FOLIATIONS, C^*-ALGEBRAS AND INDEX THEORY

EXAM EXERCICES

Exercise 1. Let $\{A_n, \tau_n, \delta_i^n, \sigma_i^n, i \in \{0, 1, ..., n\}, n \in \mathbb{N}\}$ be a cyclic object in an abelian category. Check that rows of the cyclic bicomplex (*) are complexes and the square with arrows b_2', N_2, b_2, N_1 is commutative.

Exercise 2. Let $M = \mathbb{R}^2$, $E^0 = E^1 = \mathbb{R}^2 \times \mathbb{C}$. Let

$$D = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}.$$

Compute $\ker(D)$, its principal symbol and check if D is an elliptic operator.

Exercise 3. Let $\mathfrak{g}$ be a Lie algebra acting by derivations on an algebra A. Let $C^* = (Hom(\bigwedge_* \mathfrak{g}, A), d)$ be a complex computing Lie algebra cohomology of $\mathfrak{g}$ with values in A. Let E be a right A-module and $E \otimes_A C^*$ be a right graded C^*-module. Suppose an operator ∇ on $E \otimes_A C^*$ of degree one such that for every $\varepsilon \in E \otimes_A C^n$ and $c \in C^*$

$$\nabla(\varepsilon c) = \nabla(\varepsilon)c + (-1)^n \varepsilon dc$$

is given. Show that ∇^2 is C^*-linear and determined by $\nabla^2 : E \to E \otimes_A C^2$, and the latter map is of the form

$$\nabla^2_{X \wedge Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

for all $X, Y \in \mathfrak{g}$.

Exercise 4. Compute the Lie algebra homology with scalar coefficients of the Lie algebra of matrices of the form

$$\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}.$$

Exercise 5. Let A be an algebra acted by a Hopf algebra H with the antipode S and a character δ. Define the convolution product $*$ and prove that for any linear functional φ on A such that for all $h \in H$, $a \in A$

$$\varphi(h(a)) = \delta(h)\varphi(a)$$

the following formula holds

$$\varphi(h(a)b) = \varphi(a(\delta * S)(h)(b)),$$

for all $h \in H$, $a, b \in A$.

Exercise 6. Let X, Y, Z be left invariant vector fields on the Lie group $\mathrm{SL}_2(\mathbb{R})$ such that

$$[X, Y] = Z, \quad [X, Z] = -2X, \quad [Y, Z] = 2Y.$$

Consider the compact quotient manifold $M = \Gamma \backslash \mathrm{SL}_2(\mathbb{R})$ where Γ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. Show that the images of vector fields X, Z under the canonical projection $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{M}$ span a distribution tangent to a foliation of codimension one with a non-zero Godbillon-Vey class.