_1),\, d_{g_1} f_1 \otimes d_{g_1} f_2 \rangle$$

hence the thesis. □

Definition 7.2. *When one of the conditions of proposition (7.1) is verified* (G, Π) *is called a* ***Poisson Lie group****.*

Remarks 7.3.

- For a Poisson Lie group (G, Π) we have $\Pi(e) = 0$. In fact $\Pi(ee) = 2\Pi(e)$.
- $$0 = \Pi(e) = \Pi(gg^{-1}) = L_{g,*}\Pi(g^{-1}) + R_{g^{-1},*}\Pi(g)$$

  so

  $$\Pi(g^{-1}) = -\operatorname{Ad}_{g^{-1},*} \Pi(g)$$

  This means that the inverse $g \mapsto g^{-1}$ is not a Poisson map, but anti-Poisson.
- Another equivalent condition is

  $$L_X L_Y \Pi = 0, \quad \forall\, X \text{ right invariant, and } Y \text{ left invariant}$$

  and additionally $\Pi(e) = 0$.

  This obviously suggests what if $\Pi(e) \neq 0$? We have

  $$\Pi(g_1 g_2) = L_{g_1}\Pi(g_2) + R_{g_2}\Pi(g_1) + L_{g_1} R_{g_2} \Pi(e)$$

  what is called an **affine Poisson structure on** G.

Let us move on to the infinitesimal description of the Poisson Lie groups. Consider

$$\eta\colon G \to \Lambda^2 \mathfrak{g}$$

given by right translating the Poisson tensor

$$\eta(g) = R_{g^{-1},*}\Pi(g)$$

(obviously $\eta(e)=0$). Now

$$\begin{aligned}\eta(g_1g_2) &= R_{(g_1g_2)^{-1},*}\Pi(g_1g_2)\\ &= R_{g_1^{-1},*}R_{g_2^{-1},*}(L_{g_1,*}\Pi(g_2)+R_{g_2,*}\Pi(g_1))\\ &= R_{g_1^{-1},*}\Pi(g_1)+\mathrm{Ad}_{g_1}R_{g_2^{-1},*}\Pi(g_2)\\ &= \eta(g_1)+\mathrm{Ad}_{g_1}\eta(g_2)\end{aligned}$$

i.e. Π multiplicative $\implies$ η is a cocycle of G with values in $\Lambda^2\mathfrak{g}$.

Define now

$$\delta\colon \mathfrak{g}\to\Lambda^2\mathfrak{g}$$

to be its derivative at e, i.e.

$$\delta(X):=\frac{d}{dt}\eta(e^{tX})|_{t=0}$$

What are the properties of δ coming from the fact that Π is Poisson and multiplicative ?

Proposition 7.4.

1. Π *multiplicative* $\implies$

$$\delta([X,Y])=\mathrm{ad}_X\,\delta(Y)-\mathrm{ad}_Y\,\delta(X)$$

2. Π *Poisson* $\implies$

$${}^t\delta\colon\Lambda^2\mathfrak{g}^*\to\mathfrak{g}$$

satisfies Jacobi identity.

Proof. 1.

$$\begin{aligned}\eta(e^{tX}e^{tY}) &= \eta(e^{tX})+\mathrm{Ad}_{e^{tX}}\,\eta(e^{tY})\\ \eta(e^{tY}e^{tX}) &= \eta(e^{tY})+\mathrm{Ad}_{e^{tY}}\,\eta(e^{tX})\\ \eta(e^{tX}e^{tY})-\eta(e^{tY}e^{tX}) &= \eta(e^{tX})-\eta(e^{tY})+\mathrm{Ad}_{e^{tX}}\,\eta(e^{tY})-\mathrm{Ad}_{e^{tY}}\,\eta(e^{tX})\end{aligned}$$

2.

Lemma 7.5. *Let* $\xi_1,\xi_2\in\mathfrak{g}^*$. *Choose* $f_1,f_2\in C^\infty(G)$ *such that* $d_ef_i=\xi_i$, $i=1,2$. *Then*

$${}^t\delta(\times_1,\xi_2)=d_e\{f_1,\,f_2\}$$

Proof.

$$\begin{aligned}\{f_1,\,f_2\}(g) &= \langle\Pi(g),\,d_gf_1\otimes d_gf_2\rangle\\ &= \langle\eta(g),\,R^{\otimes2}_{g,*}(d_gf_1\otimes d_gf_2)\rangle\end{aligned}$$

Take $g=e^{tX}$ and the derivative at $t=0$.

$$\underbrace{\frac{d}{dt}\{f_1,\,f_2\}(e^{tX})|_{t=0}|_{t=0}}_{\langle X,\,d_e\{f_1,f_2\}\rangle}=\underbrace{\frac{d}{dt}\langle\eta(g^{tX}),\,R^{\otimes2}_{e^{tX},*}(d_{e^{tX}}f_1\otimes d_{e^{tX}}f_2)\rangle|_{t=0}}_{\langle\frac{d}{dt}\eta e^{tX}|_{t=0},\,d_ef_1\otimes d_ef_2\rangle}$$

$$=\langle\delta(X),\,d_ef_1\otimes d_ef_2\rangle=\langle X,\,{}^t\delta(d_ef_1,d_ef_2)\rangle$$

Thus the claim. Remark that this proves indirectly independence of the right hand side from choices. □

Now the statement follows easily from

$$\mathrm{Jac}_{{}^t\delta}(\xi_1,\xi_2,\xi_3)=d_e\,\mathrm{Jac}_{\{-,-\}}(f_1,f_2,f_3)$$

□

7.2 Lie bialgebras

Definition 7.6. *A **Lie bialgebra** is a pair* $(\mathfrak{g}, \delta)$ *where* $\mathfrak{g}$ *is a Lie algebra and* $\delta\colon \mathfrak{g} \to \Lambda^2\mathfrak{g}$ *is such that*

1. ${}^t\delta$ *satisfies Jacobi identity (coJacobi:* $\mathrm{cyclic}((\delta \otimes \mathrm{id})\delta(X)) = 0$*)*

2. $\delta([X, Y]) = \mathrm{ad}_X\, \delta(Y) - \mathrm{ad}_Y\, \delta(X)$

We have just proven that the tangent space of a Poisson Lie group has a Lie bialgebra structure. To what extend is the converse true ?

Example 7.7. $\mathfrak{g}$ abelian Lie algebra. Thus any $\delta\colon \mathfrak{g} \to \Lambda^2\mathfrak{g}$ such that ${}^t\delta$ satisfies Jacobi identity gives a Lie bialgebra. Choose a nontrivial one. Therefore $\mathfrak{g}^*$ is a non trivial Lie algebra, which implies that $\mathfrak{g}$ itself has a non trivial Poisson linear structure ($\mathfrak{g} \cong \mathfrak{g}^{**}$).

Take $\Gamma \in \mathfrak{g}$ a lattice under which the Poisson structure is not invariant. Take a Lie group $H = \mathfrak{g}/G$. Then $\mathrm{Lie}(H) = \mathfrak{g}$ is a Lie bialgebra which does not integrate to a Poisson Lie group structure.

The point here is

Lemma 7.8. *Given a 1-cocycle* $\mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ *there is a unique 1-cocycle* $\eta\colon G \to \mathfrak{g} \wedge \mathfrak{g}$*, where* G *is the connected simply connected Lie group integrating* $\mathfrak{g}$ *(i.e.* $\mathrm{Lie}(G) = \mathfrak{g}$*).*

Basically this is all you need to prove

Theorem 7.9 (Drinfel'd)**.** *The correspondence* $G \mapsto \mathfrak{g}$ *gives you a 1:1 correspondence between Lie bialgebras and Poisson Lie groups.*

Given any Poisson Lie group (G, Π) consider its Lie bialgebra $(\mathfrak{g}, \delta)$. Then $(\mathfrak{g}^*, {}^t[-,-])$ is a Lie bialgebra. Therefore it integrates to a unique connected, simply connected Poisson Lie group G^* called the **dual Poisson Lie group** of G.

Lie bialgebras form a category. Morphisms are those homomorphisms which respect δ

$$\begin{array}{ccc} \mathfrak{g} \wedge \mathfrak{g} & \xrightarrow{\chi \otimes \chi} & \mathfrak{g}' \wedge \mathfrak{g}' \\ {\scriptstyle \delta}\uparrow & & \uparrow{\scriptstyle \delta'} \\ \mathfrak{g} & \xrightarrow[\chi]{} & \mathfrak{g}' \end{array}$$

Proposition 7.10. *Given a Lie bialgebra* $(\mathfrak{g}, \delta)$*, the vector space* $\mathfrak{g}^*$ *has a canonical Lie bialgebra structure. The cobracket* δ' *being dual to bracket* $[-,-]$ *in* $\mathfrak{g}$*, and the bracket* $[-,-]'$ *in* $\mathfrak{g}^*$ *being dual to* δ*.*

Definition 7.11. $\mathfrak{g}^*$ *is called dual bialgebra of* $\mathfrak{g}$*.*

Examples 7.12.

1. Any Lie algebra with $\delta = 0$.

2. Dual of previous, $\mathfrak{g}^*$ as vector space, $[-,-] = 0$, $\delta' = [-,-]^*_{\mathfrak{g}}$.

3. $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $X^+, X^-, H \in \mathfrak{sl}(2, \mathbb{C})$

$$[H, X^{\pm}] = \pm 2X^{\pm}, \quad [X^+, X^-] = H$$

$$\delta(X^{\pm}) = X^{\pm} \wedge H, \quad \delta(H) = 0$$

$$\text{cyclic}(\delta \otimes \text{id}) \circ \delta(X^{\pm}) = \text{cyclic}(\delta \otimes \text{id})(\underbrace{X^{\pm} \wedge H}_{X^{\pm} \otimes H - H \otimes X^{\pm}})$$

$$= \text{cyclic}(X^{\pm} \otimes H - H \otimes X^{\pm}) \otimes H = 0$$

(co-Jacobi identity). Now check the 1-cocycle condition

$$\delta([a, b]) = a \cdot \delta(b) - b \cdot \delta(a)$$

We have

$$\delta([H, X^{\pm}]) \stackrel{?}{=} H \cdot (X^{\pm} \wedge H) - X^{\pm} \cdot \delta(H)$$

$$LHS = \pm 2\delta(X^{\pm}) = \pm 2X^{\pm} \wedge H$$

$$RHS = [H, X^{\pm}] \wedge H + X^{\pm} \wedge [H, H] = \pm 2X^{\pm} \wedge H$$

Similarly

$$\delta([X^{+}, X^{-}]) \stackrel{?}{=} X^{+} \cdot \delta(X^{-}) - X^{-} \cdot \delta(X^{+})$$

$$LHS = \delta(H) = 0$$

$$RHS = X^{+} \cdot X^{-} \wedge H - X^{-} \cdot X^{+} \wedge H$$

$$= [X^{+}, X^{-}] \wedge H + X^{-} \wedge [X^{+}, H] - [X^{-}, X^{+}] \wedge H - X^{+} \wedge [X^{-}, H]$$

$$= H \wedge H - 2X^{-} \wedge X^{+} - 2X^{+} \wedge X^{-} = 0$$

4. Let $\mathfrak{g}$ be a $\mathbb{C}$ simple Lie algebra with fixed bilinear, nondegenerate, symmetric form $(-,-)$ on $\mathfrak{g}$ (and on $\mathfrak{g}^*$). choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ ($n = \dim \mathfrak{h}$ is the rank of $\mathfrak{g}$). Choose a simple roots $\alpha_1, \ldots, \alpha_n \in \mathfrak{h}^*$. This gives a decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$$

where $\mathfrak{n}_\pm$ are nilpotent and $\mathfrak{h}$ abelian. Let $X_i^{\pm}$, H_i be the corresponding Chevalley generators and $A = [a_{ij}]$ the Cartan matrix

$$a_{ij} = \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

Recall

$$[H_i, H_j] = 0, \quad [H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm}, \quad [X_i^{+}, X_i^{-}] = \delta_{ij} H_j$$

The following cobracket

$$\delta(H_i) = 0, \quad \delta(X_i^{\pm}) = d_i X^{\pm} \wedge H_i,$$

where d_i symmetrize $[a_{ij}]$, i.e. $d_i a_{ij} = a_{ij} d_j$, gives the structure of a Lie bialgeba.

Definition 7.13. *This example is called a **standard Lie bialgebra** structure on* $\mathfrak{g}$.

Remark. There exist other structures, and all standard structures are equivalent up to conjugation.

7.3 Manin triples

Definition 7.14. *Let $\mathfrak{g}$ be a Lie algebra with a non degenerate invariant symmetric bilinear form. Let $\mathfrak{g}_+$ and $\mathfrak{g}_-$ be Lie subalgebras such that*

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

*as vector spaces, and such that $\mathfrak{g}_+, \mathfrak{g}_-$ are maximal isotropic subspaces of $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is called a **Manin triple**.*

Using the form we can identify

$$\mathfrak{g}_- \cong (\mathfrak{g}_+)^*, \quad \mathfrak{g}_+ \cong (\mathfrak{g}_-)^*$$

In particular $\dim \mathfrak{g}_+ = \dim \mathfrak{g}_-$.

Theorem 7.15.

1. *Suppose $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple. Let*

$$[-,-] \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, \quad [-,-]_+ = [-,-]|_{\mathfrak{g}_+ \otimes \mathfrak{g}_+}, \quad [-,-]_- = [-,-]|_{\mathfrak{g}_- \otimes \mathfrak{g}_-}$$

   Put

$$\delta_+ =^t [-,-]_- \colon (\mathfrak{g}_-^*) = \mathfrak{g}_+ \to \Lambda^2 \mathfrak{g}_+, \quad \delta_- =^t [-,-]_+ \colon (\mathfrak{g}_+^*) = \mathfrak{g}_- \to \Lambda^2 \mathfrak{g}_-$$

   Then $(\mathfrak{g}_+, \delta_+)$ and $(\mathfrak{g}_-, \delta_-)$ are Lie bialgebras.

2. *Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra. Define on $\mathfrak{g} \oplus \mathfrak{g}^*$*

$$\langle X + \xi,\, Y + \eta \rangle := \xi(Y) + \eta(X)$$

$$[X + \xi, Y + \eta] = \langle [X, Y] + \operatorname{ad}^*_X \xi - \operatorname{ad}^*_Y \eta,\, [\xi, \eta] + \operatorname{ad}^*_\xi X - \operatorname{ad}^*_\eta Y \rangle$$

   Then $\mathfrak{g} \oplus \mathfrak{g}^*$ with this form and bracket is a Manin triple.

Proof. Let us rewrite the cocycle condition in a Lie bialgebra

$$\langle \delta([X, Y]),\, \xi \otimes \eta \rangle = \langle [X, Y],\, [\xi, \eta] \rangle$$

Indeed,

$$\begin{aligned}
\langle \delta([X, Y]),\, \xi \otimes \eta \rangle &= \langle \operatorname{ad}_X \delta(Y) - \operatorname{ad}_Y \delta(X),\, \xi \otimes \eta \rangle \\
&= -\langle \delta(Y),\, \operatorname{ad}^*_X(\xi \otimes \eta) \rangle + \langle \delta(X),\, \operatorname{ad}^*_Y(\xi \otimes \eta) \rangle \\
&= -\langle \delta(Y),\, \operatorname{ad}^*_X(\xi) \otimes \eta + \xi \otimes \operatorname{ad}^*_X \eta \rangle + \langle \delta(X),\, \operatorname{ad}^*_Y(\xi) \otimes \eta + \xi \otimes \operatorname{ad}^*_Y \eta \rangle \\
&= \langle Y,\, [\operatorname{ad}^*_X \xi, \eta] + [\xi, \operatorname{ad}^*_X \eta] \rangle - \langle X,\, [\operatorname{ad}^*_Y \xi, \eta] + [\xi, \operatorname{ad}^*_Y \eta] \rangle \\
&= \langle \operatorname{ad}^*_\eta Y,\, \operatorname{ad}^*_X \xi \rangle - \langle \operatorname{ad}^*_\xi Y,\, \operatorname{ad}^*_X \eta \rangle - \langle \operatorname{ad}^*_\eta X,\, \operatorname{ad}^*_Y \xi \rangle + \langle \operatorname{ad}^*_\xi X,\, \operatorname{ad}^*_Y \eta \rangle
\end{aligned}$$

Invariance of bilinear form is equivalent to

$$[\xi, X] = \operatorname{ad}^*_\xi X - \operatorname{ad}^*_X \xi$$

$$\langle [\xi, X],\, \eta \rangle = (\xi, [X, \eta]) = -\langle \operatorname{ad}^*_X \xi,\, \eta \rangle, \quad \forall\, \eta$$

$$\langle [\xi, X],\, Y \rangle = \langle \xi,\, [X, Y] \rangle = -\langle \operatorname{ad}^*_X \xi,\, Y \rangle, \quad \forall\, Y$$

Therefore

$$\begin{aligned}\langle \delta([X,Y]),\, \xi\otimes\eta\rangle &= \langle [X,Y],\, [\xi,\eta]\rangle\\ &= -\langle X,\, [Y,[\xi,\eta]]\rangle\\ &= -\langle X,\, [\eta,[Y,\xi]] + [\xi,[\eta,Y]]\rangle \qquad \text{(from Jacobi identity)}\\ &= -\langle X,\, [\eta, \operatorname{ad}^*_Y \xi - \operatorname{ad}^*_\xi Y] + [\xi, \operatorname{ad}^*_\eta Y - \operatorname{ad}^*_Y \eta]\rangle\\ &= \langle \operatorname{ad}^*_\eta Y,\, \operatorname{ad}^*_X \xi\rangle - \langle \operatorname{ad}^*_\xi Y,\, \operatorname{ad}^*_X \eta\rangle - \langle \operatorname{ad}^*_\eta X,\, \operatorname{ad}^*_Y \xi\rangle + \langle \operatorname{ad}^*_\xi X,\, \operatorname{ad}^*_Y \eta\rangle\end{aligned}$$

and this is formula obtained before. This proves (1). □

Proposition 7.16. *Let* $(\mathfrak{g},\delta)$ *be a Lie bialgebra,* $(D\mathfrak{g},[-,-])$ *Lie algebra in* $\mathfrak{g}\oplus\mathfrak{g}^*$*. Then*

$$\delta\colon D\mathfrak{g}\to \Lambda^2 D\mathfrak{g}$$

given by

$$\delta(X+\xi) = \delta(X) +^t [-,-](\xi)$$

is a Lie cobracket.

Example 7.17. $\mathfrak{g}$ complex simple Lie bialgebra, $\mathfrak{g}\hookrightarrow \mathfrak{g}\oplus\mathfrak{g}$ diagonal embedding. Fix $\mathfrak{h}$ Cartan subalgebra and choice of positive roots.

$$\mathfrak{g} = \mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$$

$$\mathfrak{b}_\pm = \mathfrak{h}\oplus\mathfrak{n}_\pm$$

$$S := \{(x,y)\in \mathfrak{b}_+\oplus\mathfrak{b}_- : x|_\mathfrak{h} = -y|_\mathfrak{h}\}$$

Let on $\mathfrak{g}\oplus\mathfrak{g}$

$$\langle (a,b),\, (c,d)\rangle = \langle a,\, c\rangle - \langle b,\, d\rangle$$

Then $(\mathfrak{g}\oplus\mathfrak{g}, \mathfrak{g}, S)$ is a Manin triple.

Example 7.18. With the notation as before $(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{b}_+, \mathfrak{b}_-)$ is a Manin triple.

Chapter 8

Poisson actions

Recall some notations. Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra, $L_g, R_g \colon G \to G$ left and right translations with derivatives $L_{g,*} \colon T_hG \to T_{gh}G$, $R_{g,*} \colon T_hG \to T_{hg}G$.

Let (G, Π) be a Poisson Lie group, i.e.

$$\Pi(g_1 \cdot g_2) = L_{g_1,*}\Pi(g_2) + R_{g_2,*}\Pi(g_1)$$

and let $\eta \colon G \to \Lambda^2 T_eG = \Lambda^2\mathfrak{g}$ be

$$\eta(g) = R_{g^{-1},*}\Pi(g).$$

Then η is a 1-cocycle of G with respect to adjoint action on $\Lambda^2\mathfrak{g}$, i.e.

$$\eta(g_1g_2) = \eta(g_1) + \mathrm{Ad}_{g_1}\,\eta(g_2)$$

Let $\delta \colon \mathfrak{g} \to \Lambda^2\mathfrak{g}$,

$$\delta(X) = \frac{d}{dt}\eta(e^{tX})\big|_{t=0}$$

Then $(\mathfrak{g}, \delta)$ is a Lie bialgebra

$$\begin{aligned} (\mathfrak{g}, [-,-]) &\quad \text{is Lie} \\ (\mathfrak{g}^*, {}^t\delta) &\quad \text{is Lie} \end{aligned}$$

satisfying compatibility

$$\delta([X, Y]) = \mathrm{ad}_X\,\delta(Y) - \mathrm{ad}_Y\,\delta(X).$$

The Lie algebra $\mathfrak{g}^*$ integrates to a (unique) connected (simply connected) Poisson Lie group G^*. Furthermore on $\mathfrak{g} \oplus \mathfrak{g}^*$ we have the following Lie bracket

$$[X + \xi, Y + \eta] = ([X, Y] + \mathrm{ad}^*_X\,\eta - \mathrm{ad}^*_Y\,\xi, [\xi, \eta] + \mathrm{ad}^*_\xi\,Y - \mathrm{ad}^*_\eta\,X)$$

and Lie cobracket

$$\delta_D(X + \xi) = \delta(X) + \delta^*(\xi)$$

This makes $\mathfrak{g} \oplus \mathfrak{g}^*$ a Lie bialgebra, which is called **Drinfeld double of a Lie algebra $\mathfrak{g}$**. It integrates to (a unique sonnected, simply connected) Poisson Lie group DG called **Drinfeld double of a Lie group G**.

8.1 Poisson actions

Definition 8.1. *Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra. Let (M, Π) be a Poisson manifold, together with an infinitesimal action, i.e. a Lie algebra morphism*

$$\rho\colon \mathfrak{g} \to \mathfrak{X}(M)$$

Then ρ is called an ***infinitesimal Poisson action*** *if*

$$L_{\rho(X)}\Pi = \rho^{\wedge 2}(\delta(X)), \quad \forall\, X \in \mathfrak{g} \tag{8.1}$$

Remark 8.2.

1. Π is not invariant under an infinitesimal Poisson action. If the infinitesimal action is effective it is invariant if and only if $\delta = 0$.
2. To be precise this could be considered an infinitesimal left Poisson action. An infinitesimal right Poisson action is then a Lie algebra antihomomorphism such that (8.1) is verified.

Let now $\phi\colon G \times M \to M$ be a Lie group action. Let us fix the following notations

$$\phi(g, x) = g \cdot x$$

$$\forall\, g \in G, \quad \phi_g\colon M \to M, \quad x \mapsto g \cdot x$$

$$\forall\, x \in M, \quad \phi_x\colon G \to M, \quad g \mapsto g \cdot x$$

Remark that

$$\phi_{g \cdot x} = \phi_x R_g, \quad \phi_g \phi_x = \phi_x L_g.$$

For $f \in C^\infty(M)$ let $\theta_f\colon M \to \mathfrak{g}^*$ be defined by

$$\theta_f(x) = d_g f(g \cdot x)\big|_{g=e}$$

If we have only the infinitesimal action we can define equivalently

$$\langle \theta_f, Y \rangle = \rho(Y) f$$

Theorem 8.3 (Semonov-Tian-Shanskii)**.** *Let (G, Π_G) be a connected, simply connected, Poisson Lie group with Lie bialgebra $(\mathfrak{g}, \delta)$. Let (M, Π_M) be a Poisson manifold. Let $\phi\colon G \times M \to M$ be a Lie group action with infinitesimal map $\rho\colon \mathfrak{g} \to \mathfrak{X}(M)$. then following are equivalent*

1. *ϕ is a Poisson map with respect to the product Poisson structure.*
2. $$\Pi_M(g \cdot x) = \phi_{g,*}\Pi_M(x) + \phi_{x,*}\Pi_G(g), \quad \forall\, x \in M,\ g \in G$$
3. $$\rho(X)\{f, g\} - \{\rho(X)f, g\} - \{f, \rho(X)g\} = \langle [\theta_f, \theta_g], X \rangle \quad \forall\, X \in \mathfrak{g},\ f, g \in C^\infty(M)$$
4. *ρ is an infinitesimal Poisson action, i.e.*
$$L_{\rho(X)}\Pi_M = \rho^{\wedge 2}(\delta(X)), \quad \forall X \in \mathfrak{g}$$

Proof. (1) $\iff$ (2) by definition of product Poisson structure ϕ is Poisson iff for all $f_1, f_2 \in C^\infty(M)$, $g \in G$, $x \in M$

$$\{f_1 \circ \phi,\, f_2 \circ \phi\}_{G\times M}(g,x) = \{f_1,\, f_2\}_M(g \cdot x)$$

But the left hand side equals

$$\{f_1 \circ \phi_x,\, f_2 \circ \phi_x\}_G(g) + \{f_1 \circ \phi_g,\, f_2 \circ \phi_g\}_M(x)$$

$$= \langle \Pi_G(g),\, \phi_x^* d_{g\cdot x} f_1 \wedge \phi_c^* d_{g\cdot x} f_2\rangle + \langle \Pi_M(x),\, \phi_g^* d_{g\cdot x} f_1 \wedge \phi_g^* d_{g\cdot x} f_2\rangle$$
$$= \langle \phi_{x,*}\Pi_G(g),\, d_{g\cdot x} f_1 \wedge d_{g\cdot x} f_2\rangle + \langle \phi_{g,*}\Pi_M(x),\, d_{g\cdot x} f_1 \wedge d_{g\cdot x} f_2\rangle$$

and the right hand side is

$$\langle \Pi_M(g\cdot x),\, d_{g\cdot x} f_1 \wedge d_{g\cdot x} f_2\rangle$$

Hence (1) $\iff$ (2).
(3) $\iff$ (4)

$$L_{\rho(X)}\Pi_M = \rho^{\wedge 2}(\delta(X)) = \langle \delta(X),\, \theta_f \wedge \theta_g\rangle$$
$$\iff \langle L_{\rho(X)}\Pi_M,\, df \wedge dg\rangle = \langle \rho^{\wedge 2}(X),\, df \wedge dg\rangle$$
$$\iff L_{\rho(X)}\langle \Pi_M,\, df \wedge dg\rangle = \langle \Pi_M,\, (L_{\rho(X)df}) \wedge dg\rangle - \langle \Pi_M,\, df \wedge L_{\rho(X)} dg\rangle = \langle \delta(X),\, \theta_f \wedge \theta_g\rangle$$
$$\iff \rho(X)\{f,\, g\} - \{\rho(X)f,\, g\} - \{f,\, \rho(X)g\} = \langle X,\, [\theta_f, \theta_g]\rangle$$

because

$$\langle \delta(X),\, \theta_f \wedge \theta_g\rangle = \langle X,\, [\theta_f, \theta_g]\rangle.$$

(2) $\implies$ (4) by applying $\phi_{g^{-1},*}$ to both sides of (2) we have

$$\phi_{g^{-1},*}\Pi_M(g\cdot x) = \Pi_M(x) + \phi_{g^{-1},*}\phi_{x,*}\Pi_G(g)$$

$$\phi_{g^{-1},*}\Pi_M(g\cdot x) = \Pi_M(x) + \phi_{x,*}L_{g^{-1},*}\Pi_G(g)$$

Now let $g = e^{tX}$, $X \in \mathfrak{g}$ and differentiate with respect to t at $t = 0$

$$\frac{d}{dt}\phi_{e^{-tX},*}\Pi_M(e^{tX}x)\big|_{t=0} = L_{\rho(X)}\Pi_M$$

$$\frac{d}{dt}\phi_{x,*}L_{e^{-tX},*}\Pi_G(e^{tX})\big|_{t=0} = \phi_{x,*}\frac{d}{dt}L_{e^{-tX},*}\Pi_G(e^{tX})\big|_{t=0}$$

(4) $\implies$ (2) Prove that

$$\phi_{e^{-tX},*}\Pi_M(e^{tX}\cdot x) = \Pi_M(x) + \phi_{e^{-tX},*}\phi_{x,*}\Pi_G(e^{tX})$$

Then prove that derivatives $\frac{d}{dt}$ at $t = 0$ are equal.

$$\begin{aligned}
\phi_{e^{-tX},*}\Pi_M(e^{tX}\cdot x) &= \phi_{e^{-tX},*}(L_{\rho(X)}\Pi_M)(e^{tX}x) \\
&= \phi_{e^{-tX},*}(\rho^{\wedge 2}(\delta(X)))(e^{tX}x) \\
&= \phi_{e^{-tX},*}\phi_{e^{tX}x,*}[(L_X\Pi_G)(e)] \\
&= \phi_{e^{-tX},*}\phi_{x,*}R_{e^{tX},*}[(L_X\Pi_G)(e)] \\
&= \phi_{x,*}L_{e^{-tX},*}R_{e^{tX},*}[(L_X\Pi_G)(e)] \\
&= \phi_{x,*}\operatorname{Ad}_{e^{-tX}}[(L_X\Pi_G)(e)].
\end{aligned}$$

On the other hand

$$\frac{d}{dt}(\Pi_M(x) + \phi_{x,*}L_{e^{-tX},*}\Pi_G(e^{tX})) = \phi_{x,*}\frac{d}{dt}L_{e^{-tX},*}\Pi_G(e^{tX})$$
$$= \phi_{x,*}\,\mathrm{Ad}_{e^{-tX}}(L_X\Pi_G(e)).$$

Therefore the two sides coincide for any $x \in M$ in an open neighbourhood of $e \in G$. Being G connected any open neighbourhood generates it and the theorem is proven. $\square$

Remark 8.4.

1. ϕ does not preserve Π_M unless $\Pi_G = 0$. Neither ϕ_g nor ϕ_x are in general Poisson maps.

2. The multiplicity condition (2) is often referred to as Π_M being covariant with respect to Π_G.

3. $m\colon G \times G \to G$ is a left Poisson action on the Poisson Lie group itself. As a special case of the previous statement neither left nor right translations are Poisson maps.

4. Another way of stating the infinitesimal Poisson action condition is

$$d_\Pi(\rho(X)) = \rho^{\wedge 2}(\delta(X))$$

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\rho} & \mathfrak{X}(M) \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle d_\Pi} \\ \Lambda^2\mathfrak{g} & \xrightarrow[\rho^{\wedge 2}]{} & \mathfrak{X}^2(M) \end{array}$$

Thus ϕ looks like some sort of intertwining operator between differentials. In fact δ can be extended to a degree 1 derivation of $\Lambda^\bullet\mathfrak{g}$, simply by letting

$$\delta(X_1 \wedge \cdots \wedge X_n) = \sum_{i=1}^{n}(-1)^i X_1 \wedge \cdots \wedge \delta(X_i) \wedge \cdots \wedge X_n$$

The coJacobi condition on δ implies $\delta^2 = 0$. This turns $\Lambda^\bullet\mathfrak{g}$ into a differential Gerstenhaber algebra $(\Lambda^\bullet\mathfrak{g}, \wedge, [-,-])$. The infinitesimal action condition shows that $\rho\colon \mathfrak{g} \to \mathfrak{X}(M)$ with its natural extension $\rho^\wedge\colon \Lambda^\bullet\mathfrak{g} \to \mathfrak{X}^\bullet(M)$ provides a morphism of differential Gerstenhaber algebras.

5. Right hand side of (4) does not depend on Π_M. Consider Π_M and Π'_M such that (G, Π_G) acts in a Poisson way on both. Then $L_{\rho(X)}(\Pi_M - \Pi'_M) = 0$. Thus $\Pi_M - \Pi'_M$ is an invariant bivector (not necessarily Poisson).

We can give a slightly different look on conditions (3)-(4).

$$\theta\colon \Omega^1(M) \to C^\infty(M; \mathfrak{g}^*) \in \mathfrak{X}(M) \otimes C^\infty(M; \mathfrak{g}^*)$$

Recall the Poisson coboundary introduced in (4.1).

$$d_\Pi\colon \mathfrak{X}^p(M) \to \mathfrak{X}^{p+1}(M), \quad d_\Pi(P) = [\Pi, P]$$

$$(d_\Pi X)(df, dg) = (L_X\Pi)(df, dg) = X\{f,\, g\} - \{Xf,\, g\} - \{f,\, Xg\}$$

Therefore LHS of (3) can be rewritten as

$$(d_\Pi \theta)(df, dg)$$

and phrased with suitable conventions as

$$(d_\Pi \theta - \tfrac{1}{2}[\theta, \theta])(df, dg) = 0$$

i.e. θ satisfies a Maurer-Cartan type of equation.

Proposition 8.5. *Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra with an infinitesimal Poisson action $\rho\colon \mathfrak{g} \to \mathfrak{X}(M)$ on the Poisson manifold (M, Π_M). Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$,*

$$C^\infty(M)^{\mathfrak{h}} = \{f \in C^\infty(M) : \rho(X)f = 0 \quad \forall\, X \in \mathfrak{h}\},$$

$$\mathfrak{h}^\perp = \{\xi \in \mathfrak{g}^* : \langle \xi,\, x\rangle = 0 \quad \forall\, X \in \mathfrak{h}\}.$$

Then

1. *If $\mathfrak{h}^\perp$ is a Lie subalgebra then $C^\infty(M)^{\mathfrak{h}}$ is a Poisson subalgebra.*
2. *If $C^\infty(M)^h$ is a Poisson subalgebra and $\{\theta_f : f \in C^\infty(M)^{\mathfrak{h}}\}$ span $\mathfrak{h}^\perp$, then $\mathfrak{h}^\perp$ is a Lie subalgebra.*

Proof. Let $f, g \in C^\infty(M)^{\mathfrak{h}}$. This means that for any $X \in \mathfrak{h}$, $\rho(X)f = 0 = \rho(X)g$. Using condition (3)

$$\rho(X)\{f,\, g\} = \{\rho(X)f,\, g\} + \{f,\, \rho(X)g\} + \langle [\theta_f, \theta_g],\, X\rangle = 0$$

is equivalent to $[\theta_f, \theta_g] \in \mathfrak{h}^\perp$. Now simply remark that for the case of an infinitesimal action θ_f is defined via

$$\langle \theta_f,\, Y\rangle = \rho(Y)f \quad \forall\, Y \in \mathfrak{g}$$

Therefore f is invariant if and only if $\theta_f \in \mathfrak{h}^\perp$ and $\mathfrak{h}^\perp$ is generated by such elements. Thus the statement. □

Corollary 8.6. *If $\mathfrak{h} \setminus M$ is a smooth manifold then it posesses a Poisson structure and $p\colon M \to \mathfrak{h} \setminus M$ is a Poisson map.*

Corollary 8.7. *If we have a global action and a closed connected subgroup H such that $\mathfrak{h}^\perp$ is a Lie subalgebra then the same holds true for $H \setminus M$.*

8.2 Poisson homogeneous spaces

Definition 8.8. *A* ***Poisson homogeneous space*** *is a Poisson manifold (M, Π_M) together with a transitive Poisson action of a Poisson Lie group.*

Remark 8.9. The covariance condition is

$$\Pi_M(g \cdot x) = \phi_{g,*}\Pi_M(x) + \phi_{x,*}\Pi_G(g)$$

When H is homogeneous for a given $x \in M$ this formula allows to compute Π_M at all points from $\Pi_M(x)$, i.e. Π_M is uniquely determined by its value at one fixed point.

Homogeneous G-spaces are of the form G/H for a closed Lie subgroup H. We will show how, and why, such description does not work any more at the Poisson level. First we need to describe properties of subgroups of Poisson Lie group.

Definition 8.10. *A Lie subgroup H of a Poisson Lie group G is called a* ***Poisson Lie subgroup*** *if it is a Poisson submanifold. It is called a* ***coisotropic subgroup*** *if it is a coisotropic submanifold.*

Remark 8.11. Let $H \leq G$ be a Poisson (coisotropic) Lie subgroup and $g \in G$. Then $\mathrm{Ad}_g(H) = gHg^{-1}$ may be Poisson, coisotropic or none of the above.

Proposition 8.12. *Let H be a connected Lie subgroup of a Poisson Lie group (G, Π_G).*

1. *H is a Poisson Lie subgroup if and only if $\mathfrak{h}^\perp$ is an ideal in $\mathfrak{g}^*$.*
2. *H is coisotropic if and only if $\mathfrak{h}^\perp$ is a Lie subalgebra.*

Proof. H is a Poisson submanifold if and only if

$$I_H = \{f \in C^\infty(G) : f|_H = 0\}$$

is a Poisson ideal. Being $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ spanned by covectors $d_e f$, $f \in I_H$, I_H is a Poisson ideal implies $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is an ideal. The converse is true due to connectedness.

The second statement is proved analogously, but now we request that I_H is a Poisson subalgebra. $\mathfrak{h}^\perp$ is still spanned by $d_e f$, $f \in I_H$, therefore the thesis. □

Poisson homogeneous spaces $\phi\colon G \times M \to M$ contain a number of special cases.

1. Invariant Poisson structures ($\Pi_G = 0$)
2. Affine Poisson structures ($M = G$)
3. Non symplectic covariant (i.e. $\Pi_G \neq 0$) Poisson structures, which include
   (a) "Highly singular" covariant Poisson structures ($\exists\, x_0\ \Pi_M(x_0) = 0$)
   (b) Quotients by coisotropic subgroups
   (c) Quotients by Poisson Lie subgroups

   Furthermore $(a) = (b) \supset (c)$.

Some relevant examples of Poisson Lie groups:

- (G, Π_G) any Poisson Lie group. Drinfeld double DG has a natural Poisson Lie structure. $G, G^* \hookrightarrow DG$ (if it can be embedded) is a Poisson Lie subgroup.
- G complex semisimple Lie group. K compact real form with standard Poisson structure. Then $DK = G$. Furthermore, as the standard Poisson structure is defined via simple roots any Dynkin diagram embedding sorresponds to a Poisson Lie group. In particular to each node there corresponds a distinct Poisson embedding

  $$\mathrm{SU}(2) \subset \mathrm{SU}(n)$$

  Remark though that SL_2 triples not corresponding to simple roots are not Poisson Lie subgroups. For example

  $$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

are Poisson Lie subgroups, but

$$\begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix}$$

is not.

Exercise 8.13. *Classify Poisson Lie subgroups of* $\mathrm{SU}(2)$*.*

Hint: Compute the dual Lie bialgebra. Classify ideals in this 3-dimensional Lie algebra, distinguishing between 2-dimensional ideals and 1-dimensional ideals. Check which of them is the $\perp$ of a Lie algebra, and you have that the only pair $(\mathfrak{h}, \mathfrak{h}^\perp)$ such that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{su}(2)$ and $\mathfrak{h}^\perp$ is a Lie ideal in $\mathfrak{su}(2)^*$ is when $\mathfrak{h} = \langle H \rangle$, H being the Cartan diagonal element. Therefore the only connected Poisson-Lie subgroup is $\mathbb{S}^1$ diagonally embedded in $\mathrm{SU}(2)$ and the disconnected ones are its discrete subgroups.

Exercise 8.14. *Classify Poisson Lie subgroups of* $\mathrm{SL}(n, \mathbb{C})$ *with respect to the standard structure.*

It requires some work. A good start is to look at the first pages of [70].

Coisotropy condition is much weaker. For example let $H \leq G$ be a Lie subgroup of codimension 1. Then H is coisotropic. In fact $\dim \mathfrak{h}^\perp = 1$ and therefore $\mathfrak{h}^\perp$ is a Lie algebra, $[X, X] = 0$.

Let M be a Poisson homogeneous space. Fix $x \in M$

$$T_x M \simeq \mathfrak{g}/\mathfrak{h}_x, \quad \mathfrak{h}_x \text{ - stabilizer of } x$$

Proposition 8.15. *For any* $v \in \Lambda^2 \mathfrak{g}/\mathfrak{h}_x$

$$L_x := \{X + \xi : X \in \mathfrak{g},\ \xi \in \mathfrak{h}_x^\perp,\ (\xi \otimes \mathrm{id})(v) = X + \mathfrak{h}_x\}$$

is a Lagrangian subspace of the double.

Proof.

$$\langle X + \xi, Y + \eta \rangle = (\xi \otimes \eta + \eta \otimes \xi)(v) = 0$$

so L_x is isotropic. Surjectivity follows from surjectivity of $X + \xi \to X$, which implies maximality. □

Theorem 8.16. *For any* $x \in M$ *let* L_x *be the Lagrangian subspace in* $D\mathfrak{g}$*. Then:*

1. L_x *is a Lie subalgebra in* $D\mathfrak{g}$
2. $L_{gx} = gL_x$ *where* gL_x *is given by the adjoint action of* G *in* $D\mathfrak{g}$
3. *There is a bijection between Poisson* G*-homogeneous structures on* M *and* G*-equivariant maps from* M *to the set of Lagrangian subalgebras such that if* $x \in M$ *then* $L_x \cap \mathfrak{g} = \mathfrak{h}_x$*.*

Remark 8.17. Let $D\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ be a Drinfeld double, $G \times D\mathfrak{g} \to D\mathfrak{g}$ adjoint action

$$\mathrm{Ad}_g(X + \xi) = \mathrm{Ad}_g X + \mathrm{Ad}^*_{g^{-1}} \xi\, i_{R_{g^{-1},*}\Pi(g)} + \mathrm{Ad}^*_{g^{-1}} \xi$$

$\mathcal{L}(D\mathfrak{g})$ is an algebraic variety; the set of Lagrangian subalgebras of the double. The adjoint action of G passes to an action on this variety

$$G \times \mathcal{L}(D\mathfrak{g}) \to \mathcal{L}(D\mathfrak{g})$$

Then theorem (8.16) says that on $\mathcal{L}(D\mathfrak{g})$ orbits are "models" for Poisson homogeneous spaces ([?]).

Proposition 8.18. *Let M be a Poisson homogeneous space of (G, Π_G). For $x_0 \in M$ the following are equivalent:*

1. $\Pi_M(x_0) = 0$
2. $\phi_{x_0} : G \to M$ *is a Poisson map*
3. H_{x_0} *(stabilizer* $= \{g \in G : gx_0 = x_0\}$*) is coisotropic;* $M \simeq G/H_{x_0}$

Proof. (1) $\implies$ (2) Take the same x_0

$$\Pi_M(gx_0) = \underbrace{\phi_{g,*}\Pi_M(x_0)}_{=0} + \phi_{x_0,*}\Pi_G(g)$$

Therefore ϕ_{x_0} is Poisson.

(2) $\implies$ (1) Let ϕ_{x_0} be a Poisson map

$$\Pi_M(x_0) = \Pi_M(ex_0) = \phi_{x_0,*}\Pi_G(e) = 0$$

(2) $\iff$ (3) We have already proven (3) $\implies$ (2). Furthermore we know that $\phi_{x_0} : G \to M$ is Poisson if and only if $\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G$ is constant along the fibers of ϕ_{x_0} (proposition (3.20)).

Lemma 8.19. $\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G$ *is constant along all fibers if and only if*

$$\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G\big|_{\phi_{x_0}^{-1}(x_0) = H_{x_0}} = 0$$

Proof. Let $\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G$ be constant along all fibers. Then it is constant when restricted to H_{x_0}. But $e \in H_{x_0}$ and

$$\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G(e) = 0$$

due to $\Pi_G(e) = 0$. Therefore it is 0 on all H_{x_0}.

Let $\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G\big|_{H_{x_0}} = 0$. Take $g, g' \in G$ on the same fiber of ϕ_{x_0}. Then there exists $h \in H_{x_0}$ such that $g' = gh$. Now

$$\begin{aligned}
\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G(g') &= \langle \Pi_G(gh),\, d_{gh}(\phi_{x_0}^* f_1) \otimes d_{gh}(\phi_{x_0}^* f_2) \rangle \\
&= \langle L_{g,*}\Pi_G(h) + R_{h,*}\Pi_G(g),\, d_{gh}(\phi_{x_0}^* f_1) \otimes d_{gh}(\phi_{x_0}^* f_2) \rangle \\
&= \{L_g^* \phi_{x_0}^* f_1,\, L_g^* \phi_{x_0}^* f_2\}(h) + \{R_h^* \phi_{x_0}^* f_1,\, R_h^* \phi_{x_0}^* f_2\}(g) \\
&= \underbrace{\{\phi_{x_0}^* (\phi_g \circ f_1),\, \phi_{x_0}^* (\phi_g \circ f_2)\}(h)}_{=0 \text{ by hypothesis}} + \{\phi_{hx_0}^* (\phi_g \circ f_1),\, \phi_{hx_0}^* (\phi_g \circ f_2)\}(g) \\
&= \{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}(g)
\end{aligned}$$

because $h \in H_{x_0} \implies \phi_{hx_0} = \phi_{x_0}$. □

Now we want to show

$$\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G\big|_{H_{x_0}} = 0 \iff H_{x_0} \text{ is coisotropic.}$$

$\phi_{x_0}^* f_1$ is constant along H_{x_0}, so

$$(\phi_{x_0}^* f_1)(e) = c + f', \quad f' \in I_{H_{x_0}} = \{f \in C^\infty(G) : f|_{H_{x_0} = 0}\}$$

$$\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G = \{f_1' + c,\, f_2' + c\}_G = \{f_1',\, f_2'\}_G$$

Remember that

$$\{\phi_{x_0}^* f_1,\, \phi_{x_0}^* f_2\}_G(h) = \langle \Pi_G(h),\, d_h\phi_{x_0}^* f_1 \otimes d_h\phi_{x_0}^* f_2\rangle$$

We want to prove that $\operatorname{im}\#_\Pi \subseteq N^*H$. The point is that we can restrict to a neighbourhood of identity (due to multiplicativity and connectedness). There choose $h = e^{tH}$. It is enough to show that

$$\frac{d}{dt}\langle \Pi_G(h),\, d_h\phi_{x_0}^* f_1 \otimes d_h\phi_{x_0}^* f_2\rangle|_{t=0} = 0$$

because we know that it is 0 at e. But this equals

$$\langle \delta(H),\, d_e\phi_{x_0}^* f_1 \otimes d_e\phi_{x_0}^* f_2\rangle$$

and $d_e\phi_{x_0}^* f_1 \in \mathfrak{h}^\perp$, and in fact generates it. It is 0 if and only if $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}$, that is precisely when H_{x_0} is coisotropic. □

Proposition 8.20. *Let G be a Poisson Lie group. Let K be a Poisson Lie subgrouop and let H' be a coisotropic subgroup. Then $H = K \cap H'$ is coisotropic in K and*

$$i\colon K/K\cap H' \to G/H'$$

is a Poisson embedding.

Proof.

$$I_K := \{f \in C^\infty(G) : f|_K = 0\}$$

I_K is a Lie ideal with respect to $\{-,\, -\}$ and $I_{H'}$ is a Lie subalgebra with respect to $\{-,\, -\}$.

$$I_{H'\cap K} = I_{H'} + I_K$$

$$f = f_1 + f_2$$

Take $f' \in I_K$

$$\{f',\, f_1 + f_2\} = \underbrace{\{f',\, f_1\}}_{?} + \underbrace{\{f',\, f_2\}}_{\in I_K}$$

so $I_{H'\cap K}$ is not in general a Lie ideal.

Take $l_1, l_2 \in I_{H'\cap K}$

$$\{l_1 + l_2,\, f_1 + f_2\} = \underbrace{\{l_1,\, f_1\}}_{\in I_{H'}} + \underbrace{\{l_1,\, f_2\}}_{\in I_K} + \underbrace{\{l_2,\, f_1\}}_{\in I_K} + \underbrace{\{l_2,\, f_2\}}_{\in I_K} \in I_{H'} + I_K$$

Therefore $I_{H'\cap K}$ is a Lie subalgebra with respect to $\{-,\, -\}$. The second statement follows from the fact that in this diagram everything is Poisson

$$\begin{array}{ccc} K & \longrightarrow & G \\ \downarrow & & \downarrow \\ K/H'\cap K & \longrightarrow & G/H' \end{array}$$

□

The following example is carried out in all details in [20].

Example 8.21. Take SU(n) with standard Poisson Lie structure

$$\begin{aligned} \delta(H_i) &= 0 \\ \delta(E_i) &= H_i \wedge E_i \\ \delta(F_i) &= H_i \wedge F_i \end{aligned}$$

E_i, F_i simple roots, $i = 1, \dots, n$.

Then $\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n-1))$, $(a, A) \mapsto \begin{pmatrix} A & 0 \\ O & a \end{pmatrix}$ is a Poisson Lie subgroup of SU(n). For every $k \in \{1, \dots, n\}$, $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ gives a Poisson Lie subgroup $K_k := \mathrm{S}(\mathrm{U}(k)\times\mathrm{U}(n-k)) \hookrightarrow \mathrm{SU}(n)$. In particular $\mathrm{SU}(n)/K_{n-1} = \mathbb{C}P^{n-1}$ with covariant Poisson structure.

Now take $H' = K_{n-1}$, $K = K_k$, $k = 1, \dots, n-2$

$$H' \cap K \cong \mathrm{U}(k) \times \mathrm{U}(n-k-1)$$

$$H'/H' \cap K \simeq \mathbb{C}P^{n-k-1}$$

with the same Poisson structure.

Therefore we get

$$* \hookrightarrow \mathbb{C}P^1 \hookrightarrow \dots \hookrightarrow \mathbb{C}P^{n-2} \hookrightarrow \mathbb{C}P^{n-1}$$

This gives all symplectic foliation of $\mathbb{C}P^{n-1}$. Now change things a little bit. There exists a family $\sigma_c \in \mathrm{SU}(n)$ such that $\mathrm{Ad}_{\sigma_c} H'$ is coisotropic. We want to study

$$\mathbb{C}P_c^{n-1} \simeq \mathrm{SU}(n)/\operatorname{Ad}_{\sigma_c} H'$$

Now $H' \cap K_k$ changes.

Proposition 8.22. *The embedding*

$$H'/H' \cap K_k \hookrightarrow \mathbb{C}P_c^{n-1}$$

is an embedding of

$$\mathbb{S}^{2k-1} \times \mathbb{S}^{2(n-k)-1} \hookrightarrow \mathbb{C}P_c^{n-1}$$

In particular, when $k = 1$, this gives

$$\mathbb{S}^{2n-3} \hookrightarrow \mathbb{C}P_c^{n-1}$$

where odd spheres have the standard Poisson structure.

Recall that we have defined a Lie bracket on $\Omega^1(M)$ (where M is Poisson)

$$[\alpha, \beta] = L_{\#_\Pi(\alpha)}\beta - L_{\#_\Pi(\beta)}\alpha - d(\Pi(\alpha, \beta)) \tag{8.2}$$

What happens to this bracket when $M = G$ is a Poisson-Lie group ?

Theorem 8.23 (Dazord-Karasev-Weinstein)**.** *The left (resp. right) invariant 1-forms on a Poisson Lie group (G, Π_G) form a Lie subalgebra with respect to (8.2). Furthermore this induces a Lie bracket on $\mathfrak{g}^*$ isomorphic to ${}^t\delta$.*

Proof. Let α, β be left invariant 1-forms. Let $X \in \mathfrak{X}^1(G)$ be a left invariant vector field. We will prove that $[\alpha, \beta]$ is left invariant by proving that $\langle X, [\alpha,\beta]\rangle$ is constant for any such X.

$$\langle X, [\alpha,\beta]\rangle = \langle X, L_{\#_\Pi(\alpha)}\beta - L_{\#_\Pi(\beta)}\alpha\rangle - \langle X, d(\Pi(\alpha,\beta))\rangle$$

Let's look at the second summand

$$\begin{aligned}\langle X, d(\Pi(\alpha,\beta))\rangle &= L_X(\Pi(\alpha,\beta))\\ &= (L_X\Pi)(\alpha,\beta) + \Pi(L_X\alpha,\beta) + \Pi(\alpha, L_X\beta)\\ &= (L_X\Pi)(\alpha,\beta) - \langle \#_\Pi, L_X\alpha\rangle + \langle \#_\Pi(\alpha), L_X\beta\rangle\end{aligned}$$

Now consider first summand

$$\begin{aligned}\langle X, L_{\#_\Pi(\alpha)}\beta\rangle &= \underbrace{L_{\#_\Pi(\alpha)}\langle X, \beta\rangle}_{=0 \text{ because } \langle X,\beta\rangle \text{ is constant}} - \langle [\#_\Pi(\alpha), X], \beta\rangle\\ &= -\langle [\#_\Pi(\alpha), X], \beta\rangle\\ &= \langle L_X(\#_\Pi(\alpha)), \beta\rangle\\ &= \langle i_{L_X\Pi}\alpha + \#_\Pi(\alpha), \beta\rangle\end{aligned}$$

because $\#_\Pi(\alpha) = i_\Pi(\alpha)$, $[L_X, i_\Pi] = i_{L_X\Pi}$.
Therefore

$$\langle X, L_{\#_\Pi(\alpha)}\beta\rangle = (L_X\Pi)(\alpha,\beta) - \langle \#_\Pi(\beta), L_X\alpha\rangle$$

Similarly

$$\langle X, L_{\#_\Pi(\alpha)}\beta\rangle = -(L_X\Pi)(\alpha,\beta) - \langle \#_\Pi(\alpha), L_X\beta\rangle$$

Now substitute

$$L_X\Pi(\alpha,\beta) - \langle \#_\Pi(\beta), L_X\alpha\rangle + L_X\Pi(\alpha,\beta) + \langle \#_\Pi(\alpha), L_X\beta\rangle$$

$$-L_X\Pi(\alpha,\beta) + \langle \#_\Pi(\beta), L_X\alpha\rangle - \langle \#_\Pi(\alpha), L_X\beta\rangle = (L_X\Pi)(\alpha,\beta)$$

Lemma 8.24. *If Π is a Poisson Lie bracket on G then for any X left invariant vector field $L_X\Pi$ is left invariant.*

Proof. If X is left invariant on G its flow are right translations

$$\begin{aligned}(L_X\Pi)(g) &= \frac{d}{dt}R_{e^{-tX},*}\Pi(ge^{tX})|_{t=0}\\ &= \frac{d}{dt}(R_{e^{-tX},*}L_{g,*}\Pi(e^{tX}) + R_{e^{-tX},*}R_{e^{tX},*}\Pi(g))|_{t=0}\\ &= L_{g,*}\frac{d}{dt}R_{e^{tX},*}\Pi(e^{tX})|_{t=0}\\ &= L_{g,*}(L_X\Pi(e))\end{aligned}$$

□

This proves that the bracket of left invariant 1-forms is left invariant because $\langle X, [\alpha,\beta]\rangle = (L_X\Pi)(\alpha,\beta) = \langle L_X\Pi(e), \alpha_e \wedge \beta_e\rangle$ so $L_X\Pi$ is a left invariant 2-vector field.

Now the statement follows from

$${}^t\delta(d_e f, d_e g) = d_e\{f, g\}$$

which gives the same as (8.2) computed at e.

$$\begin{aligned}[df, dg] &= L_{\#_\Pi(df)}dg - L_{\#_\Pi(dg)}df - d(\Pi(df, dg)) \\ &= L_{X_f}dg - L_{X_g}df - d\{f,\, g\} \\ &= d\langle X_f,\, dg\rangle - d\langle X_g,\, df\rangle - d\{f,\, g\} \\ &= d\{f,\, g\} + d\{f,\, g\} - d\{f,\, g\} \\ &= d\{f,\, g\}\end{aligned}$$

Left invariant 1-forms evaluated at e give you all of $\mathfrak{g}^*$ and therefore you can say

$${}^t\delta(\xi_1, \xi_2) = [df_1, df_2](e), \text{ where } \xi_1 = d_e f_1,\ \xi_2 = d_e f_2.$$

□

Exercise 8.25. *Consider the standard Poisson Lie group structure on* SU(2)*. Then* $\mathfrak{su}(2)$ *has a basis*

$$E_1 = \tfrac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}, \quad E_2 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \quad E_3 = \tfrac{1}{2}\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}$$

Now $\delta\colon \mathfrak{su}(2) \longrightarrow \mathfrak{su}(2) \wedge \mathfrak{su}(2)$

$$\begin{aligned}\delta(E_1) &= 0 \\ \delta(E_2) &= E_1 \wedge E_2 \\ \delta(E_3) &= E_1 \wedge E_3\end{aligned}$$

Prove that this defines a Lie bialgebra, that is δ *is a 1-cocycle*

$$\delta([X, Y]) = \mathrm{ad}_X\, \delta(Y) - \mathrm{ad}_Y\, \delta(X), \quad \mathrm{ad}_X = \mathrm{ad}_X \otimes 1 - 1 \otimes \mathrm{ad}_X$$

and

$${}^t\delta\colon \mathfrak{su}(2)^* \wedge \mathfrak{su}(2)^* \longrightarrow \mathfrak{su}(2)^*$$

satisfies Jacobi identity. Check it is enough to verify the cocycle conditions for $(X, Y) = (E_1, E_2),\ (E_1, E_3),\ (E_2, E_3)$.

Prove that for ${}^t\delta = [-, -]$

$$\begin{aligned}[e_1, e_2] &= e_2 \\ [e_1, e_3] &= e_3 \\ [e_2, e_3] &= 0\end{aligned}$$

define a Lie algebra structure.

Use the Killing form

$$\langle A,\, B\rangle = \mathrm{im}(\mathrm{Tr}(AB))$$

to identify $\mathfrak{su}(2)^*$ *with*

$$\left\{\begin{pmatrix} x & a + ib \\ 0 & -x\end{pmatrix} : x, a, b \in \mathbb{R}\right\}$$

Therefore the connected simply connected dual group

$$\mathrm{SB}(2) = \left\{\begin{pmatrix} x & z \\ 0 & x^{-1}\end{pmatrix} : x \in \mathbb{R}_{>0},\ z \in \mathbb{C}\right\} \cong \mathbb{R} \ltimes \mathbb{C}$$

Now let us describe all $(\mathrm{SU}(2), \Pi_G)$ *Poisson homogeneous space structures on* $\mathbb{S}^2$*. Let* Π_1, Π_2 *be Poisson homogeneous bivectors on* $\mathbb{S}^2$*. Then*

1. $\Pi_1 - \Pi_2$ *is* $\mathrm{SU}(2)$*-invariant (general)*

2. $\Pi_1 - \Pi_2$ *is Poisson (because of dimension 2)*

3. On $\mathbb{S}^2$ *there is a "unique" invariant symplectic form* ω_0 *corresponding to bivector* Π_0*.*

$$\Pi_1 - \Pi_2 = f\Pi_0$$

but being $\Pi_1 - \Pi_2$ $\mathrm{SU}(2)$*-invariant,* $f =$ *constant,* $\Pi_1 - \Pi_2 = C\Pi_0$*. Therefore we have a Poisson pencil of invariant Poisson structures on* $\mathbb{S}^2$

$$c\Pi_0 + \Pi_1$$

Choose as Π_1 *the quotient with respect to the Poisson Lie subgroup of diagonal matrices. It is explicitely given by*

$$\begin{aligned}
\{x_1, x_2\} &= (1 - x_1)x_3 \\
\{x_2, x_3\} &= (1 - x_1)x_1 \\
\{x_3, x_1\} &= (1 - x_1)x_2 = (1 - x_1)\Pi_0
\end{aligned}$$

Now

$$c\Pi_0 + (1 - x_1)\Pi_0 = (\lambda - x_1)\Pi_0, \quad \lambda \in \mathbb{R}$$

Prove that $\lambda \mapsto -\lambda$ *is a Poisson isomorphism.*

In the corresponding symplectic foliation 0-dimensional leaves are given by

$$\{x_1^2 + x_2^2 + x_3^2 = 1\} \cap \{x_1 = \lambda\}$$

There are 3 cases

- $\lambda > 1$ *no 0-dimensional leaves,* $(\lambda - x_1)\Pi_0$ *is symplectic*
- $\lambda = 1$ *(corresponding to* Π_1 *the quotient by Poisson Lie group)* $\{N\}$ *is a 0-dimensional leaf, and* $\mathbb{S}^2 \setminus \{N\}$ *is a 2-dimensional leaf*
- $0 \leq \lambda < 1$ *(corresponding to* Π_1 *the quotient by Poisson Lie subgroup)* $\mathbb{S}^1$*-family of* 0*-dimensional leaves, two distinct 2-dimensional leaves.*

If $\lambda > 1$ *they all have different symplectic volume, thus they are not symplectomorphic. If* $0 \leq \lambda < 1$ *they are not unimodular. The modular class is* $x_2\delta_{x_3} - x_3\delta_{x_2}$*. Using 2-Poisson cohomology it is possible to show that* $\mathbb{S}^2_\lambda \ncong \mathbb{S}^2_{\lambda'}$ *for* $\lambda \neq \lambda'$ *in* $[0, 1]$*.*

8.3 Dressing actions

Take $\xi \in \mathfrak{g}^*$, and denote by ξ^L the associated left invariant 1-form and by ξ^R the associated right invariant 1-form, i.e.

$$\xi^L(g) = L^*_{g^{-1}}\xi \in T^*_g G; \qquad \xi^R(g) = R^*_{g^{-1}}\xi \in T^*_g G\,.$$

Definition 8.26. *Define* $\lambda, \rho \colon \mathfrak{g}^* \to \mathfrak{X}(G)$

$$\lambda(\xi) := \#_\Pi(\xi^L)$$

$$\rho(\xi) := -\#_\Pi(\xi^R)$$

Lemma 8.27. *λ is a Lie algebra morphism, ρ is a Lie algebra antimorphism.*

Proof.

$$\begin{aligned}\lambda([\xi_1,\xi_2]) &= \#_\Pi([\xi_1,\xi_2]^L)\\ &= \#_\Pi([\xi_1^L,\xi_2^L])\\ &= [\#_\Pi(\xi_1^L),\#_\Pi(\xi_2^L)]\end{aligned}$$

$$\begin{aligned}\rho([\xi_1,\xi_2]) &= \#_\Pi([\xi_1,\xi_2]^R)\\ &= \#_\Pi([\xi_1^R,\xi_2^R])\\ &= -[\#_\Pi(\xi_1^R),\#_\Pi(\xi_2^R)]\end{aligned}$$

□

Therefore λ defines an infinitesimal left action of $\mathfrak{g}^*$ on G and ρ defines an infinitesimal right action of $\mathfrak{g}^*$ on G. These are called **infinitesimal dressing actions**.

Exercise 8.28. *Prove that the inversion map $S\colon g \mapsto g^{-1}$ intertwines left and right infinitesimal dressing actions, i. e. $S_* \circ \lambda = \rho$.*

Definition 8.29. *If the dressing action can be integrated to a global action of G^* on G, the Poisson Lie group G is said to be **complete**.*

We recall that the notion of Poisson-Lie group is self dual, therefore the above defines also the left and right infinitesimal dressing actions of $\mathfrak{g}$ on the dual Poisson-Lie group G^*.

Proposition 8.30. *Locally symplectic leaves of G coincide with the orbits of the left (or right) dressing action. If the Poisson Lie group is complete then the symplectic leaves coincide with such orbits.*

Proof. By definition left dressing vector fields are hamiltonian vector fields. They are tangent to leaves. Therefore locally orbits are contained in leaves.

On the other hand values of the left dressing vector fields at any $\mathfrak{g} \in G$ span the tangent space to the leaf through g. Therefore orbits and leaves coincide locally. If the action is global consider the whole orbit $\mathcal{O} \in S$ and $T_p\mathcal{O} = \mathrm{im}_{\#_{\Pi,p}} = T_pS$ for all p. Thus $\mathcal{O}$ is a Poisson submanifold of S and therefore $\mathcal{O} = S$. □

Dressing action is the most powerful tool for computing the symplectic foliation of Poisson Lie group.

Proposition 8.31. *Taking the derivative at e of left (resp. right) infinitesimal dressing action you get (resp. minus) the coadjoint action of $\mathfrak{g}^*$ on $\mathfrak{g}$.*

Theorem 8.32 (Semonov-Tian-Shansky,[])**.** *Left and right dressing actions are Poisson actions.*

How can one integrate the dressing action ? Recall the Drinfeld double $D\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$. Then locally (around $e \in DG$)

$$DG|_U = GG^*|_U$$

For any $d \in U$ denote with d_G its component in G, and with d_{G^*} its component in G^*, such that $d = d_G d_{G^*}$.

Proposition 8.33. *The local action given by this splitting*

$$\mathfrak{g}^* \cdot \mathfrak{g} := (\mathfrak{g}^* \mathfrak{g})_G$$

is a local left action of G^* on G integrating the infinitesimal dressing action λ.

The proof relies on a characterisation of the dressing action we could not give. Whenever $DG = GG^*$ holds globally you have the **global dressing action**.

Example 8.34. Standard Poisson Lie structure on K compact. $DK = G$ complex semisimple in which K compact real form. $DG = KAN_+$ Iwasawa decomposition is a global splitting of the double. Therefore symplectic leaves on K are orbits of an AN_+ action.

Take $(G, \Pi = 0)$. Then $G^* \simeq \mathfrak{g}^*$ abelian Lie group with Lie-Poisson bracket. The dressing action of G on G^* is given by

$$\lambda\colon \mathfrak{g} \to \mathfrak{X}(G^*)$$
$$X \mapsto \#_{\mathrm{LP}}(X^L)$$

where X^L is identified with an invariant 1-form on G^* (remark that $T_e G^* = \mathfrak{g}^*$, $T_e^* G^* = \mathfrak{g}^{**} = \mathfrak{g}$).

$$\langle \underbrace{\#_{\mathrm{LP}}(X^L)}_{\in \Omega^1_{\mathrm{inv}}(G^*)}, \underbrace{Y^L}_{\in \mathfrak{X}_{\mathrm{inv}}(G^*)} \rangle(\xi) = \{X^L, Y^L\}(\xi) = \langle \xi, [X, Y]\rangle = \langle \mathrm{ad}^*_X \xi, Y\rangle$$

Therefore $\#_{\mathrm{LP}}(X^L)$ as vector field is the same as $-\mathrm{ad}^*_X$. Thus locally it is given by coadjoint action of G on $\mathfrak{g}^*$. But this action is global. We recover the result that symplectic leaves for the Lie-Poisson structure are orbits of the coadjoint action.

How to integrate the dressing action ? Recall that the Drinfeld double is a Lie bialgebra on $D\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ which integrates to a Poisson Lie group DG. Then locally around $e \in DG$ we have

$$DG|_U = GG^*|_U$$

Let $d \in U \subseteq DG$

$$d := d_G \cdot d_{G^*}$$

with obvious notation.

Proposition 8.35. *The local action given by the splitting*

$$DG|_U = GG^*|_U$$

as

$$\mathfrak{g}^* \cdot \mathfrak{g} := (\mathfrak{g}^* \mathfrak{g})_G$$

integrates the infinitesimal dressing action.

Remark 8.36. When you have a global splitting of the double, you have a global dressing action.

Examples 8.37.

1. K compact with standard Poisson Lie structure. Then $G = KAN_+$ (Iwasawa decomposition) is the double.

2. $(G, \Pi = 0)$, $(G^* = \mathfrak{g}^*, \Pi_{\mathrm{PL}})$. Then the dressing action of G on G^* is the coadjoint action.

Theorem 8.38. *Let $g \in G$ (around e). The leaf through g locally is the image of the double coset $G^* g G^*$ under the natural projection*

$$DG \to DG/G^* \cong G$$

If the dressing action is global they are exactly those.

Chapter 9

Quantization

9.1 Introduction

The purpose will be here to give a definition of quantization and estabilish a vocabulary given us the link between two languages: Poisson geometry and noncommutative algebras. Something like

classical	semiclassical	quantum
manifold	Poisson manifold	noncommutative algebra
group	Poisson Lie group	noncommutative Hopf algebra
point	0-leaf	character

Of course to state all this correctly we need to be very precise on the setting in which we will work. Apart from some preliminaries we will content ourselves to deal with the group case where, for a number of reasons and still with a high degree of attention on details, such dictionary behaves particularly well (i.e. is a functor).

Let us start with a general definition of quantization. On the formal level that will first require from us some definitions. We will work over the field $k = \mathbb{C}$. Basically all what follows work on any field of characteristic 0 and a not so trivial part still holds in characteristic p.

Let us denote with $\mathbb{C}[[\hbar]]$ the ring of formal power series in an indeterminant $\hbar$ with coefficients in $\mathbb{C}$. The algebraic structure here is obvious:

$$\sum_{n\geq 0} a_n\hbar^n + \sum_{n\geq 0} b_n\hbar^n = \sum_{n\geq 0}\sum_{n\geq 0}(a_n + b_n)\hbar^n$$

$$\left(\sum_{n\geq 0} a_n\hbar^n\right) \cdot \left(\sum_{n\geq 0} b_n\hbar^n\right) = \sum_{n\geq 0}\left(\sum_{p+q=n} a_p b_q\right)\hbar^n$$

This is a ring with unit 1. Invertible elements are exactly those power series with $a_0 \neq 0$ (check this as an exercise).

Let now M be a $\mathbb{C}[[\hbar]]$-module. For every $x \in M$ define

$$\kappa(x) := \max\{k : x \in \hbar^k M\}$$

Define for every $x, y \in M$

$$d(x, y) := 2^{-k(x-y)}$$

Lemma 9.1. *d is a pseudo metric on M.*

This metric induces a topology on M which is called the **$\hbar$-adic topology**. A $\mathbb{C}[[\hbar]]$-module is called **torsion free** if the multiplication by $\hbar$ is an injective map.

Proposition 9.2. *Let M be a topological $\mathbb{C}[[\hbar]]$-module. Then there exists a $\mathbb{C}$-vector space M_0 such that $M \cong_{\mathbb{C}} M_0[[\hbar]]$ if and only if M is Hausdorff, complete, $\hbar$-torsion free.*

Proof. If $M \cong M_0[[\hbar]]$ then one simply applies definitions.

In the opposite direction let M be Hausdorff, complete, torsion free. Let $M_0 := M/\hbar M$. Take $\pi\colon M \to M_0$. Choose a section $\sigma\colon M_0 \to M$ and define

$$\widetilde{\sigma}\colon M_0[[\hbar]] \to M$$

$$\sum_{n\geq 0} \hbar^n m_n \mapsto \sum_{n\geq 0} \hbar^n \sigma(m_n)$$

This $\widetilde{\sigma}$ is well defined on formal power series because of completeness. In fact

$$\sum_{n=0}^{N} \hbar^n \sigma(m_n)$$

is a Cauchy sequence in N, therefore we have a well defined

$$\lim_{N\to\infty} \sum_{n=0}^{N} \hbar^n \sigma(m_n)$$

This $\widetilde{\sigma}$ is injective as a consequence of $\hbar$-torsion freeness. In fact

$$\sum_{n\geq 0} \hbar^n \sigma(m_n) = 0 \implies \pi\left(\sum_{n=0}^{N} \hbar^n \sigma(m_n)\right) = 0$$

$$\implies m_0 = 0 \implies \hbar \sum_{n=1}^{N} \hbar^{n-1} \sigma(m_n) = 0$$

Now divide by $\hbar$ and repeat the argument.

$\widetilde{\sigma}$ is injective because of Hausdorffness. □

A module M over $\mathbb{C}[[\hbar]]$ of this form is called a **topologically free module**.

Take A to be a topologically free $\mathbb{C}[[\hbar]]$-algebra (completed tensor product). Then being $\hbar A$ an ideal $A/\hbar A$ is an algebra over $\mathbb{C}$.

Definition 9.3. *A* ***quantization*** *of an algebra A_0 is a topological free $\mathbb{C}[[\hbar]]$-algebra A such that $A/\hbar A$ is commutative.*

Proposition 9.4. *Let A be a quantization of A_0. Then A_0 is a Poisson algebra.*

Proof. Take $a, b \in A_0$, $\bar{a}, \bar{b} \in A$ respective lifts (i.e. $a = \bar{a} \mod \hbar$, $b = \bar{b} \mod \hbar$). Remark that $[\bar{a}, \bar{b}] \in \hbar A$ from the commutativity of A_0. Define

$$\{a,\, b\} := \frac{[\bar{a}, \bar{b}]}{\hbar} \mod \hbar$$

This is well defined

$$\frac{[\bar{a} + \hbar u, \bar{b} + \hbar v]}{\hbar} = \frac{[\bar{a}, \bar{b}]}{\hbar} + \frac{\hbar[u, \bar{b}] + \hbar[\bar{a}, v]}{\hbar} + \frac{\hbar^2[u, v]}{\hbar} = [\bar{a}, \bar{b}] \mod \hbar$$

□

In fact, when you have a Lie group, then you have two algebraic objects to describe with: $F[G]$ and $U(\mathfrak{g})$. What is their relation ?

$U(\mathfrak{g})$ is a Hopf algebra (cocommutative). The "right" choice of $F[G]$ is a Hopf algebra:

- G affine algebraic group and $\mathbb{C}[G]$ algebra of regular functions (sheaf of Hopf algebras wnen you do not have affine)
- K compact group and $R[K]$ algebra of representative functions (matrix elements of irreducible representations)
- G Lie group and $\mathbb{C}_f[G]$ algebra of formal functions

If you consider everything as real objects you have a Hopf-*-algebras. $(H, m, \Delta, \varepsilon, S)$ is a Hopf-*-algebra if $*\colon A \to A$ is an involution, i.e.

$$
\begin{aligned}
(ab)^* &= b^* a^* \\
(\lambda a)^* &= \bar{\lambda} a^* \\
&\text{and} \\
\Delta(a^*) &= (\Delta a)^* \\
(a \otimes b)^* &= a^* \otimes b^*
\end{aligned}
$$

(this implies $(* \circ S)^2 = \mathrm{id}$). Then $U(\mathfrak{g})$ and $F[G]$ can be seen as Hopf-*-algebras.

9.2 Duality

Take $X \in U(\mathfrak{g})$. Then it defines a left invariant differential operator on G. Take $f \in F[G]$

$$(Xf)(e) = \langle X,\, f \rangle$$

$$\langle \Delta X,\, f_1 \otimes f_2 \rangle = \langle X,\, f_1 f_2 \rangle$$

It gives you a nondegenerate pairing of Hopf-*-algebras. In general it is a map

$$\langle -,\, - \rangle \colon A \otimes B \to \mathbb{C}$$

such that

$$
\begin{aligned}
\langle a,\, b \rangle = 0 \;\forall\, a \in A &\implies b = 0 \\
\langle a,\, b \rangle = 0 \;\forall\, b \in B &\implies a = 0 \\
\langle 1,\, b \rangle &= \varepsilon(b) \\
\langle a,\, 1 \rangle &= \varepsilon(a) \\
\langle a_1 a_2,\, b \rangle &= \langle a_1 \otimes a_2,\, \Delta b \rangle \\
\langle \Delta a,\, b_1 \otimes b_2 \rangle &= \langle a,\, b_1 b_2 \rangle \\
\langle S(a),\, b \rangle &= \langle a,\, S(b) \rangle \\
\langle a^*,\, b \rangle &= \overline{\langle a,\, S(b)^* \rangle}
\end{aligned}
$$

So you have a pair of Hopf-*-algebras in nondegenerate duality. More structure when (G, Π) is a Poisson-Lie group, $F[G]$ is a Poisson algebra such that multiplication $m\colon G \times G \to G$ satisfies

$$\{f_1 \circ m,\, f_2 \circ m\}_{G \times G} = \{f_1,\, f_2\}_G \circ m$$

Definition 9.5. ***Poisson Hopf algebra*** *is defned by condition*

$$\{\Delta f_1,\, \Delta f_2\}_{G\times G} = \Delta\{f_1,\, f_2\}_G$$

From our point of view it will be better to start with the infinitesimal description, i.e. universal enveloping algebra level. Let us first see what happens at the universal enveloping algebra of a Lie bialgebra.

Definition 9.6. *A* ***coPoisson Hopf algebra*** *is a pair $(U, \widehat{\delta})$, where U is a Hopf algebra and the linear map $\widehat{\delta}\colon U \to U \otimes U$ is such that*

$$\widehat{\delta}(ab) = (\Delta a)\widehat{\delta}(b) + \widehat{\delta}(a)(\Delta b)$$

and the dual map $\delta^*\colon U^* \otimes U^* \to U^*$ is a Poisson bracket.

Proposition 9.7. *Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra and $U = U(\mathfrak{g})$ its universal enveloping algebra. Then there exists unique $\widehat{\delta}\colon U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ such that*

$$\widehat{\delta}|_{\mathfrak{g}} = \delta$$

In particular $U(\mathfrak{g})$ has a canonical coPoisson Hopf algebra structure.

Proof. The formula

$$\widehat{\delta}(ab) = (\Delta a)\widehat{\delta}(b) + \widehat{\delta}(a)(\Delta b)$$

plus $\widehat{\delta}|_{\mathfrak{g}} = \delta$ defines $\widehat{\delta}$ uniquely on all of $U(\mathfrak{g})$ once you have checked

$$\delta([a, b]) = [\Delta a, \delta(b)] + [\delta(a), \Delta b], \quad \forall\, a, b \in \mathfrak{g}$$

which is equivalent to the 1-cocycle condition

$$[\Delta, a] = \mathrm{ad}_a \text{ on } \mathfrak{g} \otimes \mathfrak{g}.$$

□

Definition 9.8. *A topologically free Hopf algebra H over $\mathbb{C}[[\hbar]]$ is a* ***quantized universal enveloping algebra*** *if*

$$H/\hbar H \cong U(\mathfrak{g})$$

for some Lie algebra $\mathfrak{g}$.

Proposition 9.9. *Let H be a quantized universal enveloping algebra. Then $\mathfrak{g}$ has a Lie bialgebra structure defined by*

$$\delta(X) = \frac{\Delta\overline{X} - \Delta^{\mathrm{op}}\overline{X}}{\hbar} \mod \hbar$$

where $\overline{X}$ is any lifting of $X \in \mathfrak{g}$ to H.

Proof. $\Delta\overline{X} - \Delta^{\mathrm{op}}\overline{X} \in \hbar H$ because $U(\mathfrak{g})$ is cocommutative and therefore

$$\frac{\Delta\overline{X} - \Delta^{\mathrm{op}}\overline{X}}{\hbar} \in H$$

$\delta(X)$ as defined does not depend on the choice of $\overline{X}$

$$\frac{\Delta(\overline{X} + \hbar u) - \Delta^{\mathrm{op}}(\overline{X} + \hbar v)}{\hbar} = \frac{\Delta\overline{X} - \Delta^{\mathrm{op}}\overline{X}}{\hbar} + \alpha, \quad \alpha \in \hbar H$$

Modulo $\hbar$ one obtains

$$\frac{\Delta\overline{X} - \Delta^{\mathrm{op}}\overline{X}}{\hbar} \mod \hbar$$

$\delta(X)$ is skewsymmetric (clear) and belongs to $\mathfrak{g} \otimes \mathfrak{g}$. $\delta(X) \in \mathfrak{g} \otimes \mathfrak{g}$ if and only if its two components are primitive elements.

$$\begin{aligned}(\Delta \otimes \mathrm{id})\delta(X) &= \left[\frac{1}{\hbar}(\Delta \otimes \mathrm{id})(\Delta\overline{X} - \Delta^{\mathrm{op}}\overline{X})\right] \mod \hbar \\ &= \left[\frac{1}{\hbar}(\mathrm{id} \otimes \Delta - \mathrm{id} \otimes \Delta^{\mathrm{op}})\Delta\overline{X} + \sigma_{23}(\Delta \otimes \mathrm{id} - \Delta^{\mathrm{op}} \otimes \mathrm{id})\Delta\overline{X}\right] \\ &= (\mathrm{id} \otimes \delta)\Delta X + \sigma_{23}(\delta \otimes \mathrm{id})\Delta X\end{aligned}$$

CoJacobi identity for δ follows from coassociativity. Cocycle condition follows from Δ being an algebra morphism. □

So for us a **quantum group** will be the following set of data. A pair $F_\hbar[G]$ (quantum functions algebra), $U_\hbar(\mathfrak{g})$ (quantum universal enveloping algebra) of topological Hopf algebras over $\mathbb{C}[[\hbar]]$ together with a nondegenerate Hopf pairing

$$\langle -, - \rangle \colon F_\hbar[G] \times U_\hbar(\mathfrak{g}) \to \mathbb{C}[[\hbar]]$$

The pairing gives you $U_\hbar(\mathfrak{g})$ as "dual" of $F_\hbar[G]$ and vice-versa. You can start with one of the two legs and construct the other. On the way you have some choices. Many technical problems containing remarkable details.

This "pairing" contains the $(X^L f)(e)$ kind of pairing, i.e. the interpretation of $U(\mathfrak{g})$ as differentiable distributions supported at e. But it contains something completely different.

9.3 Local, global, special quantizations

The discussion in the preceeding section was about local quantization. Their main advantage is that they are well suited to capture relations between the classical, semiclassical, and quantum properties (we will see some examples of these relations in more details later). However they miss part of the relevant information, or at least of the full geometry. For example local quantization does not allow to specialize the deformation parameter to complex values $\neq 0$. Being $(\hbar)$ the only maximal ideal in the local ring $\mathbb{C}[[\hbar]]$ they can describe only the limit $\hbar \to a$. But we know of some relevant parts of the theory of quantum groups staying out of this range. This is the case, for example, of the theory of quantum groups at roots of unity, which links quantum groups to 3-manifold invariants and Lie algebras in characteristic p.

Let us denote with $\mathbb{C}(q)$ the field of rational functions in the variable q.

Definition 9.10. *Let A_q be a $\mathbb{C}(q)$-Hopf algebra. An **integer form** (resp. **rational form**) of A_q is a $\mathbb{Z}[q, q^{-1}]$-Hopf subalgebra (resp. $\mathbb{Q}[q, q^{-1}]$) $\mathcal{A}$ of A_q such that*

$$\mathcal{A} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) = A_q$$

(resp. $\mathcal{A} \otimes_{\mathbb{Q}[q,q^{-1}]} \mathbb{C}(q) = A_q$)

Definition 9.11. *Given a $\mathbb{C}(q)$-Hopf algebra A_q together with an integer form $\mathcal{A}$ a **specialization** of A_q to the complex number λ is*

$$A_\lambda := \mathcal{A} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$$

where the tensor product is taken with respect to $\varphi \colon \mathbb{Z}[q, q^{-1}] \to \mathbb{C}$, $\varphi(q) = \lambda$.

In this way starting from a $\mathbb{C}(q)$-Hopf algebra we obtain a $\mathbb{C}$-Hopf algebra.

Example 9.12. Let $\mathfrak{g}$ be a finitely dimensional complex simple Lie algebra. Then $U_q(\mathfrak{g})$ is the associative $\mathbb{Q}(q)$-algebra with generators $X_i^{\pm}$, $K_i^{\pm 1}$, $1 \leq i \leq n$ and relations

$$\begin{aligned} K_i K_j &= K_j K_i \\ K_i K_i^{-1} &= K_i^{-1} K_i = 1 \\ K_i X_j^+ K_i^{-1} &= q_i^{a_{ij}} X_j^+ \\ K_i X_j^- K_i^{-1} &= q_i^{-a_{ij}} X_j^- \\ [X_i^+, x_j^-] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \end{aligned}$$

$$\sum_{r=0}^{1-a_ij} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_q (X_i^{\pm})^{1-a_{ij}} X_j^{\pm} (X_i^{\pm})^r = 0, \text{ if } i \neq j$$

together with the Hopf algebra structure

$$\begin{aligned} \Delta_q K_i^{\pm 1} &= K_i^{\pm} \\ \Delta_q X_i^+ &= X_i^+ \otimes K_i + 1 \otimes X_i^+ \\ \Delta_q X_i^- &= X_i^- \otimes K_i + K_i^{-1} \otimes X_i^+ \end{aligned}$$

$$\begin{aligned} S_q(K_i) &= K_i^{-1} \\ S_q(X_i^+) &= -X_i^+ K_i^{-1} \\ S_q(X_i^-) &= -K_i X_i^- \end{aligned}$$

$$\begin{aligned} \varepsilon_q(K_i) &= 1 \\ \varepsilon_q(X_i^{\pm}) &= 0 \end{aligned}$$

where $[a_{ij}]$ is the Cartan matrix of $\mathfrak{g}$, $q_i = q^{d_i}$, and d_i are positive integers such that $[d_i a_{ij}]$ is symmetric,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

$$(q;q)_n = (1-q) \cdot \ldots \cdot (1-q^n)$$

are the q-binomial coefficients.

Remark 9.13.

- If we have a relation $xy = qyx$, then there is a following formula using q-binomial coefficients
$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}$$
- It is not true that $U_q(\mathfrak{g}) \otimes_{\mathbb{Q}(q)} \mathbb{C}(q) = U(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C}(q)$. For example in $U(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C}(q)$ you do not have that many invertibles.
- Let $q = e^h$, $K_i = e^{d_i h H_i}$. This defines a local quantization $U_h(\mathfrak{g})$ of the standard bialgebra structure on $\mathfrak{g}$. To be precise you have, after modding out relations, take closure in the h-adic topology.

- Examples of ambiquities in choices of integer form. You can declare

$$\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} = \widehat{H}_i \text{ to belong to } A_q$$

or

$$\frac{K_i^2 - 1}{q_i^2 - q_i} \text{ to belong to } A_q$$

Choose between $K_i^{\pm 1} X_i^{\pm}$ or $X_i^{\pm}$. Connected to choice of a lattice in between weight and root lattice, which is equivalent to choice in between different groups with the same Lie algebra.

Definition 9.14. *Let $F_q(\mathrm{GL}_n(\mathbb{C}))$ be the $\mathbb{C}(q)$-algebra generated by $t_{ij}, \det_q^{-1}$, $1 \leq i, j \leq n$ with relations*

$$\begin{aligned}
t_{ki}t_{kj} &= qt_{kj}t_{ki}, \quad i < j \\
t_{ik}t_{jk} &= qt_{jk}t_{ik}, \quad i < j \\
t_{il}t_{kj} &= t_{kj}t_{il}, \quad l < k,\ j < l \\
t_{ij}t_{kk} - t_{kl}t_{ij} &= (q - q^{-1})t_{il}t_{kj}, \quad l < k,\ j < l \\
\det{}_q &= \sum_{\sigma \in \Sigma_n} (-1)^{l(\sigma)} t_{1\sigma(1)} \dots t_{n\sigma(n)}
\end{aligned}$$

together with the Hopf algebra structure

$$\begin{aligned}
\Delta t_{ij} &= \sum_{k=1}^{n} t_{ik} \otimes t_{kj} \\
\varepsilon(t_{ij}) &= \delta_{ij} \\
S(t_{ij}) &= (-q)^{i-j} \xi_{i_c}^{j_c} \det{}_q^{-1}
\end{aligned}$$

where

$$\xi_{i_c}^{j_c} = \sum_{\substack{\sigma \in \Sigma_n \\ i_1, \dots, i_{n-1} \in [1, \dots, n] \setminus \{i_c\} \\ \sigma(i_1), \dots, \sigma(i_{n-1}) \in [1, \dots, n] \setminus \{j_c\}}} (-q)^{l(\sigma)} t_{i_1\sigma(1)} t_{j\sigma(1)} \dots t_{i_{n-1}\sigma(n-1)} t_{j\sigma(n-1)}$$

Here apparently there is no need to use the machinery of $\mathbb{C}(q)$-algebras and integer forms to specialize the parameter to complex values. This is why often in this context one does not mention integer forms. Still they are relevant in the duality between $F_g[G]$ and $U_q(\mathfrak{g})$.

Definition 9.15. *Let G be an affine algebraic complex Poisson group. A* ***global quantized function algebra on*** *G is a $\mathbb{C}(q)$-Hopf algebra A_q together with an integer form $\mathcal{A}$ such that $A_{q=1} \cong F[G]$ as Hopf algebras.*

Another good aspect of global quantization is that it provides you with genuine (non topological) Hopf algebras.

9.4 Real structures

The usual approach to real structures is to consider $\mathbb{C}$-Hopf algebras endowed with a *-structure.

Definition 9.16. *A Hopf-*-algebra is a Hopf algebra over* $\mathbb{C}$ *endowed with the unital, involutive, antimultiplicative morphism* $*\colon A \to A$ *such that* Δ *and* ε *are *-homomorphisms.*

One can then prove that $* \circ S = S^{-1} \circ *$.

Proposition 9.17. *Let* G *be a complex algebraic group with Lie algebra* $\mathfrak{g}$*. Then there is a 1-1 correspondence between*

1. *real forms of* G
2. *Hopf-* structures on* $U(\mathfrak{g})$
3. *Hopf-* structures on* $F[G]$

Definition 9.18. *A* ***real quantum group*** *is a global quantized function algebra with a compatible *-structure.*

Example 9.19. Consider the example of $F_q[\mathrm{GL}_n(\mathbb{C})]$. Fix on it the *-structure given by

$$t_{ij}^* = S(t_{ji})$$

This gives you what is called the unitary $F_q[\mathrm{U}(n)]$ (compact form of $F_q[\mathrm{GL}_n(\mathbb{C})]$).

Example 9.20. Let $0 < q < 1$. Consider the *-algebra generated by α, γ $(= t_{11,t_{22}})$ subject to relations

$$\begin{aligned} \alpha\gamma &= q\gamma\alpha \\ \alpha\gamma^* &= q\gamma^*\alpha \\ \gamma^*\gamma &= \gamma\gamma^* \\ \alpha\alpha^* + q^2\gamma\gamma^* &= 1 \\ \alpha^*\alpha + \gamma^*\gamma &= 1 \end{aligned}$$

together with the Hopf-algebra structure

$$\begin{aligned} \Delta\alpha &= \alpha \otimes \alpha - q\gamma^* \otimes \gamma \\ \Delta\gamma &= \gamma\alpha + \alpha^*\gamma \end{aligned}$$

$$\begin{aligned} \varepsilon(\alpha) &= 1 \\ \varepsilon(\gamma) &= 0 \end{aligned}$$

$$\begin{aligned} S(\alpha) &= \alpha^* \\ S(\gamma) &= -q\gamma \end{aligned}$$

This is called the (standard) quantum $\mathrm{SU}_q(2)$; $F_q[\mathrm{SU}(2)]$.

Example 9.21. Let $0 < q < 1$. Consider the *-algebra generated by v, n subject to relations

$$\begin{aligned} vv^{-1} = v^{-1}v &= 1 \\ vn &= qnv \\ nn^* &= qn^*n \\ vn^* &= qn^*v \end{aligned}$$

together with the Hopf-algebra structure

$$\begin{aligned}
\Delta v &= v \otimes v \\
\Delta n &= v^* \otimes n + n \otimes 1 \\
\varepsilon(v) &= 1 \\
\varepsilon(n) &= 0 \\
S(v) &= v^{-1} \\
S(n) &= -qn \\
S(n^*) &= -q^{-1} n^*
\end{aligned}$$

This is called the (standard) quantum $E_q(2)$; $F_q[E(2)]$.

Example 9.22. Consider now the *-algebra generated by v, n subject to relations

$$\begin{aligned}
vv^{-1} = v^{-1}v &= q \\
vn - nv &= q(1-v)^2 \\
[n, n^*] &= in
\end{aligned}$$

The Hopf algebra structure as before. This is called the non standard quantum $E_q(2)$.

9.5 Dictionary

In the following we would like to set up a whole dictionary

classical	semiclassical	quantum
algebraic group	Poisson algebraic group	quantum group
compact group	Poisson compact group	compact quantum group
Lie algebra	Lie bialgebra	quantum universal enveloping agebra
	Poisson dual	quantum duality principle
	Poisson double	quantum double construction
point	0-leaf	character

It is known in examples that quantum groups have few characters (classical points). Why is it so ?

Proposition 9.23. *Let $A_\hbar$ be a local quantization of $A_0 = (F[M], \Pi)$. There is an injective map between set of characters of $A_\hbar$ (i.e. maps $\varepsilon\colon A_\hbar \to \mathbb{C}[[\hbar]]$ such that $\varepsilon([A_\hbar, A_\hbar]) = 0$) and 0-leaves of the Poisson bivector* Π.

Proof. Let ε be the character of $A_\hbar$. Then ε defines a character of A_0. Thus there exists $x_0 \in M$ such that $\varepsilon(f) = f(x_0)$ for all $f \in A_0$.

Now

$$\begin{aligned}
&\varepsilon([a,b]) = 0 \quad \forall\, a, b \in A_\hbar \\
\Longrightarrow\ &\varepsilon(\{f_1,\, f_2\}) = 0 \quad \forall\, f_1, f_2 \in A_0 \\
\Longrightarrow\ &\underbrace{\{f_1,\, f_2\}(x_0)}_{\langle \Pi(x_0),\, d_{x_0} f_1 \wedge d_{x_0} f_2 \rangle} = 0
\end{aligned}$$

Thus if A_0 is an algebra of functions on a smooth manifold such that $d_{x_0} f$ generate $\Omega^1_{x_0}(M)$ we have $\Pi(x_0) = 0$. □

Example 9.24. $F_q[\mathrm{SU}(2)]$ (here *-characters - looking for real points)

$$\underbrace{\varepsilon(\alpha\gamma)}_{\varepsilon(\alpha)\varepsilon(\gamma)} = \underbrace{\varepsilon(q\gamma\alpha)}_{q\varepsilon(\gamma)\varepsilon(\alpha)} \implies (1-q)\varepsilon(\alpha)\varepsilon(\gamma) = 0$$

Thus $\varepsilon(\alpha) = 0$ or $\varepsilon(\gamma) = 0$ and so on. We end up with

$$\varepsilon(\alpha) = t, \quad \varepsilon(\alpha^*) = t^{-1}$$

This is just an issue of a more general situation. In principle you would like to have a correspondence between primitive ideals of $F_\hbar[G]$ and symplectic foliation of (G, Π). For example if we take $U_q(\mathfrak{g}) = F_q[\mathfrak{g}^*]$ then by orbit method we obtain a homeomorphism between primitive ideals in $U_q(\mathfrak{g})$ and coadjoint orbits of G on $\mathfrak{g}^*$. It would be nice to have a "quantum orbit method". In fact it works for compact quantum groups.

9.6 Quantum subgroups

Let H be a closed or algebraic subgroup of G.

$$I_H = \{f \in F[G] : f|_H = 0\}$$

is a Hopf ideal and

$$F[G]/I_H \cong F[H]$$

as Hopf algebras. To put it another way

$$H \text{ subgroup of } G \iff F[G] \to F[H] \quad \text{Hopf algebra epimorphism}$$

Alternatively thinking at the infinitesimal level

$$\mathfrak{h} \text{ subalgebra of } \mathfrak{g} \iff U(\mathfrak{h}) \to U(\mathfrak{g}) \quad \text{Hopf algebra monomorphism}$$

It is therefore natural to say

Definition 9.25. *A **quantum subgroup** of a (global, local, special) quantized algebra of functions is a topological Hopf algebra epimorphism*

$$F_q[G] \to F_q[H]$$

Therefore quantum subgroups correspond to Hopf ideals in $F_q[G]$.

9.7 Quantum homogeneous spaces

Let B be a unital *-algebra and let A be a Hopf-*-algebra.

Definition 9.26. *A *-algebra homomorphism $\delta\colon B \to b \otimes A$ is a **right coaction** if*

$$(\mathrm{id} \otimes \Delta) \circ \delta = (\delta \otimes \mathrm{id}) \circ \delta$$

$$(\mathrm{id} \otimes \varepsilon) \circ \delta = \mathrm{id}$$

*B is called A-**right quantum space**.*

Which right coactions correspond to homogeneous actions ? Here we mean $A = F[G]$, $B = F[X]$, δ dual of action $\phi\colon G \times X \to X$.

Definition 9.27. *Two right quantum spaces (B, δ), (B', δ') are* ***equivalent*** *if and only if there exists $\Phi\colon B \to B'$ *-algebra isomorphism such that*

$$\delta' \circ \Phi = (\Phi \otimes \mathrm{id})\delta \tag{9.1}$$

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \downarrow & & \downarrow \\ B \otimes A & \longrightarrow & B' \otimes A \end{array}$$

Modifying the following definition replacing the identity in (9.1) by a *-algebra morphism $\Psi\colon A \to A'$

$$\delta' \circ \Phi = (\Phi \otimes \Psi) \circ \delta$$

gives the definition of **equivariant map** of quantum spaces on different Hopf algebras.

Proposition 9.28. *Let (B, δ) be A-right quantum space. There is a 1:1 correspondence between *-algebra homomorphisms $\widetilde{\varepsilon}\colon B \to \mathbb{C}$ and *-algebra homomorphisms $i\colon B \to A$ such that*

$$\Delta \circ i = (i \otimes \mathrm{id}) \circ \delta$$

The correspondence is given by

$$i_{\widetilde{\varepsilon}} = (\widetilde{\varepsilon} \otimes \mathrm{id}) \circ \delta$$
$$\widetilde{\varepsilon} = \varepsilon \circ i$$

Proof. Say $\widetilde{\varepsilon}\colon B \to \mathbb{C}$ is given. Define

$$i_{\widetilde{\varepsilon}} := (\widetilde{\varepsilon} \otimes \mathrm{id}) \circ \delta$$

We have

$$\begin{aligned} \Delta \circ i_{\widetilde{\varepsilon}} &= \Delta \circ (\widetilde{\varepsilon} \otimes \mathrm{id}) \circ \delta \\ &= (\widetilde{\varepsilon} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \circ \delta \qquad (\Delta \text{ is } \mathbb{C}\text{-linear}) \\ &= (\widetilde{\varepsilon} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\delta \otimes \mathrm{id}) \circ \delta \qquad (\delta \text{ is a coaction}) \\ &= (i_{\widetilde{\varepsilon}} \otimes \mathrm{id}) \circ \delta \end{aligned}$$

hence $i_{\widetilde{\varepsilon}}$ verifies the required identity. Furthermore we then have

$$\varepsilon \circ i_{\widetilde{\varepsilon}} = (\widetilde{\varepsilon} \otimes \varepsilon) \circ \delta = (\widetilde{\varepsilon} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varepsilon) \circ \delta = \widetilde{\varepsilon}$$

Say $i\colon B \to A$ is given. Let $\widetilde{\varepsilon} = \varepsilon \circ i$. Then

$$(\widetilde{\varepsilon} \otimes \mathrm{id}) \circ \delta = ((\varepsilon \circ i) \otimes \mathrm{id}) \circ \delta = (\varepsilon \otimes \mathrm{id}) \circ (i \otimes \mathrm{id}) \circ \delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta \circ i = i$$

□

Thus for any A-right quantum space (B, δ) such that B has a character there exists an equivariant map between (B, δ) and a subalgebra $(i_\varepsilon(B), \Delta|_{i_\varepsilon(B)})$ of A.

What is $i_\varepsilon(B)$ in usual language ? Take a G-space X. Fix $x_0 \in X$. Then consider

$$F[X] \to F[G], \quad f \mapsto \widetilde{f_{x_0}}$$

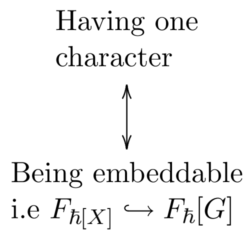
where $\widetilde{f_{x_0}}(g) := f(gx_0)$. When X is a classical homogeneous space we have that this map is injective.

Definition 9.29. *An* ***embeddable*** *quantum homogeneous space is an A-right quantum space* (B, δ) *with a *-homomorphism* $\widetilde{\varepsilon}\colon B \to \mathbb{C}$ *such that* $i_{\widetilde{\varepsilon}}$ *is injective.*

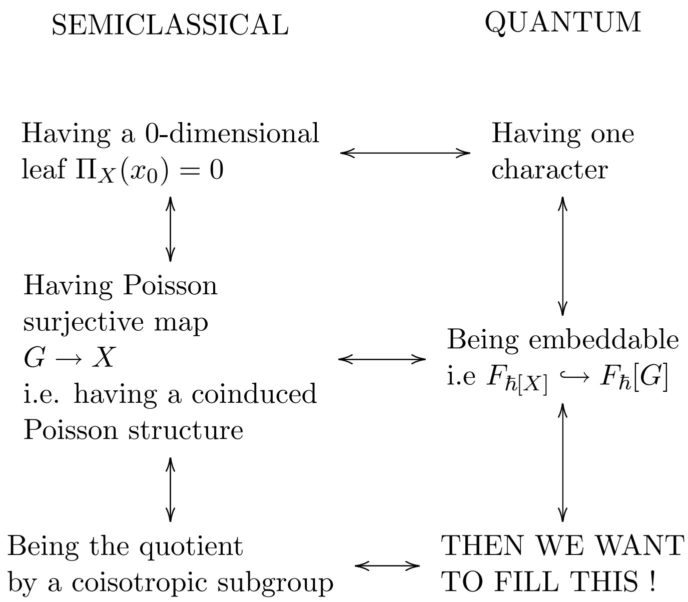
Identifying (B, δ) with $(i_{\widetilde{\varepsilon}}(B), \Delta|_{i_{\widetilde{\varepsilon}}(B)})$ we can equivalently declare an embeddable quantum homogeneous space to be a *-subalgebra and right coideal of $F_q[G]$.

Remark 9.30. This is not the most general fdefinition of quantum homogeneous space. In fact it requires B to have a character, which is in noncommutative algebras something not so trivial.

Let us understand this from the point of view of semiclassical limit. Everything above can be rephrased on $\mathbb{C}[\hbar]$-Hopf-*-algebras. Now we have



But we have seen already this at the semiclassical level



Before going into this we want to understatnd the relation betwwen quantum subgroups and embeddable quantum homogeneous spaces.

Proposition 9.31. *Let* $F_q[G] = A$ *be a quantum group and let* $F_q[H]$ *be a quantum subgroup with defining ideal* I_H*, i.e.*

$$F_q[H] = F_q[G]/I_H, \quad p_H\colon F_q[G] \to F_q[H]$$

If our quantum group is real require also $I_H^* = I_H$*. Define*

$$B_H := \{b \in A : (p_H \otimes \mathrm{id})\Delta b = 1 \otimes b\} = B^{\mathrm{co}I_H}$$

Then B_H *is a *-subalgebra and right coideal of A. Furthermore* B_H *is* S^2*-invariant and* $p_H(b) = \varepsilon(b)1$ *for all* $b \in B$*.*

Proof. Remark that

$$y \in B_H \otimes A \quad \Longleftrightarrow \quad (p_H \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})y = 1 \otimes y$$

Take $b \in B_H$. We want to show that $\Delta b \in B_H \otimes A$.

$$\begin{aligned}(p_H \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ \Delta b &= (p_H \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \circ \Delta b \\ &= (id \otimes \Delta) \circ \underbrace{(p_H \otimes \mathrm{id})\Delta b}_{1 \otimes b} \\ &= 1 \otimes \Delta b\end{aligned}$$

Now we will prove that B_H is S^2-invariant. In fact S^2 is a Hopf algebra automorphism

$$\begin{aligned}(p_H \otimes \mathrm{id}) \circ \Delta \circ S^2(b) &= (p_H \otimes \mathrm{id}) \circ (S^2 \otimes S^2) \circ \Delta b \\ &= ((p_H \circ S^2) \otimes S^2) \circ \Delta b \\ &= (S^2 \otimes S^2) \circ (1 \otimes b) \\ &= 1 \otimes S^2 b\end{aligned}$$

Lastly, apply $\mathrm{id} \otimes \varepsilon$ to $(p_H \otimes \mathrm{id}) \circ \Delta b = 1 \otimes b$ to prove $p_H(b) = \varepsilon(b)1$. □

We would like to check whether all quantum homogeneous spaces are of this form. We have a necessary condition, S^2-invariance. Is it always verified ?

Example 9.32. Consider on the standard $F_q[E(2)]$

$$z = \lambda v + n, \quad \bar{z} = \bar{\lambda} v^* + n^*, \quad \lambda \in \mathbb{C},\ |\lambda| = 1$$

$B =$ *-subalgebra generated by $z, \bar{z}$. Then B coincides with polynomials in z and $\bar{z}$.

$$z\bar{z} = q^2 \bar{z} z + (1 - q^2)$$

$$z^* = \bar{z}$$

Furthermore B is a coideal, $\Delta B \subset A \otimes B$.

$$\Delta z = v \otimes z + n \otimes 1$$

$$\Delta \bar{z} = v^* \otimes \bar{z} + n^* \otimes 1$$

But B is not S^2-invariant unless $\lambda = 0$.

$$S(z) = \lambda S(v) + S(n) = \lambda v^* - q^{-1} n$$

$$S^2(z) = \lambda v - q^{-2} n$$

Thus $z, S^2(z) \in B$, so $n \in B$, which is not true if $\lambda \neq 0$.

Example 9.33. Similarly consider $F_q[\mathrm{SU}(2)]$. Take

$$K := s(\gamma\alpha + \alpha^*\gamma^*) + (1 - s^2)\gamma^*\gamma$$

$$L := s(\alpha^2 - q\gamma^{*2}) + (1 - s^2)\alpha\gamma^*$$

One can check that:

1. The *-subalgebra generated by K and L is isomorphic to the universal *-algebra on these two generators and relations

$$\begin{aligned} K &= K^* \\ LK &= q^2 KL \\ LL^* + K^2 &= (1-s^2)K + s^2 \\ LL^* + qK^2 &= (1-s^2)q^2K + s^2, \qquad s \in [0,1] \end{aligned}$$

2. This *-subalgebra is always a right coideal and therefore is an embeddable quantum homogeneous space

3. This *-subalgebra is a quotient by a quantum subgroup if and only if $s = 1$.

We are looking for a quantum analogue of a coisotropic subgroup.

9.8 Coisotropic creed

When $A_\hbar$ is a quantization of (M, Π) then one-sided ideals in $A_\hbar$ should correspond to coisotropic submanifolds. The motivation for this comes from characterization

Poisson submanifold N	$\longleftrightarrow$	I_N is an ideal and a Poisson ideal	$\longleftrightarrow$	$I_{q,N}$ is an ideal (two-sided ideal)

Coisotropic submanifold N	$\longleftrightarrow$	I_N is an ideal and a Poisson subalgebra	$\longleftrightarrow$	SOMETHING WEAKER BUT STRONGER THAN BEING A SUBALGEBRA

Proposition 9.34. *Let $A_\hbar$ is a quantization of M. Take I to be a right ideal in M. Then $I_0 = I/\hbar I$ is an ideal in A_0 and a Poisson subalgebra in M.*

Proof. Let $i \in I$, $f \in A_\hbar$

$$f * i = fi + \hbar\{f, i\} + \ldots \in I$$

$$[f * i]_{\hbar I} = fi \in I/\hbar I$$

To be precise, take $f \in A_0$, $i \in I_0$. Take any lift $\bar{f} \in A_\hbar$, $\bar{i} \in I$.

$$\bar{f} = f + O(\hbar), \quad \bar{i} = i + O(\hbar)$$

Then

$$\begin{aligned} \bar{f} * \bar{i} = fi + O(\hbar) \quad &\Longrightarrow \quad [\bar{f} * \bar{i}] = f_0 i_0 \in A_0 I_0 \\ \bar{i} * \bar{f} = fi + O(\hbar) \quad &\Longrightarrow \quad [\bar{i} * \bar{f}] = i_0 f_0 \in A_0 I_0 \end{aligned}$$

But now

$$\bar{f} * \bar{i} - \bar{i} * \bar{f} \in \hbar A_\hbar \notin I$$

so we cannot define $\{f, i\} \in I_0$. Still what we have is the following. Let $i, j \in I_0$. Take $\bar{i}, \bar{j} \in I$ lifting i, j.

$$\begin{aligned} \bar{i} * \bar{j},\ \bar{j} * \bar{i} \in I \quad &\Longrightarrow \quad [\bar{i}, \bar{j}] \in \hbar I \\ &\Longrightarrow \{i, j\} \in I_0 \end{aligned}$$

so I_0 is a Poisson subalgebra. ∎

We will stick to this creed and declare the following

Definition 9.35. *Let A be (*)-Hopf algebra. A* ***right (real) coisotropic quantum subgroup*** *C is a coalgebra and A-right module C such that there exists surjective linear map $p\colon A \to C$, which is a morphism of coalgebras and right A-module (endowed with an involution σ such that $p \circ (* \circ S) = \sigma \circ p$).*

Proposition 9.36. *C is a right (real) coisotropic quantum subgroup if and only if there exists $I_C \subseteq A$, which is a $((* \circ S)$-invariant) two sided coideal and right ideal such that*

$$p\colon A \to A/I_C \cong C$$

Remark 9.37. All Poisson subgroups can be quantized in a context of functorial quantization, but it is not known in such context whether all coisotropic subgroups can be quantized.

Proposition 9.38.

1. *Let C be a coisotropic quantum subgroup of A with defining ideal I. Then*

$$B_C := \{a \in A : (p \otimes \mathrm{id}) \circ \Delta b = p_{I_C}(1) \otimes B\}$$

is an embeddable quantum homogeneous space of A.

2. *Let B be an embeddable quantum homogeneous space. Then*

$$I_B := \{(b - \varepsilon(b)1) : b \in B\}$$

is a right ideal and two sided coideal of A.

Is this

coisotropic quantum subgroups	$\longleftrightarrow$	embeddable quantum homogeneous spaces

a bijective correspondence ? Is it true that quotient by quantum subgroups are characterized by S^2-invariance ? Almost.

Let B be a right coideal subalgebra. Take

$$AB^+ := B \cap \ker \varepsilon = \{b - \varepsilon(b)1 : b \in B\}$$

In general $B \subseteq A^{\mathrm{co}A/AB^+}$ but not necessarily equal. If the antipode is bijective and we restrict to left faithfully flat right coideal subalgebras and left faithfully coflat coisotropic quantum subgroups, then in that case S^2-invariance corresponds to quotient by a coisotropic quantum subgroup.

Bibliography

[1] Abouqateb A. and Boucetta M., *The modular class of a regular Poisson manifold and the Reeb class of its symplectic foliation.* C.R.Math Acad. Sci. Paris **337**, 61-66 (2003).

[2] Aminou R. and Kosmann-Schwarzbach Y., *Bigébres de Lie doubles et carrés*, Ann. Inst. H.Poincaré, Phys.Teor. **49A**, 461–478 (1988).

[3] Belavin A.A. e Drinfel'd V.G., *Solutions of the classical Yang–Baxter equation for simple Lie algebras,* Funct. Anal. Appl. **16**, 159–180 (1982).

[4] Benayed M., *Central extensions of Lie bialgebras and Poisson–Lie groups,* Journ. Geom. Phys. **16**, 301–304 (1995).

[5] Benayed M., *Lie bialgebras real cohomology,* Journ. Lie Theory **7**, 287–292 (1997).

[6] Bonechi F., Ciccoli N. and Tarlini M., *Non commutative instantons on the 4–sphere from quantum groups,* Commun. Math. Phys. **226**, 419–432 (2002).

[7] Bonechi F., Ciccoli N. and Tarlini M., *Quantum even spheres* Σ_q^{2n} *from Poisson double suspension,* Commun. Math. Phys. **234**, 449–459 (2003).

[8] Bonechi F., Ciccoli N., L. Dąbrowski and Tarlini M., *Bijectivity of the canonical map for the non commutative instanton bundle,* Journ. Geom. Phys. **51**, 71–81 (2004).

[9] Bordermann M. *Nondegenerate invariant bilinear forms on non associative algebras* Acta Math. Univ. Comen. **66**, 151–201 (1997).

[10] Brylinski J.–L., *A differential complex for Poisson manifolds,* J. Diff. Geom. **28**, 93–114 (1998).

[11] J.–L. Brylinski e E. Geztler., *The homology of algebras of pseudo-differential symbols and the non commutative residue,* K-theory **1**, 385-402 (1987).

[12] Bursztyn H. and Radko O., *Gauge equivalence of Dirac structures and symplectic groupoids,* Ann. Inst. Fourier (Grenoble) **53**, 309–337 (2003).

[13] Bursztyn H. and Weinstein A., *Poisson geometry and Morita equivalence,* `math.SG/0402347`.

[14] Bursztyn H. and Weinstein A., *Picard groups in Poisson geometry*, Moscow J. Math.....

[15] Canas da Silva A. e Weinstein A., *Geometric models for Noncommutative Algebras,* Berkeley Mathematics Lectures vol. **10**, American Mathematical Society, Providence 1999.

[16] A. Cattaneo and G. Felder, *Coisotropic submanifolds in Poisson geometry, branes and Poisson σ–models*, Lett. Math. Phys. **69** (2004), 157–175.

[17] Chari V. and Pressley A., A guide to quantum groups, Cambridge University Press, Cambridge 1994.

[18] Ciccoli N. and Gavarini F., *Quantum duality principle for coisotropic subgroups and quotients*, Adv. Math. **199**, 104–135 (2006).

[19] Ciccoli N. e Guerra L., *Lagrangian subalgebra of the double* $Sl(2, \mathbb{R})$, Geom. Ded. in stampa.

[20] Ciccoli N. and Sheu A.J.–L., *Covariant Poisson structures on complex Grassmannians,* Comm. Anal. Geom. at press.

[21] Crainic M., *Differentiable and algebroid cohomology, van Est isomorphisms and characteristic classes,* Comment. Math. Helv. **78**, 681–721 (2003).

[22] Crainic M. and Fernandeś R.L., *Integrability of Poisson brackets,* Journ. Diff. Geom. **66**, 71–137 (2004).

[23] Crainic M. and Fernandeś R.L., *Rigiditiy and flexibility in Poisson geometry,* Trav. Math. **16**, 53–68 (2005).

[24] Courant T., *Dirac manifolds,* Trans. Amer. Math. Soc. **319**, 631–661(1990).

[25] De Smedt V. *Existence of a Lie bialgebra structure on every Lie algebra,* Lett. Math. Phys. **31**, 225–231 (1994).

[26] M. S. Dijkhuizen and M. Noumi, *A family of quantum projective spaces and related q-hypergeometric orthogonal polynomials*, Trans. Amer. Math. Soc. **350** (1998), 3269–3296.

[27] M.S. Dijkhuizen, M. Noumi and T. Sugitani, *Multivariable Askey–Wilson polynomials and quantum complex Grassmannians*, in "Special functions, q–series and related topics", eds. M.E.H. Ismail et al., 167–177, Fields. Inst. Comm. **14**, AMS (1997).

[28] Drinfel'd V.G., *Hamiltonian Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equation*, Sov. Math. Dokl. **27**, 68–71 (1983).

[29] Drinfel'd V.G., *On Poisson homogeneous spaces of Poisson-Lie groups*, Teor. Math. Phys. **95**, 226–227 (1993).

[30] E. Dahl e B. Enriquez, *Homologie cyclique et de Hochschild de certaines èspaces homogènes quantiques,* K-theory **6**, 499–517 (1992).

[31] Delorme P., *Classification des triples de Manin pour les algébres de Lie réductives complexes.* J. Algebra **246**, 97–174 (2001).

[32] Etingof P. e Varchenko A., *Geometry and classification of solutions of the classical dynamical Yang–Baxter equation*, Comm. Math. Phys. **192**, 77–120 (1998).

[33] Evens S. and Lu J.–H., *Poisson harmonic forms, Kostant harmonic forms and the* $\mathbb{S}^1$*–equivariant cohomology of* K/T, Adv. Math. **142**, 171–220 (1999).

[34] S. Evens and J.–H. Lu, *On the variety of Lagrangian subalgebras I.*, Ann. Sci. Ecole Norm. Sup. Paris **34** (2001), 631–668.

[35] S. Evens and J.–H. Lu, *On the variety of Lagrangian subalgebras II.*, Ann. Sci. Ecole Norm. Sup. Paris, to appear.

[36] Evans S., Lu J.H. and Weinstein A., *Transverse measures, the modular class and a cohomology pairing for Lie algebroids,* Quart. J. Math. Oxford **50**, 417–436 (1999).

[37] Etingof P. e Varchenko A., *Geometry and classification of solutions of the classical dynamical Yang–Baxter equation*, Comm. Math. Phys. **192**, 77–120 (1998).

[38] P. Feng e B. Tsygan, *Hochschild and cyclic homology of quantum groups,* Comm. Math. Phys. **140**, 481–521 (1990).

[39] Feldvöss J., *Existence of triangular Lie bialgebra structures,* Jour. Pure Appl. Alg. **134**, 1–14 (1999).

[40] R.L. Fernandeś, *Connections in Poisson Geometry I: Holonomy and Invariants,* J. of Differential Geometry **54**, (2000) 303-366.

[41] Fernandes R. L., *Lie algebroids, holonomy and characteristic classes,* Adv. Math. **170**, 119–179 (2002).

[42] P. Foth and J.–H. Lu, *A Poisson structure on compact symmetric spaces,* Commun. Math. Phys. **251** (2004), 557-566.

[43] B. Fresse, *Théorie des opérades de Koszul et homologie des algèbres de Poisson,* preprint 1995.

[44] Ginzburg V., *Momentum mappings and Poisson cohomology,* Int. J. Math. **7**, 329–358 (1996).

[45] V. Ginzburg, *Grothendieck groups of Poisson vector bundles*, J. Sympl. Geom. **1**, 121–169 (2002).

[46] Gomez X., *Classification of 3–dimensional Lie bialgebras,* Journ. Math. Phys. **41**, 4939–4956 (2000).

[47] Hawkins E., *Noncommutative rigidity,* Commun. Math. Phys. **246**, 211–235 (2004).

[48] Ibort A. and Martínez Torres D, *A new construction of Poisson manifolds*, Journ of Symp. Geom. **2**, 83–107 (2003).

[49] E. Karolinsky, *The classification of Poisson homogeneous spaces of compact Poisson Lie groups*, Mathematical Physics, Analysis, and Geometry, **3** (1996), 272-289.

[50] S. Khoroshkin, A. Radul, and V. Rubtsov, *A family of Poisson structures on hermitian symmetric spaces*, Comm. Math. Phys. **152** (1993), 299-315.

[51] Koszul J.–L., *Crochét de Schouten–Nijenhuis et cohomologie,* Astèrisque hors série 257–271 (1985).

[52] Liu Z.J. e Qian M., *Generalized Yang-Baxter equations, Koszul operators and Poisson-Lie groups*, J. Diff. Geom. **35**, 399–414 (1992).

[53] Liu Z. J., Weinstein A. and Xu P, *Manin triples for Lie bialgebroids,* J. Diff. Geom. **45**, 547–574 (1997).

[54] J. H. Lu, *Multiplicative and affine Poisson structures on Lie groups*, Ph. D. thesis, Univ. of California, Berkeley, 1990.

[55] Lu J.–H., *Poisson homogeneous spaces and Lie algebroids associated to Poisson actions,* Duke Math. J. **86** (1997), 261–304.

[56] Lu, J.–H. *Coordinates on Schubert cells, Kostant's harmonic forms, and the Bruhat Poisson structure on G/B*, Transform. Groups **4** (1999).

[57] Lu J.–H., *Classical dynamical r–matrices and homogeneous Poisson structures on G/H and K/T*, Comm. Math. Phys. **212**, 337–370 (2002).

[58] Lu J.–H. e Weinstein A., *Poisson-Lie groups, dressing transformations and Bruhat decompositions*, J. Diff. Geom. **31**, 501–526 (1990).

[59] Michaelis W., *Lie coalgebras,* Adv. Math. **38**, 1–54 (1980).

[60] Michaelis W. *A class of infinite–dimensional Lie bialgebras containing the Virasoro algebra,* Adv. Math. **107**, 365–392 (1994).

[61] Montaldi J, Ortega J.–P. and Ratiu T, *The relation between local and global dual pairs* Math. Res. Lett. **11**, 355-363 (2004).

[62] Natsume and Olsen

[63] Papadopoulo G., *Homologies associées aux variétés de Poisson,* Math. Ann. **318**, 397–416 (2000).

[64] G. B. Podkolzin and L. I. Vainerman, *Quantum Stiefel manifolds and double cosets of quantum unitary group,* Pac. J. Math. **138** (1999), 179–199.

[65] D. Roytenberg, *Poisson cohomology of $SU(2)$–covariant "necklace" Poisson structures on* $\mathbb{S}^2$, Journ. Nonlin. Math. Phys. **9**, 347–356 (2002).

[66] Ševera P. and Weinstein A., *Poisson geometry with a 3–form background,* Prog. Theo. Phys. Suppl. **144**, 145–154 (2001).

[67] A. J.–L. Sheu, *Compact quantum groups and groupoid C*-algebras*, J. Func. Anal. **144** (1997), 371-393.

[68] A. J.–L. Sheu, *Groupoid approach to quantum projective spaces*, Contemp. Math. **228** (1998), 341–350.

[69] A. J.–L. Sheu, *Covariant Poisson structures on complex projective spaces,* Comm. Anal. Geom. **10** (2002), 61–78.

[70] J. Stokman, *The quantum orbit method for generalized flag manifolds*, Math. Res. Lett. **10** (2003), 469–481.

[71] B. Tsygan, *Formality conjecture for chains,* in "Differential Topology, infinite–dimensional Lie algebras and applications" 261–274, Amer. Math. Soc. Transl. Ser. 2 **194**, AMS, Providence RI 1999.

[72] Vaisman I., Lectures on the geometry of Poisson manifolds, Progr. Math. **118**, Birkhaüser (1994).

[73] Weinstein A., *The local structure of Poisson manifolds,* J. Diff. Geom. **18**, 523–557 (1983).

[74] Weinstein A., *Symplectic groupoids, geometric quantization and irrational rotation algebras,* in Symplectic geometry, groupoids and integrable systems, Séminaire Sud-Rhodaniene á Berkeley, Springer 1991.

[75] Weinstein A., *The modular automorphism group of a Poisson manifold,* Journ. Geom. Phys. **23**, 379–394 (1997).

[76] Weinstein A. and Xu P., *Extensions of symplectic groupoids and quantization,* Journ. für Reine Angew. Math. **417**, 159–189 (1991).

[77] Van den Bergh, M. *Noncommutative homology of some three-dimensional quantum spaces.* K-Theory **8** (1994), 213–230.

[78] Xu P., *Morita equivalence of Poisson manifolds,* Comm. Math. Phys., **142**, 493–509 (1991).

[79] Xu, Ping, *Gerstenhaber algebras and BV-algebras in Poisson geometry,* Comm. Math. Phys. **200** (1999), 545–560.

[80] Xu P., *Dirac submanifolds and Poisson involutions,* Ann. Sci. École Norm. Sup. **36**, 403–430 (2003).

[81] V. Dolgushev, *A formality theorem for chains,* Adv. Math. at press.

[82] Hawkins E., *The structure of noncommutative deformations,* `math.QA/0504232`.

[83] Kosmann-Schwarzbach Y., *Lie bialgebras, Poisson Lie groups and dressing transformations*, preprint Centre de Mathématiques - Ecole Polytechnique, n. 22/1997.

[DaSo] P. Dazord and D. Sondaz, *Groupes de Poisson affines*, in 'Symplectic Geometry, Groupoids, and Integrable Systems', P. Dazord and A. Weinstein (Eds.), Springer-Verlag, 1991.

[84] Stachura P., *Double Lie algebras and Manin triple,* preprint `math.QA/9912???`.

[85] Zakrzewski S., *Poisson structures on the Poincaré groups,* Comm. Math. Phys,. ??????? Poisson structures on Poincar group. Comm. Math. Phys. 185 (1997), no. 2, 285–311