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ON BOREL SETS BELONGING TO EVERY INVARIANT CCC σ-IDEAL ON $2^{\mathbb{N}}$

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ABSTRACT. Let I_{ccc} be the σ-ideal of subsets of the Cantor group $2^{\mathbb{N}}$ generated by Borel sets which belong to every translation invariant σ-ideal on $2^{\mathbb{N}}$ satisfying the countable chain condition (ccc). We prove that I_{ccc} strongly violates ccc. This generalizes a theorem of Balcerzak-Rosłanowski-Shelah stating the same for the σ-ideal on $2^{\mathbb{N}}$ generated by Borel sets $B \subseteq 2^{\mathbb{N}}$ which have perfectly many pairwise disjoint translates. We show that the last condition does not follow from $B \in I_{ccc}$ even if B is assumed to be compact. Various other conditions which for a Borel B imply that $B \in I_{ccc}$ are also studied. As a consequence we prove in particular that:

- If A_n are Borel sets, $n \in \mathbb{N}$, and $2^{\mathbb{N}} = \bigcup_n A_n$, then there is n_0 such that every perfect set $P \subseteq 2^{\mathbb{N}}$ has a perfect subset Q a translate of which is contained in A_{n_0}.
- CH is equivalent to the statement that $2^{\mathbb{N}}$ can be partitioned into $\aleph_1$ many disjoint translates of a closed set.

1. INTRODUCTION

A *σ-ideal* on an uncountable Polish space X is a family $I \subseteq \mathcal{P}(X)$ which is closed under taking subsets and countable unions. Throughout the paper we assume that I is *proper*, i.e., $X \notin I$, contains all singletons and every set from I is covered by a Borel set from I. We say that a σ-ideal I on X is *ccc* if there is no uncountable family of disjoint Borel subsets of X outside I.

Our starting point is a question stated in [2] by Balcerzak, Rosłanowski and Shelah. Looking at σ-ideals that are *not* ccc they asked what the reasons for the failure of ccc there could be. They considered properties (M) and (D), introduced and investigated earlier by Balcerzak in [1].

We say that a σ-ideal I in a Polish space X has *property (M)* if there is a Borel function $f : X \to 2^{\mathbb{N}}$ such that all fibers of f are not in I.

If $(G, +)$ is a Polish abelian group, following [22] we denote by $\mathcal{F}_0(G)$ the family of all Borel sets $B \subseteq G$ such that there exists a perfect set $P \subseteq G$ with $(B + x) \cap (B + y) = \emptyset$ for $x, y \in P$ and $x \neq y$. We say that an invariant σ-ideal I on G has property (D) if there is a Borel set $B \in \mathcal{F}_0(G) \setminus I$.

Balcerzak (see [1]) observed that (D) implies (M) for every invariant σ-ideal I on G and posed the problem whether the converse holds true. Bukovsky reformulated the Balcerzak's question about (M) $\Rightarrow$ (D) by considering the σ-ideal I_0 on $2^{\mathbb{N}}$

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(where $2^{\mathbb{N}}$ is considered with the coordinatewise addition modulo 2 and referred to as the *Cantor group*) generated by $\mathcal{F}_0(2^{\mathbb{N}})$ and asking if I_0 has property (M). The positive answer to the latter is the content of [2, Theorem 2.1].

A σ-ideal I on a Polish abelian group G is *translation invariant* (shortly: *invariant*), if

$$\forall x \in G\ \forall A \subseteq G\ (A \in I\ \Rightarrow\ x + A \in I).$$

The idea behind (D) is to single out a particular property (namely, $\mathcal{F}_0$) of a Borel subset of, say, $2^{\mathbb{N}}$ that prevents it from being a member of *any* invariant ccc σ-ideal and then to ask whether a failure of ccc of an invariant σ-ideal I (even in the strong form of (M)) is always witnessed by an I-positive Borel set with the property under consideration. Balcerzak, Rosłanowski and Shelah gave the negative answer in the case of property $\mathcal{F}_0$. Generalizing this result we prove that the idea indicated above cannot be accomplished by *any* property. Namely, in Section 2 we show (see Theorem 2.1) that the σ-ideal I_{ccc} generated by the Borel subsets of $2^{\mathbb{N}}$ which belong to every invariant ccc σ-ideal on $2^{\mathbb{N}}$ has property (M). This is done by adapting the main idea of a simplified proof of the Balcerzak-Rosłanowski-Shelah theorem, due to Solecki [22]. The point is that it is actually easier to prove that I_{ccc} has (M) than it is to prove that so has I_0. In effect, our argument provides a still simpler proof of the latter as well.

In Section 3 we consider the question which Borel sets belong to I_{ccc}. In particular, we give an example (see Theorem 3.8) of a compact set $C \in I_{ccc} \setminus \mathcal{F}_0(2^{\mathbb{N}})$ (in fact no two different translates of C are disjoint). An example of a compact subset of $\mathbb{R}$ which is not in $\mathcal{F}_0(\mathbb{R})$ but belongs to every invariant ccc σ-ideal on $\mathbb{R}$ is the standard middle 1/3 Cantor set (see Remark 3.10).

Finally, in Section 4 we indicate some links between the subject of this paper and that of covering $2^{\mathbb{N}}$ (or, more generally, an uncountable Polish abelian group) by less than $\mathfrak{c}$ many translates of its closed nowhere dense subset. The latter has been recently dealt with by many authors (see [6], [7], [8], [9], [19], [15]). We prove (see Theorem 4.3) that CH is equivalent to the statement that $2^{\mathbb{N}}$ can be partitioned into $\aleph_1$ many disjoint translates of a closed set.

2. Property (M)

Let I_{ccc} be the σ-ideal on $2^{\mathbb{N}}$ generated by Borel sets which belong to every invariant ccc σ-ideal on $2^{\mathbb{N}}$.

Theorem 2.1. *I_{ccc} has property (M).*

Proof. Following Solecki [22], for each $n \in \mathbb{N}$ we fix a partition of $2^{\mathbb{N}}$ into an F_σ-set A_n^0 of measure 1 and a dense G_δ-set A_n^1. Next we define $f : (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $f((x_n))(i) = 0$ if $x_i \in A_i^0$ and $f((x_n))(i) = 1$ if $x_i \in A_i^1$. Using the canonical topological group isomorphism we view f as a function with domain $2^{\mathbb{N}}$. Our goal is to show that the fibers of f are not only outside I_0, as shown in [22], but actually even outside I_{ccc}.

To that end, for each $y \in 2^{\mathbb{N}}$ we shall find an invariant ccc σ-ideal I_y on $2^{\mathbb{N}}$ such that $f^{-1}(y) \notin I_y$.

So fix $y \in 2^{\mathbb{N}}$. Let $Y_0 = \{n \in \mathbb{N} : y(n) = 0\}$ and $Y_1 = \{n \in \mathbb{N} : y(n) = 1\}$.

First assume that $Y_0 \neq \emptyset$ and $Y_1 \neq \emptyset$. Using the canonical topological group isomorphism identify $2^{\mathbb{N}}$ with the product group $G = G_0 \times G_1$ where $G_0 = (2^{\mathbb{N}})^{Y_0}$ and $G_1 = (2^{\mathbb{N}})^{Y_1}$. Then $f^{-1}(y) = \prod_{n\in Y_0} A_n^0 \times \prod_{n \in Y_1} A_n^1$ and it suffices to find a

G-invariant ccc σ-ideal I_y on G such that $f^{-1}(y) \notin I_y$. Let $\mathcal{N}(G_0)$ be the σ-ideal of null sets in G_0 (with respect to the product of ordinary measures on $2^{\mathbb{N}}$) and let $\mathscr{M}(G_1)$ be the σ-ideal of meager subsets of G_1. Note that $\prod_{n\in Y_0} A^0_n \notin \mathcal{N}(G_0)$ (in fact it is of measure 1) and $\prod_{n\in Y_1} A^1_n \notin \mathscr{M}(G_1)$ (in fact it is comeager). It follows that $f^{-1}(y) \notin \mathcal{N}(G_0) \otimes \mathscr{M}(G_1)$, where $\mathcal{N}(G_0) \otimes \mathscr{M}(G_1)$, the Fubini product of $\mathcal{N}(G_0)$ and $\mathscr{M}(G_1)$, is the σ-ideal generated by the Borel sets $B \subseteq G_0 \times G_1$ with $\{x \in G_1 : B_x \notin \mathscr{M}(G_1)\} \in \mathcal{N}(G_0)$. By a theorem of Gavalec [12] (see also [11] for a more general result), the σ-ideal $\mathcal{N}(G_0) \otimes \mathscr{M}(G_1)$ is ccc. As it is also easy to see that it is G-invariant, it suffices to let $I_y = \mathcal{N}(G_0) \otimes \mathscr{M}(G_1)$.

If $Y_0 = \emptyset$ or $Y_1 = \emptyset$, then it suffices to let $I_y = \mathscr{M}(G_1)$ in the first case and $I_y = \mathcal{N}(G_0)$ in the second case.

□

The following corollary was originally proved by Balcerzak, Rosłanowski and Shelah [2, Theorem 2.1] and a simplified proof was found by Solecki [22]. Our approach, though closely related to [22], provides a still simpler proof of it.

Corollary 2.2. I_0 has property (M).

Proof. As, clearly, $I_0 \subseteq I_{ccc}$, the corollary is an immediate consequence of Theorem 2.1.

□

Remark 2.3. Exactly as in [22], Theorem 2.1 can be generalized to the case of the countable product of compact, metric, uncountable abelian groups.

3. Which Borel sets are in I_{ccc}?

For a cardinal $0 < \kappa \leq \aleph_1$ we say that a subset A of an abelian, uncountable group G is *κ-small*, if there is an uncountable set $P \subseteq G$, called *a witness* (of κ-smallness of A), such that $|(g + A) \cap P| < \kappa$ for every $g \in G$ (cf. [6] and [8]). If, moreover, G is Polish and there exists a perfect witness, then we say that A is *perfectly κ-small*. Clearly, $\emptyset$ is the only 1-small subset of G and if $0 < \lambda < \kappa \leq \aleph_1$ and A is λ-small, then it is also κ-small with the same witness. We say that A is small (perfectly small respectively) if it is $\aleph_1$-small (perfectly $\aleph_1$-small, respectively).

Perfectly small subsets of $\mathbb{R}$ have been studied by Darji and Keleti [6] who proved (see [6, Theorem 2.5]) that every compact subset of $\mathbb{R}$ with packing dimension less than 1 is perfectly small. An example of such a set is the standard middle 1/3 Cantor set.

Following an idea from [6] let us consider the action of G on the product group G^κ by coordinate-wise translations, i.e., the function $F_\kappa : G \times G^\kappa \to G^\kappa$ defined by:

$$F_\kappa(g, \langle g_\alpha : \alpha < \kappa\rangle) = \langle g + g_\alpha : \alpha < \kappa\rangle.$$

For a set $P \subseteq G$ let

$$IS_\kappa(P) = \{\langle g_\alpha : \alpha < \kappa\rangle \in P^\kappa : \forall\alpha, \beta\ (\alpha \neq \beta \Rightarrow g_\alpha \neq g_\beta)\}$$

be the set of all injective sequences of length κ of elements of P.

The following simple lemma gives useful characterizations of smallness (cf. [6, Lemma 2.3]).

Lemma 3.1. For a non-zero cardinal $\kappa \leq \aleph_1$ and subsets A and P, $|P| \geq \aleph_1$, of an uncountable abelian group G the following are equivalent:

(1) P is a witness that A is κ-small,
(2) $\bigcap_{\alpha<\kappa}(-p_\alpha + A) = \emptyset$ for every κ many different elements $p_\alpha \in P$,
(3) $F_\kappa[G \times A^\kappa] \cap IS_\kappa(P) = \emptyset$.

Proof. Note that P is *not* a witness that A is κ-small iff for a certain $g \in G$ there are $p_\alpha \in P$, $\alpha < \kappa$, with $\alpha \neq \beta \Rightarrow p_\alpha \neq p_\beta$ such that

(A) $\{p_\alpha : \alpha < \kappa\} \subseteq g + A$.

But (A) is equivalent to:

(B) $-g \in \bigcap_{\alpha<\kappa}(-p_\alpha + A)$.

and also to:

(C) $\langle p_\alpha : \alpha < \kappa\rangle \in F_\kappa[\{g\} \times A^\kappa]$.

□

In particular, by (2), A is 2-small with a witness of cardinality λ iff there are λ many pairwise disjoint translations of A. Likewise, a Borel subset A of a Polish abelian group G is perfectly 2-small iff $A \in \mathcal{F}_0(G)$.

Also, A is $\aleph_0$-small ($\aleph_1$-small respectively) iff there is an uncountable set $P \subseteq G$ such that the family $\{g + A : g \in P\}$ is point-finite (point-countable respectively). In the terminology of Kubiś [14] and Mátrai [16], condition (3) says that P is *C-homogeneous* where C is the complement of $F_\kappa[G \times A^\kappa]$ in G^κ.

Given a group G and a non-empty set $A \subseteq G$, the supremum of cardinalities λ such that there are λ many pairwise disjoint translations of A is sometimes called (see [3]) the *packing index* of A and denoted by $\mathrm{ind}_P(A)$. Therefore, A is 2-small iff $\mathrm{ind}_P(A)$ is uncountable. The packing indices of analytic subsets of Polish group have been studied by Banakh, Lyaskovska and Repovs [3].

The following result sheds some light on interesting (and to some extent still unclear) connections between the various notions of smallness introduced above. The particular case of $n = 2$ is implicit in [3, Theorem 1] (cf. Remark 3.3).

Proposition 3.2. Let A be a subset of a Polish, abelian group G.

(1) Assume that G is σ-compact. If A is F_σ and n-small for a certain n, $0 < n < \aleph_0$, then A is perfectly n-small.
(2) If A is analytic and $\aleph_0$-small with a non-meager witness, then A is perfectly $\aleph_0$-small.
(3) Let $\aleph_1 < \lambda \leq \mathfrak{c}$. It is consistent with ZFC that if A is analytic and $\aleph_0$-small with a witness of cardinality λ, then A is perfectly $\aleph_0$-small.

Proof. In order to deal with κ-small sets, where $0 < \kappa \leq \aleph_0$, let $C = G^\kappa \setminus F_\kappa[G \times A^\kappa]$. By Lemma 3.1(3), in each case assuming the existence of a C-homogeneous set with certain properties we want to prove the existence of a perfect one.

In case (1) ($\kappa = n$), G and A being σ-compact and F_n being continuous, C is a G_δ subset of G^n and the existence of an uncountable C-homogeneous set implies the existence of a perfect one, by a theorem of Kubiś [14, Corollary 2.3].

In case (2) and (3) ($\kappa = \aleph_0$), A being analytic, C is a co-analytic subset of $G^{\aleph_0}$, and the existence of a perfect C-homogeneous set is guaranteed by theorems 1.2 and 4.10 of Mátrai [16] respectively. □

As mentioned above, the particular case of 2-small and perfectly 2-small subsets of Polish groups has been studied by Banakh, Lyaskovska and Repovs [3] in the context of packing indices and earlier by Balcerzak [1] in the context of property (D). The following characterization of compact perfectly 2-small sets (equivalently: compact members of $\mathcal{F}_0(G)$) is implicit in [3] (see [3, Lemma 1] and the proof of [3, Theorem 1]; see also [1, Proposition 3.1])

Remark 3.3. For a compact subset C of a Polish abelian group G the following are equivalent:

(1) C is perfectly 2-small,
(2) $C - C$ does not contain a neighbourhood of the neutral element of G,
(3) $C - C$ has empty interior.

Leaving aside interrelations between κ-small and perfectly κ-small sets (see, however, Proposition 3.13), in the next two results we show that all Borel subsets of $2^{\mathbb{N}}$ with any of the properties under consideration, possibly apart from ($\aleph_1$-)smallness, are in the σ-ideal I_{ccc}. The statement that every small Borel subset of $2^{\mathbb{N}}$ is in I_{ccc} is independent from ZFC.

Recall that $S \subseteq 2^{\mathbb{N}}$ is a *Sierpiński set* if S is uncountable but $|S \cap B| \leq \aleph_0$ for every null-set $B \subseteq 2^{\mathbb{N}}$.

Proposition 3.4. Let B be a Borel subset of $2^{\mathbb{N}}$.

(1) If B is $\aleph_0$-small, then $B \in I_{ccc}$.
(2) Under MA + $\neg$CH, if B is small, then $B \in I_{ccc}$.
(3) If there exists a Sierpiński set $S \subseteq 2^{\mathbb{N}}$, then every Borel null-set $B \subseteq 2^{\mathbb{N}}$ is small with a witness S. In particular, there is one not in I_{ccc}.

Proof. To prove (1) and (2), assume that B is small and let I be an invariant ccc σ-ideal on $2^{\mathbb{N}}$. Suppose that $B \notin I$.

If B is, moreover, $\aleph_0$-small with a witness P, $|P| \geq \aleph_1$, then, by Proposition 3.4, $\{-g + B : g \in P\}$ is a point-finite family of Borel subsets of $2^{\mathbb{N}}$ not in I. But, I being ccc and P being uncountable, this is impossible by a result of Fremlin (see [10, Lemma 1.E(b)]) and this contradiction ends the proof of (1).

Analogically, if B is small with a witness P, $|P| \geq \aleph_1$, then, by Lemma 3.1, $\{-g + B : g \in P\}$ is a point-countable family of Borel subsets of $2^{\mathbb{N}}$ not in I. But under MA + $\neg$CH this is again prevented by [10, Lemma 1.E(b)] of Fremlin, since then, by a theorem of Martin and Solovay (see, e.g.[18, Theorem 11.1]), no Borel set not in I can be covered by fewer than $\aleph_2$ members of I.

The first part of (3) is clear. To prove the second one, it suffices to take a dense G_δ nullset B. □

The following result generalizes a theorem of Cichoń, Kharazishvili and Węglorz (cf. [5, Lemma 4]) stating that a perfectly small Borel subset of $\mathbb{R}$ has Lebesgue measure zero.

Theorem 3.5. Every perfectly small Borel subset of $2^{\mathbb{N}}$ is in I_{ccc}.

Proof. The idea of the proof is closely related to a reasoning described at the end of [4, Section 4] by Cichoń, Jasiński, Kamburelis and Szczepaniak.

Fix an invariant ccc σ-ideal I in $2^{\mathbb{N}}$. Assume that a Borel set $A \subseteq 2^{\mathbb{N}}$ is small with a perfect witness $P \subseteq 2^{\mathbb{N}}$.

Consider the set:

$$B = \{(x,y) \in 2^{\mathbb{N}} \times P : \; x + y \in A\}$$

The set B is Borel in $2^{\mathbb{N}} \times P$ and $B_x = P \cap (A + x)$ is countable for each $x \in 2^{\mathbb{N}}$. By a result of Recław and Zakrzewski (see [20, Theorem 2.1]) there is $y \in P$ such that $B^y \in I$ (actually, this holds for all but countably many points $y \in P$). But $B^y = A + y$ so $A \in I$ by the invariance of I.

□

Remark 3.6. The proof above shows also that every perfectly small Borel subset of a Polish uncountable abelian group G is in the σ-ideal $I_{ccc}(G)$ generated by the Borel subsets of G which belong to every invariant ccc σ-ideal on G.

An immediate consequence of Theorem 3.5 is the fact that Borel perfectly small subsets of $2^{\mathbb{N}}$ generate an invariant σ-ideal on $2^{\mathbb{N}}$. This generalizes a result of Cichoń, Jasiński, Kamburelis and Szczepaniak (see [4, Proposition 4.4]) stating that if $\mathbb{R}$ is the union of two uncountable Borel sets then $|(A + x) \cap B| = \mathfrak{c}$ for a certain $x \in \mathbb{R}$.

Corollary 3.7. $2^{\mathbb{N}}$ is not the union of any countable collection of Borel perfectly small sets. Equivalently, if A_n are Borel sets, $n \in \mathbb{N}$, and $2^{\mathbb{N}} = \bigcup_n A_n$, then there is n_0 such that every perfect set $P \subseteq 2^{\mathbb{N}}$ has a perfect subset Q a translate of which is contained in A_{n_0}.

We do not know at the moment whether the σ-ideal I_{ccc} is generated by Borel perfectly small subsets of $2^{\mathbb{N}}$. In fact, we do not even know if $I_{ccc} \neq I_0$, the σ-ideal generated by the collection $\mathcal{F}_0$ of Borel sets which have perfectly many pairwise disjoint translates. The next result yields a much simpler fact that $I_{ccc} \neq \mathcal{F}_0$.

Theorem 3.8. There exists a compact, perfectly 3-small subset C of $2^{\mathbb{N}}$ such that $C + C = 2^{\mathbb{N}}$. In particular, $C \in I_{ccc} \setminus \mathcal{F}_0$.

Proof. First we shall make the following observation.

Lemma 3.9. There are sets $A,\ S \subseteq \{0,1\}^4$ such that:

(1) $A + A = \{0,1\}^4$,
(2) $|S| = 3$ and $\bigcap_{s \in S}(s + A) = \emptyset$.

To see this, it suffices to let $A = \{\emptyset,\ \{1\},\ \{2\},\ \{3\},\ \{4\},\ \{1,2,3,4\}\}$ and $S = \{\{1\},\ \{1,2\},\ \{1,2,3,4\}\}$ where we identify subsets of $\{1,2,3,4\}$ with the corresponding elements of $\{0,1\}^4$ via their characteristic functions.

Now, to prove the theorem, let $I_k = [4k, 4k+4) \cap \mathbb{N}$, $k \in \mathbb{N}$, be consecutive 4-element intervals in $\mathbb{N}$.

Applying Lemma 3.9, for each $k \in \mathbb{N}$ choose sets $A_k,\ S_k \subseteq \{0,1\}^{I_k}$ such that

(1) $A_k + A_k = \{0,1\}^{I_k}$,
(2) $|S_k| = 3$ and $\bigcap_{s \in S_k}(s + A_k) = \emptyset$

Define $C = \prod_{k \in \mathbb{N}} A_k$ or, more precisely,

(3) $C = \{x \in 2^{\mathbb{N}} : \forall k \in \mathbb{N}\ x|I_k \in A_k\}$.

Then, by (1), we have

(4) $C + C = 2^{\mathbb{N}}$.

Next consider the set

(5) $U = \{(x_1, x_2, x_3) \in (2^{\mathbb{N}})^3 : \; \exists k \in \mathbb{N}\ \{x_1|I_k, x_2|I_k, x_3|I_k\} = S_k\}$.

Note that U is a dense open subset of $(2^{\mathbb{N}})^3$. By the Mycielski partition theorem (see [13, Theorem 19.1]), there is a perfect set $P \subseteq 2^{\mathbb{N}}$ such that

(6) $(x_1, x_2, x_3) \in U$ for any three different elements $x_i \in P$.

In view of (4), to complete the proof it is enough to show that

(7) P is a witness that C is 3-small.

So let $x_1,\ x_2,\ x_3$ be three different elements of P. Then, by (6), $(x_1, x_2, x_3) \in U$ so there is $k \in \mathbb{N}$ with $\{x_1|I_k, x_2|I_k, x_3|I_k\} = S_k$ (see (5)).

This implies (see (2)) that $(x_1|I_k + A_k) \cap (x_2|I_k + A_k) \cap (x_3|I_k + A_k) = \emptyset$ and hence (see (3)) $(x_1 + C) \cap (x_2 + C) \cap (x_3 + C) = \emptyset$.

This however, by Lemma 3.1, completes the proof of (7).

The last statement of Theorem 3.8 follows from Remark 3.3.

□

Remark 3.10. A related example is provided by the standard middle 1/3 Cantor set $C \subseteq \mathbb{R}$. By [6, Theorem 2.5], C is perfectly small so by Theorem 3.5, C is in the σ-ideal $I_{ccc}(\mathbb{R})$ (see Remark 3.6). On the other hand, C is not perfectly 2-small since, by a well-known theorem of Steinhaus, $C - C = [-1, 1]$.

From Theorem 3.5 we may obtain a strengthening (for $2^{\mathbb{N}}$) of a theorem of Elekes and Steprans [8] who proved (see [8, Theorem 1.2]), answering a question of Darji and Keleti [6], that there exists a compact subset of $\mathbb{R}$ of Lebesgue measure zero which is not perfectly small.

Proposition 3.11. There exists a compact nullset $C \subseteq 2^{\mathbb{N}}$ which is not in I_{ccc} hence, in particular, it is not perfectly small.

Proof. Any invariant ccc σ-ideal I on $2^{\mathbb{N}}$ of the form $\mathscr{M}_{\mathscr{C}^H_{\overline{n}}}$ constructed by Rosłanowski and Shelah [21, Conclusion 4.7] has the property that there is a compact nullset not in I.

□

If we do not require A to be both compact and not in I_{ccc} then we can get more.

Proposition 3.12. In every invariant, ccc σ-ideal in $2^{\mathbb{N}}$:

(1) there is a Borel set which is not in I_{ccc},
(2) there is a compact set which is not perfectly small.

Proof. To prove (1), note that no invariant, ccc σ-ideal in $2^{\mathbb{N}}$ is contained in I_{ccc}, since the latter is not ccc by Theorem 2.1.

To prove (2), we shall use the following lemma due to Balcerzak, Rosłanowski and Shelah [2, Lemma 2.2]: for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ and a set $A \subseteq \{0,1\}^m$ such that every n translates of A have non-empty intersection and likewise for $A' = \{0,1\}^m \setminus A$.

Applying the lemma inductively, choose a strictly increasing sequence $n_k \in \mathbb{N}$, $n_0 = 0$, and associated sequence of sets $A_k \subseteq \{0,1\}^{I_k}$, where $I_k = [n_k, n_{k+1}) \cap \mathbb{N}$ for $k \in \mathbb{N}$, such that

(1) every 2^{k+1} translates of A_k by elements of $\{0,1\}^{I_k}$ have non-empty intersection and likewise for A'_k.

Next, following the proof of [2, Theorem 2.1], define $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $f(x)(k) = 1$ if $x \restriction I_k \in A_k$ and $f(x)(i) = 0$ if $x \restriction I_k \in A'_k$.

Fix an invariant, ccc σ-ideal I on $2^{\mathbb{N}}$. Clearly, each of the sets $f^{-1}(y)$, where $y \in 2^{\mathbb{N}}$, is compact and at least one of them is in I. Without loss of generality assume that the set

(2) $C = \{x \in 2^{\mathbb{N}} : \forall k \in \mathbb{N}\ x \restriction I_k \in A_k\}$

is in I. We shall prove that C is not perfectly small.

To that end fix an arbitrary perfect set $P \subseteq 2^{\mathbb{N}}$. We will show (cf. [8, Theorem 1.2]) that there exists a perfect set $Q \subseteq P$ and $y \in 2^{\mathbb{N}}$ such that $y + Q \subseteq C$.

Let $T_P \subseteq \{0,1\}^{<\mathbb{N}}$ be the perfect pruned tree on $\mathbb{N}$ (see [13]) such that $P = [T_P]$, the set of all infinite branches of T_P. We shall find a perfect subtree T_Q of T_P and then define $Q = [T_Q]$.

To construct T_Q, inductively define integers $0 = k_0 < k_1 < k_2 < \dots$ and families $R_i \subseteq \{s \in T_P : \text{length}(s) = n_{k_{i+1}}\}$ such that for each $i \in \mathbb{N}$:

(3) $|R_0| = 2$, $\forall s \in R_i\ |\{t \in R_{i+1} : s \subseteq t\}| = 2$ and $\forall t \in R_{i+1}\ t \restriction n_{k_i+1} \in R_i$.

Consequently, letting $S_i = \{s \restriction [n_{k_i}, n_{k_{i+1}}) :\ s \in R_i\}$, we have $|S_i| \leq 2^{i+1}$ hence

(4) $|\{s \restriction I_k : s \in S_i\}| \leq 2^{k+1}$ for each $i \in \mathbb{N}$ and $k \in [k_i, k_{i+1})$.

It then follows from (1) and (4) that

(5) $\bigcap_{s \in S_i}((s \restriction I_k) + A_k) \neq \emptyset$ for each $i \in \mathbb{N}$ and $k \in [k_i, k_{i+1})$.

With the help of (5) define $y \in 2^{\mathbb{N}}$ in such a way that for each $i \in \mathbb{N}$ and $k \in [k_i, k_{i+1})$

(6) $y \restriction I_k \in \bigcap_{s \in S_i}((s \restriction I_k) + A_k)$,

which in turn is equivalent to

(7) $y \restriction I_k + s \restriction I_k \subseteq A_k$ for every $s \in S_i$.

Letting T_Q be the smallest tree on $\mathbb{N}$ containing $\bigcup_{i \in \mathbb{N}} R_i$ and $Q = [T_Q]$ we easily conclude from (2) and (7) that $y + Q \subseteq C$.

□

As another immediate corollary of Theorem 3.5 we get the consistency of the existence of a small set (even one with a witness of cardinality $\mathfrak{c}$) which is not perfectly small (cf. Proposition 3.4(3)).

Corollary 3.13. If I is an invariant ccc σ-ideal on $2^{\mathbb{N}}$ and there exists an I-Lusin $S \subset 2^{\mathbb{N}}$, then every Borel set from I is small with a witness S. In particular, there is one not in I_{ccc} hence not perfectly small.

With the help of Corollary 3.13 and Proposition 3.11 we get a result related to a theorem of Mátrai stating that it is consistent with ZFC that even for open sets $C \subseteq (2^{\mathbb{N}})^{\aleph_0}$ the existence of a C-homogeneous set of cardinality $\mathfrak{c}$ does not imply the existence of a perfect one (cf. [16, Theorem 1.1]).

Proposition 3.14. If there exists a Sierpiński set $S \subseteq 2^{\mathbb{N}}$, then there exists an open subset C of $(2^{\mathbb{N}})^{\aleph_1}$ with the property that S is C-homogeneous but no perfect C-homogeneous set exists.

Proof. Let $A \subseteq 2^{\mathbb{N}}$ be a compact nullset which is not perfectly small (cf. Proposition 3.11) and let $C = (2^{\mathbb{N}})^{\aleph_1} \setminus F_{\aleph_1}[2^{\mathbb{N}} \times A^{\aleph_1}]$. The function $F_{\aleph_1} :\ 2^{\mathbb{N}} \times (2^{\mathbb{N}})^{\aleph_1} \to (2^{\mathbb{N}})^{\aleph_1}$ being continuous and the set A being compact, C is an open subset of $(2^{\mathbb{N}})^{\aleph_1}$.

By Corollary 3.13 and Lemma 3.1(3), S is C-homogeneous but no perfect C-homogeneous set exists since there is no perfect witness of smallness of A. □

4. Covering $2^{\mathbb{N}}$ by $\aleph_1$ many disjoint translates of a compact set

The purpose of this section is to point out some links between the subject of this paper and that of covering $2^{\mathbb{N}}$ (or, more generally, an uncountable Polish group) by less than $\mathfrak{c}$ many translates of its compact subset. The latter has been recently dealt with by many authors (see [6], [7], [8], [9], [19], [15]).

Let us start with an obvious generalization of a simple observation of Darji and Keleti (see [6, Lemma 2]).

Lemma 4.1. Let G be an uncountable abelian group and $\aleph_1 \leq \kappa \leq \mathfrak{c}$. Then less than κ many translates of a small set with a witness of cardinality at least κ do not cover G.

Note that when $\mathfrak{c} > \aleph_1$ Lemma 4.1 prevents $2^{\mathbb{N}}$ from being covered by $\aleph_1$ many translates of any Borel perfectly small subset B. At the same time Theorem 3.5 implies that $B \in I_{ccc}$ and in fact, the smallness of a Borel set $B \subseteq 2^{\mathbb{N}}$ is, at least consistency-wise, the strongest among conditions considered in Section 3 (cf. Proposition 3.4) sufficient for this.

These remarks lead to the natural question whether in the absence of CH, at least for a compact set $A \subseteq 2^{\mathbb{N}}$, the properties of $2^{\mathbb{N}}$ *not* being covered by $\aleph_1$ many translates of A on one hand and A being a member of I_{ccc} on another, are related.

We have the following observation.

Proposition 4.2. It is consistent with the negation of CH that there is a compact null set $A \subseteq 2^{\mathbb{N}}$ such that $A \notin I_{ccc}$ but no $\aleph_1$ many translates of A cover $2^{\mathbb{N}}$. In fact, the latter is true for any null set A provided a Sierpiński set $S \subseteq 2^{\mathbb{N}}$ of size greater than $\aleph_1$ exists and CH is false.

Proof. This is an immediate consequence of Proposition 3.11 and Lemma 4.1. □

Miller [17] proved that it is consistent with the negation of CH that $2^{\mathbb{N}}$ (equivalently: any uncountable Polish space) can be partitioned into $\aleph_1$ many disjoint non-empty closed sets. In contrast to this, we have the following result.

Theorem 4.3. CH is equivalent to each of the following statements:

(1) $2^{\mathbb{N}}$ can be partitioned into $\aleph_1$ many disjoint translates of a closed set,
(2) some uncountable locally compact Polish abelian group can be,
(3) all uncountable locally compact Polish abelian groups can be.

Moreover, $\neg$CH implies that if an uncountable locally compact Polish abelian group is the union of a collection $\mathcal{A}$ of $\aleph_1$ many translates of a closed set then for every natural number n there is $x \in G$ such that x is a member of at least n elements of $\mathcal{A}$.

Proof. Let G be an uncountable locally compact Polish abelian group.

It suffices to prove that $\neg$CH implies that G cannot be covered by $\aleph_1$ many translates of a closed set in such a way that for some $n > 1$ no $x \in G$ is a member of n elements of the covering.

So assume that $\mathfrak{c} > \aleph_1$.

Let G be an uncountable locally compact Polish group. Suppose, towards a contradiction, that A a closed subset of G and for some $n > 1$ there is a covering of G by $\aleph_1$ many translates of A such that no $x \in G$ is a member of n elements of the covering. Then, by Lemma 3.1, A is n-small and by Proposition 3.2, it is perfectly small. But this implies, by Lemma 4.1, that G cannot be covered by less than $\mathfrak{c}$ many translates of A.

□

5. Open problems

Let us conclude with a list of some natural open questions.

(1) **Problem 1.** *What is the relation of I_{ccc} to the intersection of all invariant ccc σ-ideals on $2^{\mathbb{N}}$? In particular, is there a non-Borel set $A \subseteq 2^{\mathbb{N}}$ which belongs to every invariant ccc σ-ideal but is not in I_{ccc}?* (This question was pointed out to me by the referee whom I wish to thank for all his valuable remarks).

(2) **Problem 2.** *What is the relation of I_{ccc} to the σ-ideal generated by Borel perfectly small (perfectly 2-small, respectively) subsets of $2^{\mathbb{N}}$?* (cf. Theorem 3.5).

(3) **Problem 3.** *Is every Borel n-small, $1 < n < \aleph_0$, subset of a Polish abelian group G perfectly n-small?* (cf. Proposition 3.2; a closely related problem for $n = 2$ was formulated in [3] as Question 2).

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