ON THE COMPLEXITY OF THE IDEAL OF ABSOLUTE NULL SETS

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Abstract. Answering a question of Banakh and Lyaskovska, we prove that for an arbitrary countable infinite amenable group G the ideal of sets having μ-measure zero for every Banach measure μ on G is an $F_{\sigma\delta}$ subset of $\{0,1\}^G$.

1. Introduction

This note is related to a paper by T. Banakh and N. Lyaskovska [1]. Given an amenable group G, Banakh and Lyaskovska considered the ideal $\mathcal{N}$ of *absolute null* subsets of G, i.e., sets having μ-measure zero for every Banach measure μ on G (a finitely-additive, probability, left-invariant measure $\mu : \mathcal{P}(G) \to [0,1]$ defined on the family of all subsets of G; see [3]). Since each ideal on a countable infinite group G can be considered as a subspace of the Cantor set $\{0,1\}^G$ it makes sense to consider its descriptive properties. Banach and Lyaskovska asked ([1, Problem 4]) whether the ideal of absolute null subsets of the group $\mathbb{Z}$ is co-analytic. In this note we prove (see Corollary 3.1) that for an arbitrary countable infinite amenable group G the ideal $\mathcal{N}$ is in fact $F_{\sigma\delta}$. This follows from a characterisation of absolute null subsets of an arbitrary amenable group (see Proposition 2.1) based on the notion of the intersection number of Kelly [2].

2. A characterisation of absolute null sets

Following Kelly [2] we define *the intersection number* $I(\mathcal{B})$ of a family $\mathcal{B}$ of subsets of a set X to be $\inf\{i(S)/n(S)\}$ where the infimum is taken over all finite sequences $S = (S_1, \ldots, S_n)$ of (not necessary distinct) elements of $\mathcal{B}$, $n = n(S)$ is the length of S and

$$i(S) = \sup\{\sum_{i=1}^{n} \chi_{S_i}(x) : \; x \in X\}.$$

Proposition 2.1. *Let G be an amenable group and $A \subseteq G$. Then the following are equivalent:*

(1) A is absolute null.

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(2) The intersection number of the family $\{gA: g \in G\}$ is zero.

Proof. (1) $\Rightarrow$ (2): Assume that $I(\{gA : g \in G\}) = \delta > 0$. By a theorem of Kelly (see [2, Theorem 2]), there is a finitely additive probability measure m defined on $\mathcal{P}(G)$ such that $m(gA) \geq \delta$ for each $g \in G$.

Let θ be a Banach measure on G. Following the proof of Invariant Extension Theorem (see [4, Theorem 10.8]) define a function $\mu : \mathcal{P}(G) \to [0,1]$ by letting

$$\mu(B) = \int_G m(g^{-1}B)d\theta(g), \qquad \text{for} \qquad B \subseteq G.$$

It is easy to see that μ is a Banach measure on G. Moreover, we have

$$\mu(A) = \int_G m(g^{-1}A)d\theta(g) \geq \inf\{m(g^{-1}A) : g \in G\} \geq \delta > 0,$$

which shows that $A \notin \mathcal{N}$.

(2) $\Rightarrow$ (1): Let μ be an arbitrary Banach measure on G. Suppose that $\mu(A) = \epsilon > 0$. Then, since μ is left-invariant, we also have $\mu(gA) = \epsilon$ for every $g \in G$. Consequently, by [2, Proposition 1], $I(\{gA : g \in G\}) \geq \epsilon > 0$.

□

3. The Borel complexity of the ideal $\mathcal{N}$

The following corollary of Proposition 2.1 gives an answer to a question of Banakh and Lyaskovska (see [1, Problem 4]).

Corollary 3.1. *Let G be an amenable group and $A \subseteq G$. Then the following are equivalent:*

(1) A is absolute null.

(2) $\forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ \exists \bar{g} \in G^{n+1}\ \forall S \subseteq \{1, \ldots, n+1\}$

$$\frac{|S|}{n+1} > \frac{1}{k+1} \ \Rightarrow \bigcap_{i \in S} g_i A = \emptyset.$$

In particular, if G is countably infinity, then formula (2) *gives a $F_{\sigma\delta}$ definition of the ideal $\mathcal{N}$.*

Proof. It is easy to see that formula (2) simply states that $I(\{gA : g \in G\}) = 0$ so its equivalence with condition (1) was established in Proposition 2.1.

To prove the remaining part of the corollary, assume that G is countably infinity. Then it is enough to show that for fixed $n \in \mathbb{N}$, $\bar{g} \in G^{n+1}$ and $S \subseteq \{1, \ldots, n+1\}$ the family $\{A \subseteq G : \bigcap_{i \in S} g_i A = \emptyset\}$ is closed in $\mathcal{P}(G)$.

But this follows from the fact that for $A \subseteq G$ we have

$$\bigcap_{i \in S} g_i A = \emptyset \iff \forall g \in G\ \exists i \in S\ g_i^{-1} g \notin A.$$

□

4. Some open problems

Let G be an arbitrary infinite group. Following a suggestion by Taras Banakh (personal communication) let us call a set $A \subseteq G$ *Kelly null* if the intersection number of the family $\{gA : g \in G\}$ is zero; denote by $\mathscr{K}$ the collection of all Kelly null subsets of G. In view of Proposition 2.1, $\mathscr{K}$ is an ideal of subsets of G *provided* the group G is amenable. On the other hand, Proposition 5.1 of [1] implies that if G has a free subgroup of rank 2, then $\mathscr{K}$ is not an ideal; in fact G is then the union of two Kelly null sets. In any case, however, $\mathscr{K}$ contains a (possibly proper) subfamily $\mathcal{A}_{\mathscr{K}} = \{A \subseteq G : \forall K \in \mathscr{K}\ K \cup A \in \mathscr{K}\}$ which already forms an ideal.

The remarks above lead to the following problems suggested by Banakh.

Problem 1. Characterise groups G for which $\mathscr{K}$ is an ideal.

Problem 2. Characterise groups G which are finite unions of elements of $\mathscr{K}$.

Problem 3. Given a countably infinite group G find a combinatorial description of elements of the ideal $\mathcal{A}_{\mathscr{K}}$. What is its descriptive complexity? In particular, is it Borel?

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