

ON BOREL MAPPINGS AND σ -IDEALS GENERATED BY CLOSED SETS

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ABSTRACT. We obtain some results about Borel maps with meager fibers on completely metrizable separable spaces. The results are related to a recent dichotomy by Sabok and Zapletal, concerning Borel maps and σ -ideals generated by closed sets. In particular, we give a “classical” proof of this dichotomy.

We shall also show that for certain natural σ -ideals I generated by closed sets in compact metrizable spaces X , every Borel map on a Borel set in X not in I , either has a fiber not in I or else it is injective on a Borel set not in I . This is the case for the σ -ideal generated by finite-dimensional closed sets in the Hilbert cube, which provides an answer to a question asked by M. Elekes.

1. INTRODUCTION

This work is related to a paper by M. Sabok and J. Zapletal [13]. Sabok and Zapletal obtained in this paper a very interesting dichotomy concerning Borel maps and σ -ideals generated by closed sets in complete separable metric spaces, cf. Corollary 3.3. Their proof consists of two basic elements: an “open mapping theorem”, justified by some intricate forcing-related arguments, and a variation of a reasoning of Solecki [14]. We shall give another approach, using some ideas from Zakrzewski [18] and [19], to get a factorization theorem for Borel mappings a bit stronger than the open mapping theorem, which provides in effect a proof of the Sabok-Zapletal dichotomy based on a classical descriptive set theory, as presented in Kechris [3].

The factorization theorem is closely related to a dichotomy, presented in Section 4, concerning the structure of Borel mappings on complete separable metric spaces.

In Section 5 we shall show that certain natural σ -ideals I generated by closed sets in compact metrizable spaces X have the following *1-1 or constant property* property, distinguished by Sabok and Zapletal [13]: for any Borel map $f : B \rightarrow Y$ on a Borel set $B \subseteq X$ not in I , Y being

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completely metrizable separable, either f is injective on a Borel subset not in I or there is a fiber of f not in I .

The σ -ideals we consider include the one generated by closed finite-dimensional sets in the Hilbert cube, which provides in effect an answer to a question of M. Elekes raised during his seminar talk at the University of Warsaw.

We would like to thank Marcin Sabok, whose excellent seminar talks at the University of Warsaw introduced us into the subject, for his valuable comments.

2. TERMINOLOGY AND SOME BACKGROUND

2.1. Terminology and notation. All our spaces are separable metrizable. Our terminology concerning descriptive set theory follows Kechris [3].

Given a separable completely metrizable space E , we denote by $BOR(E)$ and $MGR(E)$ the collections of Borel and meager sets in E , respectively.

By a σ -ideal I on E we understand a collection of subsets of E , closed under countable unions and such that for any $A \in I$, all subsets of A are in I . We say that a σ -ideal I is *generated by closed sets* if there is a family $\mathcal{F} \subseteq I$ consisting of sets closed in E such that each element of I can be covered by countably many elements of \mathcal{F} .

We shall identify $\mathbb{N}^{\mathbb{N}}$ – the countable product of natural numbers, with the space of irrationals and $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$ – the subspace consisting of zero-one sequences, is the Cantor set.

2.2. Boolean algebras $BOR(E)/J$ and Sikorski's theorem. A key element in our proofs of the main results of this paper is the following approach, developed by Zakrzewski in [18] and [19].

Let E and F be completely metrizable separable spaces and assume that F has no isolated points. Let $f : E \rightarrow F$ be a Borel map whose all fibers $f^{-1}(y)$ are meager in E and let us consider the σ -ideal

$$(1) \quad J = \{B \in BOR(F) : f^{-1}(B) \in MGR(E)\}.$$

A minor modification of reasonings in [19] shows that the quotient Boolean algebras $BOR(F)/J$ and $BOR(F)/MGR(F)$ are isomorphic and Sikorski's theorem (see [3, Theorem 15.10]) asserts that this isomorphism is determined by a bijection of F onto itself which preserves Borel sets in both directions (cf. [18], [19]). In effect, we get a Borel isomorphism

$$(2) \quad \varphi : F \rightarrow F, \quad \forall A \in BOR(F) \quad (A \in MGR(F) \Leftrightarrow \varphi(A) \in J).$$

Let us notice that for any Borel set G comeager in E , the set $C = \varphi^{-1}(f(G))$ is comeager in F .

Indeed, for any Borel set A in F disjoint from C , $f^{-1}(\varphi(A))$ is disjoint from G , hence meager in E . Therefore, $\varphi(A) \in J$ and by (2), $A \in$

$MGR(F)$. Now, C being analytic, it is open modulo meager sets, and therefore $F \setminus C \in MGR(F)$.

2.3. Hyperspaces, function spaces and a parametrization lemma.

Given a compact space X , we shall denote by $\mathcal{H}(X)$ the space of all compact subsets of X , equipped with the Vietoris topology (see [3]). The hyperspace $\mathcal{H}(X)$ is compact and metrizable.

If the compact space X is uncountable, $\mathcal{H}(2^{\mathbb{N}}, X)$ is the space of homeomorphic embeddings of the Cantor set into X , endowed with the topology of uniform convergence (which does not depend on a specific metric in X). The completely metrizable separable space $\mathcal{H}(2^{\mathbb{N}}, X)$ is a convenient tool in some parametrization problems, cf. [10], [7] and [8].

The subject of this paper is related to a vast literature on parametrization of measurable multifunctions, cf. [6], [16], [15, Theorem 5.2.8]. However, we did not find in the literature direct references to the results we need.

In particular, the following observation will be useful in Section 4.

Lemma 2.1. *Let $W \subseteq \mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$ be a G_{δ} -set such that each vertical section $W(t)$ is dense in $2^{\mathbb{N}}$.*

Then there is a residual copy of the irrationals Z in $\mathbb{N}^{\mathbb{N}}$ and a homeomorphic embedding $\sigma : Z \times \mathbb{N}^{\mathbb{N}} \rightarrow W$ such that $\sigma(\{z\} \times \mathbb{N}^{\mathbb{N}})$ is a dense subset of $\{z\} \times 2^{\mathbb{N}}$, for each $z \in Z$.

Proof. We shall use the following result due to van Mill [9]: there are compact boundary sets $K_1 \subseteq K_2 \subseteq \dots$ in $2^{\mathbb{N}}$ such that for any sequence $C_1 \subseteq C_2 \subseteq \dots$ of closed boundary sets in $2^{\mathbb{N}}$, there is a homeomorphism $h : 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ such that $C_i \subseteq h(K_{n(i)})$, $i = 1, 2, \dots$

Let $(\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus W = \bigcup_i F_i$, where $F_1 \subseteq F_2 \subseteq \dots$ are sets closed in $\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$. Let us check that the set

$$\mathcal{L} = \{(t, h) \in \mathbb{N}^{\mathbb{N}} \times \mathcal{H}(2^{\mathbb{N}}, 2^{\mathbb{N}}) : \exists \tau \in \mathbb{N}^{\mathbb{N}} F_i(t) \subseteq h(K_{\tau(i)}), i = 1, 2, \dots\}$$

is Borel.

Indeed, for a fixed open base U_1, U_2, \dots in $2^{\mathbb{N}}$, each set

$$\begin{aligned} \mathcal{L}_{i,m} &= \{(t, h) : F_i(t) \subseteq h(K_m)\} \\ &= \bigcap_j \{(t, h) : F_i(t) \cap U_j = \emptyset \text{ or } h(K_m) \cap U_j \neq \emptyset\} \end{aligned}$$

is G_{δ} , and $\mathcal{L} = \bigcap_i \bigcup_m \mathcal{L}_{i,m}$.

The result of van Mill guarantees that each vertical section $\mathcal{L}(t)$ of the set \mathcal{L} is nonempty, and therefore, by the Yankov-von Neumann selection theorem, see [3], there is a continuous map $\lambda : Z \rightarrow \mathcal{H}(2^{\mathbb{N}}, 2^{\mathbb{N}})$ defined on a comeager G_{δ} -set Z in $\mathbb{N}^{\mathbb{N}}$ such that $\lambda(z) \in \mathcal{L}(z)$, for $z \in Z$.

The map $(z, s) \mapsto (z, \lambda(z)(s))$ is a homeomorphic embedding of $Z \times 2^{\mathbb{N}}$ into itself, $P = 2^{\mathbb{N}} \setminus \bigcup_i K_i$ is a copy of the irrationals, and for each $z \in Z$, $\lambda(z)(P)$ is a dense subset of $2^{\mathbb{N}}$ contained in $W(z)$.

It follows that $\sigma = \lambda|Z \times P$ has the required properties.

□

3. A FACTORIZATION OF BOREL MAPS AND THE SABOK-ZAPLETAL DICHOTOMY

The following factorization theorem yields easily an open mapping theorem of Sabok and Zapletal [13, Lemma 7.2].

Theorem 3.1. *Let E, F be completely metrizable separable spaces and let $f : E \rightarrow F$ be a continuous function with meager fibers.*

Then there exists a G_δ -set M comeager in E , an open continuous surjection $h : M \rightarrow \mathbb{N}^{\mathbb{N}}$ and a continuous injection $g : \mathbb{N}^{\mathbb{N}} \rightarrow F$ such that $f \upharpoonright M = g \circ h$.

Proof. Following closely Zakrzewski [18] and [19], we consider the σ -ideal

$$(1) \quad J = \{B \in BOR(F) : f^{-1}(B) \in MGR(E)\}.$$

Note that the function f being continuous and the fibers of f being meager, the space F has no isolated points, hence, as explained in Section 2.2, we get a Borel isomorphism

$$(2) \quad \varphi : F \rightarrow F, \quad \forall A \in BOR(F) \quad (A \in MGR(F) \Leftrightarrow \varphi(A) \in J).$$

Since $\varphi^{-1} \circ f : E \rightarrow F$ is Borel, there is a G_δ -set G in E such that

$$(3) \quad G \text{ is comeager in } E \text{ and } \varphi^{-1} \circ f \upharpoonright G \text{ is continuous.}$$

By (1) and the observation ending Section 2.2, the analytic set

$$(4) \quad C = \varphi^{-1}(f(G)) \text{ is comeager in } F.$$

Let G^* be a compact metrizable extension of G , let $\mathcal{K}(G^*)$ be the space of compact subsets of G^* equipped with the Vietoris topology (cf. Section 2.3), and let $\Phi : C \rightarrow \mathcal{K}(G^*)$ be defined by

$$(5) \quad \Phi(t) = \text{cl}_{G^*}(f^{-1}(\varphi(t)) \cap G),$$

where cl_{G^*} is the closure in the space G^* .

Since Φ is measurable with respect to the σ -algebra generated by analytic sets in F , both maps Φ and φ are continuous on a comeager set in C , and in effect, there is a copy D of the irrationals in C such that

$$(6) \quad D \text{ is comeager in } C, \quad \varphi \upharpoonright D \text{ and } \Phi \upharpoonright D \text{ are continuous.}$$

By (1), (3), (4) and (6), the G_δ -set

$$(7) \quad M = f^{-1}(\varphi(D)) \cap G \text{ is comeager in } E,$$

and the surjection

$$(8) \quad h = \varphi^{-1} \circ f \upharpoonright M : M \rightarrow D \text{ is continuous.}$$

By (8), (7) and (5), for $t \in D$, $h^{-1}(t) = f^{-1}(\varphi(t)) \cap M = f^{-1}(\varphi(t)) \cap G$ is dense in $\Phi(t)$, and since for each W open in G^* , $h(W \cap M) = \{t \in D : h^{-1}(t) \cap W \neq \emptyset\} = \{t \in D : \Phi(t) \cap W \neq \emptyset\}$ is open, the map h is open.

To complete the proof it is enough, by (6) and (8), to declare $g = \varphi|D$. □

The factorization theorem above leads to the following version of the Sabok-Zapletal dichotomy [13, Theorem 1.8], cf. Section 2.1.

Theorem 3.2. *Let X be a completely metrizable separable space. Let I be a σ -ideal on X generated by closed sets and having the following property:*

- (0) *for every G_δ set $U \subseteq X$ not in I and every continuous open function $h : U \rightarrow \mathbb{N}^{\mathbb{N}}$ there is a closed nowhere dense set $L \subseteq h(U)$ such that $h^{-1}(L) \notin I$.*

Then for every continuous function $f : X \rightarrow F$ with all fibers in I , F completely metrizable separable, and each analytic set $A \subseteq X$ not in I , there is a G_δ set $G \subseteq X$ not in I such that $G \subseteq A$ and f is injective on G .

Proof. The proof is a slight variation of a construction from Case 1 of Solecki's proof of [14, Theorem 1].

To begin with, we use Solecki's theorem [14, Theorem 1] to find a G_δ subset H of A not in I . Removing from H all relatively open sets intersecting H in an element of I , one gets a nonempty G_δ set $E \subseteq H$ such that each nonempty, relatively open in E set W is not in I . This implies that all subsets of E that are in I are meager in E . In particular, since all fibers of f are in I , the function $f|E : E \rightarrow Y$ satisfies the assumption of Theorem 3.1. It follows that there is a G_δ -set M comeager in E (hence nonempty relatively open subsets of M are not in I), a continuous open surjection $h : M \rightarrow \mathbb{N}^{\mathbb{N}}$ and a continuous injection $g : \mathbb{N}^{\mathbb{N}} \rightarrow F$ such that $f|M = g \circ h$.

Fix a complete metric on M . The required G_δ set G will be of the form $G = \bigcap_{n \in \mathbb{N}} \bigcup \{U_\tau : \text{lh}\tau = n\}$ for a regular Souslin scheme $\{U_\tau : \tau \in [\mathbb{N}]^{<\mathbb{N}}\}$ satisfying the following properties (cf. [14]):

- (1) U_τ is a nonempty relatively open subset of M ,
- (2) $\text{diam } U_\tau \leq 1/(\text{lh}\tau + 1)$,
- (3) $\tau \subseteq \rho$ and $\tau \neq \rho$ imply $\text{cl}_M(U_\rho) \subseteq U_\tau$,
- (4) $h(U_{\tau*n}) \cap h(U_{\tau*m}) = \emptyset$ if $n \neq m$,
- (5) $\lim_n \text{diam } U_{\tau*n} = 0$,
- (6) $\text{cl}_M(\bigcup_{n \in \mathbb{N}} U_{\tau*n}) \setminus \bigcup_{n \in \mathbb{N}} \text{cl}_M(U_{\tau*n}) \notin I$,
- (7) $\text{cl}_M(\bigcup_{n \in \mathbb{N}} U_{\tau*n}) \setminus \bigcup_{n \in \mathbb{N}} \text{cl}_M(U_{\tau*n}) \subseteq U_\tau$.

Once the construction of the system $\{U_\tau : \tau \in [\mathbb{N}]^{<\mathbb{N}}\}$ is completed, the proof that $G \notin I$ is exactly the same as in [14] (cf. also the proof of Theorem 5.1). Clearly, $G \subseteq A$ and condition (4) guarantees that the restriction $h|G$ and hence also the restriction $f|G$ are 1-1.

Let U_\emptyset be any nonempty open subset of M with $\text{diam } U_\emptyset \leq 1$ and assume that U_τ is already constructed. Note that U_τ is a G_δ subset

of X , not in I , as a nonempty open subset of M and the function $h|_{U_\tau} : U_\tau \rightarrow \mathbb{N}^{\mathbb{N}}$ is open. In particular, $W = h(U_\tau)$ is a nonempty open subset of $\mathbb{N}^{\mathbb{N}}$. By assumption (0), there is a closed nowhere dense set $L \subseteq W$ such that $K = h^{-1}(L) \cap U_\tau$ is not in I . Since h is continuous and open, K is closed and nowhere dense in U_τ .

One can find (cf. [14]) a countable discrete set $D = \{x_n : n \in \mathbb{N}\} \subseteq U_\tau$ together with pairwise disjoint open balls $B_n \subseteq U_\tau$ centered at x_n , respectively, so that $\text{cl}_M(D) = K \cup D$, $D \cap K = \emptyset$ and $h(B_n) \cap h(B_m) = \emptyset$ if $n \neq m$.

Indeed, let $\{a_n : n \in \mathbb{N}\}$ be a dense set in K . We pick the points x_n and the balls B_n inductively, additionally requiring that the distance from a_n to x_n is less than $1/(n+1)$ and $L \cup \bigcup\{\text{cl}_{\mathbb{N}^{\mathbb{N}}}(h(B_i)) : i < n\} \neq W$ for each $n \in \mathbb{N}$. At step n let $R = h^{-1}(\bigcup\{\text{cl}_{\mathbb{N}^{\mathbb{N}}}(h(B_i)) : i < n\})$. Then R is a closed subset of U_τ disjoint from K . Since K is closed and nowhere dense in U_τ , there is a point $x_n \in U_\tau \setminus (K \cup R)$ close enough to a_n . Let $y_n = h(x_n)$. Since $y_n \in W \setminus (L \cup \bigcup\{\text{cl}_{\mathbb{N}^{\mathbb{N}}}(h(B_i)) : i < n\})$, we can find an open neighborhood V of y_n such that $\text{cl}_{\mathbb{N}^{\mathbb{N}}}(V) \subseteq W \setminus (L \cup \bigcup\{\text{cl}_{\mathbb{N}^{\mathbb{N}}}(h(B_i)) : i < n\})$. Finally, let B_n be an open ball centered at x_n such that $h(B_n) \subseteq V$ and let $U_{\tau * n} = B_n$. Choosing the diameters of B_n 's sufficiently small we can arrange that conditions (2), (3) and (5) are satisfied and $\text{cl}_M(\bigcup_{n \in \mathbb{N}} U_{\tau * n}) \setminus \bigcup_{n \in \mathbb{N}} \text{cl}_M(U_{\tau * n}) = K$. □

Terminology in the following corollary is taken from [13], where this important fact was established.

Corollary 3.3 (The Sabok-Zapletal dichotomy). *Let X be a completely metrizable separable space. Let I be a σ -ideal on X generated by closed sets and such that the forcing P_I does not add Cohen reals. Then for every Borel set $B \subseteq X$ not in I and every Borel function f from B into a completely metrizable separable space F with all fibers in I there is a G_δ -set $G \subseteq B$ not in I such that $f|_G$ is 1-1.*

Proof. First, like in the proof of Theorem 3.2, we get a nonempty G_δ -set $E \subseteq B$ such that all subsets of E that belong to I are meager in E . Since f is continuous on a comeager G_δ -set in the space E , we get in effect a G_δ -set $H \subseteq B$ not in I such that $f|_H : H \rightarrow Y$ is continuous. Now the conclusion is an immediate consequence of Theorem 3.2. □

We end this section with yet another consequence of Theorem 3.1.

Remark 3.4. *Let $f : E \rightarrow F$ be as in Theorem 3.1, and let M be a comeager G_δ -set described in the assertion of this theorem. Then there is a continuous map $u : M \rightarrow M$, constant on each fiber of f , which takes open sets in M to relatively open sets in $u(M)$.*

Proof. Indeed, the map $h : M \rightarrow \mathbb{N}^{\mathbb{N}}$ being open and $\mathbb{N}^{\mathbb{N}}$ being zero-dimensional, there is a continuous selection $s : \mathbb{N}^{\mathbb{N}} \rightarrow M$, i.e., $(h \circ s)(t) = t$ for each $t \in M$. We claim that $u = s \circ h$ has the required properties.

To see this, let U be open in M , $a \in U$ and $b = u(a)$. Let V be a neighbourhood of b with $h(V)$ contained in $h(U)$ (recall that h is an open map). Then for any $x \in u(M) \cap V$ there is $y \in U$ with $h(y) = h(x)$ and therefore $u(y) = u(x) = x$. This shows that $u(M) \cap V$ is a neighbourhood of b in $u(M)$ contained in $u(U)$. \square

4. A DICHOTOMY CONCERNING BOREL MAPS

The following result is closely related to Theorem 3.1.

Theorem 4.1. *Let E, F be completely metrizable separable spaces and let $f : E \rightarrow F$ be a Borel map with meager fibers. Then one of the following two mutually exclusive possibilities holds true:*

- (A) *there is a non-meager G_δ set in E on which f is injective;*
- (B) *there is a homeomorphic embedding $u : \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \rightarrow E$ onto a comeager set in E and a continuous injection $v : \mathbb{N}^\mathbb{N} \rightarrow F$ such that $f(u(t, s)) = v(t)$ for every $t, s \in \mathbb{N}^\mathbb{N}$.*

Proof. As in Theorem 3.2, a key element of our proof will be an approach from [18] and [19] (cf. Section 2.2).

With no loss of generality we can assume that the space F has no isolated points (otherwise we replace F with the set P of condensation points of F and then replace E with its dense G_δ -subset contained in $f^{-1}(P)$).

As in the proof of Theorem 3.1, let J be the σ -ideal

$$(1) \quad J = \{B \in BOR(F) : f^{-1}(B) \in MGR(E)\},$$

$\varphi : F \rightarrow F$ a Borel isomorphism such that

$$(2) \quad \forall A \in BOR(F) \quad (A \in MGR(F) \Leftrightarrow \varphi(A) \in J),$$

cf. Section 2.2, and let G be a comeager copy of the irrationals in E such that both maps

$$(3) \quad f|G : G \rightarrow F \quad \text{and} \quad \varphi^{-1} \circ f|G : G \rightarrow F \quad \text{are continuous.}$$

We shall consider

$$(4) \quad H = \{x \in G : x \text{ is a point of condensation of } f^{-1}(f(x)) \cap G\}.$$

The set H is the projection of the set (cf. Section 2.3)

$$\{(x, K) \in E \times \mathcal{K}(E) : K \text{ is a Cantor set, } x \in K \text{ and } |f(K)| = 1\},$$

Borel in the product of E and the hyperspace $\mathcal{K}(E)$, and hence

$$(5) \quad H \text{ is analytic and } f^{-1}(f(x)) \cap H \text{ is dense in itself for } x \in H.$$

Let us notice also that

$$(6) \quad f|(G \setminus H) \text{ is countable-to-one.}$$

If $G \setminus H$ is non-meager, being coanalytic it contains a non-meager Borel set, and in effect, by (6) and a theorem of Lusin (see [3]), f is injective on a non-meager G_δ -set in E , contained in $G \setminus H$. Thus we have arrived at case (A) of the assertion.

Let us assume now that

(7) H is comeager in E .

Then, by (1), (2) and (5), (7), the set

(8) $C = \varphi^{-1}(f(H))$ is analytic and comeager in F ,

cf. a remark after (2) in Section 2.2.

Let G^* be a zero-dimensional compactification of G and let $\Phi : C \rightarrow \mathcal{H}(G^*)$, cf. Section 2.3, be defined by, cf. the proof of Theorem 3.1, (5),

(9) $\Phi(t) = \text{cl}_{G^*}(f^{-1}(\varphi(t)) \cap H)$.

The map Φ being measurable with respect to the σ -algebra generated by analytic sets in C , there is a copy D of the irrationals such that, cf. (2) and (9),

(10) $D \subseteq C$ is comeager in F and $\varphi|D, \Phi|D$ are continuous.

Let us check that, cf. (9), the set, cf. Section 2.3,

(11) $\mathcal{A} = \{(t, h) \in D \times \mathcal{H}(2^\mathbb{N}, G^*) : h(2^\mathbb{N}) = \Phi(t)\}$

is Borel.

Indeed, let V_1, V_2, \dots be an open base in G^* and let us consider the following open sets in the product $D \times \mathcal{H}(2^\mathbb{N}, G^*)$:

$\mathcal{B}_j = \{(t, h) : h(2^\mathbb{N}) \cap V_j \neq \emptyset\}$ and $\mathcal{C}_i = \{(t, h) : \Phi(t) \cap V_i \neq \emptyset\}$.

Then

$$\mathcal{A} = (D \times \mathcal{H}(2^\mathbb{N}, G^*)) \setminus \bigcup_i [(\mathcal{B}_i \setminus \mathcal{C}_i) \cup (\mathcal{C}_i \setminus \mathcal{B}_i)].$$

Moreover, by (5) and (9), each $\Phi(t)$ is a Cantor set, and therefore, for each vertical section of \mathcal{A} , cf. (11),

(12) $\mathcal{A}(t) \neq \emptyset, t \in D$.

The Yankov-von Neumann theorem (see [3]) provides a copy T of the irrationals in D and a continuous mapping $\chi : T \rightarrow \mathcal{H}(2^\mathbb{N}, G^*)$ such that, cf. (10), (12),

(13) T is comeager in F and $\chi(t) \in \mathcal{A}(t)$, for $t \in T$.

The map

(14) $k : T \times 2^\mathbb{N} \rightarrow G^*, k(t, s) = \chi(t)(s)$ is continuous.

We have that

(15) $W = k^{-1}(G)$ is G_δ in $T \times 2^\mathbb{N}$ and $k|W : W \rightarrow G$ is injective,

(16) $f \circ (k|W) = \varphi \circ \text{proj}|W$, where $\text{proj}(t, s) = t$.

Moreover, for each $t \in T$,

$$(17) \quad \chi(t)^{-1}(f^{-1}(\varphi(t)) \cap H) \subseteq W(t),$$

cf. (11), (13), (9), (15), and hence

$$(18) \quad W(t) \text{ is dense in } 2^{\mathbb{N}}, \text{ for } t \in T.$$

We shall check that

$$(19) \quad k(W) \text{ is comeager in } E,$$

$$(20) \quad k|W : W \rightarrow k(W) \text{ is a homeomorphism.}$$

By (17), $k(W) \supseteq H \setminus f^{-1}(\varphi(F \setminus T))$ and (19) follows from (7), (8), (10) and (2).

Since $k|W$ is continuous and injective, cf. (15), to get (20) we have to verify that $k(t_n, s_n) \rightarrow k(t_0, s_0)$ implies $(t_n, s_n) \rightarrow (t_0, s_0)$, whenever $(t_n, s_n) \in W$.

By (3) and (16),

$$t_n = \varphi^{-1} \circ f(k(t_n, s_n)) \rightarrow \varphi^{-1} \circ f(k(t_0, s_0)) = t_0$$

and hence, $\chi(t_n) \rightarrow \chi(t_0)$, by the continuity of χ . By (14), $\chi(t_n)(s_n) \rightarrow \chi(t_0)(s_0)$ and, $\chi(t_0)$ being a homeomorphism, $s_n \rightarrow s_0$, which completes a justification of (20).

Now, to end the proof, it is enough to take $\sigma : Z \times \mathbb{N}^{\mathbb{N}} \rightarrow W$ from Lemma 2.1, cf. (18) where we identify T with $\mathbb{N}^{\mathbb{N}}$, and to declare $u = k \circ \sigma$ and $v = \varphi|Z$, cf. (16).

Indeed, $\sigma(Z \times \mathbb{N}^{\mathbb{N}})$ is comeager in W , cf. Lemma 2.1, hence by (20), $u(Z \times \mathbb{N}^{\mathbb{N}}) = k(\sigma(Z \times \mathbb{N}^{\mathbb{N}}))$ is comeager in $k(W)$ and, by (19), $u(Z \times \mathbb{N}^{\mathbb{N}})$ is comeager in E .

□

5. CERTAIN σ -IDEALS GENERATED BY CLOSED SETS IN COMPACT SPACES

Following Sabok and Zapletal [13], we say that a σ -ideal I on a completely metrizable separable space X has the *1-1 or constant property* if it satisfies the conclusion of the Sabok-Zapletal dichotomy (see 3.3), i.e., if for every Borel set $B \subseteq X$ not in I and every Borel function f from B into a completely metrizable separable space Y with all fibers in I , there is a G_δ -set $G \subseteq B$ not in I such that $f|G$ is 1-1.

We shall describe a property of σ -ideals I on compact metrizable spaces which guarantees that I has the 1-1 or constant property (Theorem 5.1).

In particular, for such σ -ideals I , the forcing P_I introduced in the Sabok-Zapletal dichotomy (Corollary 3.3) does not add Cohen reals. Indeed, while the implication “if P_I does not add Cohen reals, then I has the 1-1 or constant property” is essentially the content of the Sabok-Zapletal dichotomy (see Theorems 3.2 and 3.3), the converse is much easier to establish. For the sake of the reader’s convenience we

shall present a short proof of this (belonging to folklore) fact in Remark 5.7 at the end of this section.

Adapting for our needs the standard terminology concerning σ -ideals of closed sets (cf. [5]) we say that a σ -ideal I of subsets of a compact metric space X generated by closed sets

- is *calibrated* if for every $F \in \mathcal{K}(X)$ (see Section 2.3) not in I and countably many compact sets $K_i \in I$, $i \in \mathbb{N}$, there is $K \in \mathcal{K}(X)$ not in I such that $K \subseteq F \setminus \bigcup_{i \in \mathbb{N}} K_i$,
- has the *covering property* if every analytic set (G_δ suffices, by Solecki's theorem ([14, Theorem 1])) not in I contains a compact set not in I .

Clearly, if a σ -ideal I has the covering property then it is calibrated. Numerous examples of σ -ideals with these properties can be found in [5].

We say that a σ -ideal I of subsets of a compact metric space X has a *coanalytic stratified calibration* if it is generated by a family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ for certain families \mathcal{F}_n satisfying the following properties for each $n \in \mathbb{N}$:

- (A) \mathcal{F}_n is a coanalytic subspace of $\mathcal{K}(X)$,
- (B) \mathcal{F}_n is hereditary (i.e., whenever A is a subset of B and B is in \mathcal{F}_n , the closure of A is in \mathcal{F}_n),
- (C) $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$,
- (D) for every $m \in \mathbb{N}$, $F \in \mathcal{K}(X)$ not in I and countably many sets $K_i \in \mathcal{F}_m$, $i \in \mathbb{N}$, there is $K \in \mathcal{K}(X) \setminus \mathcal{F}_m$ such that $K \subseteq F \setminus \bigcup_{i \in \mathbb{N}} K_i$.

Note that if I is calibrated and $I \cap \mathcal{K}(X)$ is coanalytic, then letting $\mathcal{F}_n = I \cap \mathcal{K}(X)$ for each $n \in \mathbb{N}$, one sees that I has a coanalytic stratified calibration.

A σ -ideal which has a coanalytic stratified calibration but is *not* calibrated is presented in Remark 5.5.

Theorem 5.1. *If a σ -ideal I of subsets of a compact metric space X has a coanalytic stratified calibration then it has also the 1-1 or constant property.*

Proof. Let $B \subseteq X$ be a Borel set not in I and assume that $f : B \rightarrow Y$ is a Borel function from B into a completely metrizable separable space Y with all fibers in I .

Our first objective is to show that there is a non-empty G_δ -set $M \subseteq B$ such that $f|_M : M \rightarrow Y$ is continuous and if we let $h = f|_M$, then:

- (1) every nonempty relatively open subset of M is not in I ,
- (2) $\exists n \in \mathbb{N} \forall y \in Y \text{ cl}_X(h^{-1}(y)) \in \mathcal{F}_n$,
- (3) $x \mapsto \text{cl}_X(f^{-1}(f(x)))$ is a continuous mapping from M to $\mathcal{K}(X)$.

Applying a theorem of Burgess and Hillard (see [3, Theorem 35.43]) to the (hereditary and coanalytic) family \mathcal{F} , we can assume without loss of generality that:

$$(4) \quad \forall y \in Y \exists n \in \mathbb{N} \text{cl}_X(f^{-1}(y)) \in \mathcal{F}_n.$$

Moreover (see the proof of Theorem 3.2) we shall assume that no nonempty relatively open subset of B belongs to I .

Let $\Psi : B \rightarrow \mathcal{K}(X)$ be defined by

$$(5) \quad \Psi(x) = \text{cl}_X(f^{-1}(f(x))).$$

The function Ψ is measurable with respect to the σ -algebra of X generated by analytic sets so replacing B , if necessary, by its dense G_δ -subset, we can assume that Ψ is continuous on B .

Since for each n , the family \mathcal{F}_n is coanalytic in $\mathcal{K}(X)$, the set $B_n = \Psi^{-1}(\mathcal{F}_n)$ is coanalytic in X . Note that by (4) and (B), $X = \bigcup_{n \in \mathbb{N}} B_n$, so we can fix n such that B_n is non-meager in X . Let $E \subseteq B_n$ be a G_δ -set, non-meager in X (so E is not in I).

Finally, following the proof of Corollary 3.3, we get a G_δ -set $M \subseteq E$ not in I such that $h = f|_M : M \rightarrow Y$ is continuous and no nonempty relatively open subset of M belongs to I .

Let us fix a complete metric ρ on M and a metric d on X . Subscripts will indicate that the metric notion under consideration is related to the corresponding metric (e.g., $\text{diam}_d(A)$ or $\text{dist}_d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$ for $A, B \subseteq X$).

Having defined M and h satisfying (1)–(3), we shall modify the inductive construction used in the proof of Theorem 3.2.

More precisely, we shall construct a Souslin scheme $\{U_\tau : \tau \in [\mathbb{N}]^{<\mathbb{N}}\}$ satisfying the following conditions (cf. the proof of Theorem 3.2):

- (1') U_τ is a nonempty relatively open subset of M ,
- (2') $\text{diam}_\rho U_\tau \leq 1/(\text{lh}\tau + 1)$,
- (3') $\tau \subseteq \rho$ and $\tau \neq \rho$ imply $\text{cl}_M(U_\rho) \subseteq U_\tau$,
- (4') $h(U_{\tau*n}) \cap h(U_{\tau*m}) = \emptyset$ if $n \neq m$,
- (5') $\lim_n \text{diam}_d U_{\tau*n} = 0$,
- (6') $\text{cl}_X(\bigcup_{n \in \mathbb{N}} U_{\tau*n}) \setminus \bigcup_{n \in \mathbb{N}} \text{cl}_X(U_{\tau*n}) \notin \mathcal{F}_{\text{lh}\tau}$.
- (7') $\text{cl}_X(\bigcup_{n \in \mathbb{N}} U_{\tau*n}) \setminus \bigcup_{n \in \mathbb{N}} \text{cl}_X(U_{\tau*n}) \subseteq \text{cl}_X(U_\tau)$.

Once the construction of the system $\{U_\tau : \tau \in [\mathbb{N}]^{<\mathbb{N}}\}$ is completed, we let $G = \bigcap_{n \in \mathbb{N}} \bigcup \{U_\tau : \text{lh}\tau = n\}$. Clearly, the construction guarantees that $h|_G = f|_G$ is 1-1. Moreover, the proof that $G \notin I$ from [14] requires only a minor modification.

Indeed, if $G \subseteq \bigcup_{i \in \mathbb{N}} F_i$, $F_i \in \mathcal{F}_{n(i)}$, then the Baire category theorem provides an open set W in X such that $W \cap G \neq \emptyset$ and $\text{cl}_X(W \cap G) \subseteq F_n$ for some n . But then, by (2'), W contains U_τ for some $\tau \in [\mathbb{N}]^{<\mathbb{N}}$ with $\text{lh}\tau > n$. Moreover, by (3'), (5'), (7') and the fact that $U_{\tau*n} \cap G \neq \emptyset$, for every $n \in \mathbb{N}$ we have:

$$\text{cl}_X\left(\bigcup_{n \in \mathbb{N}} U_{\tau * n}\right) \setminus \bigcup_{n \in \mathbb{N}} \text{cl}_X(U_{\tau * n}) \subseteq \text{cl}_X(U_\tau \cap G) \subseteq \text{cl}_X(W \cap G).$$

This however, taking into account (6'), contradicts the choice of τ .

So assume that U_τ is already defined. Let $F = \text{cl}_X(U_\tau)$ and $m = \text{lh}(\tau)$. Note that, by (1), $F \notin \mathcal{K}(X) \setminus I$.

Fix $C \subseteq U_\tau$ such that C is countable and dense in U_τ . Then, by (D) and (2), there is $K \in \mathcal{K}(X) \setminus \mathcal{F}_m$ such that $K \subseteq F \setminus \bigcup_{x \in C} \Psi(x)$.

We shall find (cf. the proof of Theorem 3.2) a countable set $D = \{x_n : n \in \mathbb{N}\} \subseteq C$ together with pairwise disjoint open (in M) neighbourhoods $B_n \subseteq U_\tau$ of x_n , respectively, so that:

$$\begin{aligned} (6) \quad & \text{cl}_X(D) = K \cup D \text{ and } D \cap K = \emptyset, \\ (7) \quad & h(B_n) \cap h(B_m) = \emptyset \text{ if } n \neq m. \end{aligned}$$

To that end let $\{a_n : n \in \mathbb{N}\}$ be a dense set in K .

We shall pick inductively points $x_i \in C$ and open (in M) neighbourhoods B_i, U_i of x_i , demanding that for every i and j :

$$\begin{aligned} (8) \quad & B_i \subseteq U_i \text{ and } \text{cl}_Y(h(B_i)) \subseteq h(U_i), \\ (9) \quad & h(B_i) \cap h(B_j) = \emptyset \text{ if } i \neq j, \\ (10) \quad & d(a_i, x_i) < 1/(i+1), \\ (11) \quad & \text{dist}_d(K, \bigcup \Psi(U_i)) > 0. \end{aligned}$$

At step n find $m \geq n+1$ such that $1/m < \min_{i < n} \text{dist}_d(K, \bigcup \Psi(U_i))$ (cf (11)). Pick $c \in C$ such that $d(a_n, c) < 1/m$ and let $x_n = c$ so that (10) is satisfied. Then, by the choice of K , $\Psi(c) \cap K = \emptyset$, so $\text{dist}_d(K, \Psi(x_n)) > 0$.

Let $\delta = \text{dist}_d(K, \Psi(x_n))$ and let d_H denote the Hausdorff metric in $\mathcal{K}(X)$ corresponding to d .

By the continuity of Ψ at x_n , there exists an open subset $U_n \subseteq U_\tau$ such that $x_n \in U_n$ and $d_H(\Psi(x_n), \Psi(x)) < \delta/2$ for every $x \in U_n$. It follows that $\text{dist}_d(K, \bigcup \Psi(U_n)) \geq \delta/2$, establishing (11).

Then, $a_n \in K$ and $d(a_n, x_n) < \min_{i < n} \text{dist}_d(K, \bigcup \Psi(U_i))$, imply that $x_n \notin \bigcup_{i < n} \bigcup \Psi(U_i)$ and consequently $h(x_n) \notin \bigcup_{i < n} h(U_i)$.

Hence, by (8), $h(x_n) \notin \bigcup_{i < n} \text{cl}_Y(h(B_i))$. So there is an open set $V \subseteq Y$ such that $h(x_n) \in V$ and $V \cap \bigcup_{i < n} \text{cl}_Y(h(B_i)) = \emptyset$. Finally, we choose an open subset $B_n \subseteq U_n$ such that $x_n \in B_n$, and $\text{cl}_Y(h(B_n)) \subseteq V$, thus establishing (8) and (9).

Having defined x_n, B_n and U_n satisfying (8) – (11), we let $D = \{x_n : n \in \mathbb{N}\}$ and $U_{\tau * n} = B_n$, for $n \in \mathbb{N}$. More precisely, we shrink each $U_{\tau * n}$ further, if necessary, so that all required conditions are satisfied. In particular, (6) follows from (10) and, by (6) and (5'), $K = \text{cl}_X(\bigcup_{n \in \mathbb{N}} U_{\tau * n}) \setminus \bigcup_{n \in \mathbb{N}} \text{cl}_X(U_{\tau * n})$, giving (6').

□

The following immediate corollary to Theorem 5.1 implies that many important σ -ideals discussed in the literature have the 1-1 or constant property.

Corollary 5.2. *Let X be a compact metric space and let I be a σ -ideal on X generated by compact sets.*

If I is calibrated and $I \cap \mathcal{K}(X)$ is coanalytic then I has the 1-1 or constant property.

Remark 5.3. *Examples of σ -ideals satisfying the assumptions of Corollary 5.2 and hence having the 1-1 or constant property include (see [5]):*

- *the σ -ideal generated by closed sets of uniqueness in the group \mathbb{T} of unit complex numbers,*
- *the σ -ideal generated by closed sets of extended uniqueness in \mathbb{T} ,*
- *the σ -ideal generated by closed null-sets for a subadditive capacity on a compact, metric space (in particular: the σ -ideal generated by closed null-sets of the interval $[0,1]$ or the Cantor group $2^{\mathbb{N}}$),*
- *the σ -ideal generated by closed porous sets in a compact metric space,*
- *the σ -ideal generated by closed E -smooth subsets of a compact metric space X for a Borel non-smooth equivalence relation E on X .*

Some of the σ -ideals listed above have, moreover, the covering property, some of them not (see [5]). Let us note, however, that in Corollary 5.2 the assumption that I is coanalytic is needed only to justify the use of the Burgess-Hillard theorem (see the proof of Theorem 5.1) and in fact is not needed when the function f is continuous on a compact set not in I . In particular, the definability assumption can be dropped in the case of σ -ideals with the covering property which yields the following corollary. The second part of the assertion of this corollary follows readily from the Sabok-Zapletal dichotomy, by applying in a standard way to P_I forcing arguments related to not adding Cohen reals (see [17, Theorem 3.3.2]). However, arguments based on the proof of Theorem 5.1 provide a justification avoiding forcing methods.

Corollary 5.4. *Let X be a compact metric space and let I be a σ -ideal on X generated by compact sets.*

- (1) *If I is calibrated then for every compact set $P \subseteq X$ not in I and every continuous function f from P into a completely metrizable separable space Y with all fibers in I there is a G_δ -set $G \subseteq P$ not in I such that $f|G$ is 1-1.*
- (2) *If I has the covering property then I has the 1-1 or constant property.*

Proof. To prove (1), one can follow closely the proof of Theorem 5.1, letting $B = P$ and $\mathcal{F}_n = I \cap \mathcal{K}(X)$ for each $n \in \mathbb{N}$. Note that under the present assumptions condition (2) from that proof is automatically fulfilled and we do not need additional definability assumptions about I .

To prove (2), note that given a Borel set $B \subseteq X$ not in I and a Borel function $f : B \rightarrow Y$ from B into a completely metrizable separable space Y with all fibers in I , the covering property provides readily a compact set $P \subseteq B$ not in I such that the function $f|_P$ is continuous. Then the conclusion follows from part (1). □

Remark 5.5.

Our terminology concerning dimension theory follows [1].

(A) Given a compact metrizable space X , we denote by $\mathcal{F}_n(dim)$ the collection of closed at most n -dimensional subsets of X and let $I(dim)$ be the σ -ideal of subsets of X that can be covered by countably many elements of $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n(dim)$.

Each $\mathcal{F}_n(dim)$ is a G_δ -set in the hyperspace $\mathcal{K}(X)$ (see [4, §45, IV, Theorem 4]).

Moreover, if $F \in \mathcal{K}(X) \setminus I(dim)$ and $K_i \in \mathcal{F}_n(X)$, $i \in \mathbb{N}$, we have $dim(\bigcup_{i \in \mathbb{N}} K_i) \leq n$ by the sum theorem (see [1, 1.5.3]) and by the enlargement theorem (see [1, 1.5.11]), $\bigcup_{i \in \mathbb{N}} K_i$ is contained in a G_δ -set G in X with $dim(G) \leq n$. By the addition theorem (see [1, 1.5.10]), $dim(F \setminus G) = \infty$, and $F \setminus G$ being σ -compact, again using the sum theorem we infer that for each m there is a compact set $K \subseteq F \setminus G$ with $dim(K) \geq m$.

Therefore, the σ -ideal $I(dim)$ has a coanalytic stratified calibration.

However, the σ -ideal $I(dim)$ in the Hilbert cube $[0, 1]^\mathbb{N}$ is not calibrated. Indeed, there is a compact set $F \subseteq [0, 1]^\mathbb{N}$ not in $I(dim)$ containing a zero-dimensional G_δ -set H such that $F \setminus H \in I(dim)$ (see [1, Example 5.1.7]). Clearly, the calibration property fails for F .

(B) Let $dim_{\mathbb{Z}}$ be the cohomological dimension with respect to the ring of integers (see [1, page 75]) and let us consider, replacing in (A) the covering dimension dim by the cohomological dimension $dim_{\mathbb{Z}}$, the σ -ideal $I(dim_{\mathbb{Z}})$ associated with the cohomological dimension.

Then $I(dim_{\mathbb{Z}})$ has a coanalytic stratified calibration. A verification runs analogously to that in (A), but instead of using classical results, we have to appeal to a theorem of Dobrowolski and Rubin [2] that $\mathcal{F}_n(dim_{\mathbb{Z}})$ is a G_δ -set in the hyperspace, and to counterparts for $dim_{\mathbb{Z}}$ of the enlargement and addition theorems, established respectively by Rubin and Schapiro [12] and Rubin [11].

Theorem 5.1 combined with Remark 5.5 gives the following corollary which provides an answer to a question asked by Elekes during his seminar talk at the University of Warsaw in 2009.

Corollary 5.6. *Let X be a compact metrizable space and let I be the σ -ideal of subsets of X that can be covered by countably many finite-dimensional compact sets in X . Then I has the 1-1 or constant property.*

We shall end this section with a brief remark clarifying some relations between the 1-1 or constant property of I and certain forcing properties of P_I . As was already mentioned, these relations are well-known to experts on this topic.

Remark 5.7. *Let X be a completely metrizable separable space. Let I be a σ -ideal on X generated by closed sets.*

If there is a Borel set $B \subseteq X$ not in I and a Borel function g from B into a completely metrizable separable space Y without isolated points such that

- (A) *the inverse image under g of every Borel subset $C \subseteq Y$ meager in Y is in I .*

then there is a Borel function $f : B \rightarrow \mathbb{N}^{\mathbb{N}}$ such that

- (B) *all fibers of f are in I and there is no Borel set $A \subseteq B$ not in I such that $f|_A$ is 1-1.*

Proof. Let $h : Y \rightarrow \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ be a Borel isomorphism between Y and $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that

$$\forall A \in \text{BOR}(Y) \ (A \in \text{MGR}(Y) \Leftrightarrow h(A) \in \text{MGR}(\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}})).$$

Let $f = \text{proj} \circ h \circ g$, where proj is the projection onto the first coordinate. Then, by (A), all fibers of f are in I .

Assume now that A is a Borel subset of B and $f|_A$ is 1-1. Then $\text{proj}|_h(g(A))$ is 1-1 so the set $h(g(A))$, being analytic, is meager in $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, by the Kuratowski-Ulam theorem. It follows, by (A), that $A \in I$. □

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