Union of Equational Theories: An Algebraic Approach*

Piotr Hoffman

Warsaw University, Institute of Informatics,
Banacha 2, 02-097 Warszawa, Poland
piotrek@mimuw.edu.pl

Abstract. We consider the well-known problem of deciding the union of decidable equational theories. We focus on monadic theories, i.e., theories over signatures with unary function symbols only. The equivalence of the category of monadic equational theories and the category of monoids is used. This equivalence facilitates a translation of the considered decidability problem into the word problem in the pushout of monoids which themselves have decidable word problems. Using monoids, existing results on the union of theories are then restated and proved in a succint way. The idea is then analyzed of first guaranteeing that the union is a "jointly conservative" extension and then using this property to show decidability of the union. It is shown that "joint conservativity" is equivalent to the corresponding monoid amalgam being embeddable; this allows one to apply results from amalgamation theory to this problem. Then we prove that using this property to show decidability is a more difficult matter: it turns out that even if this property and some additional conditions hold, the problem remains undecidable.

Introduction

Assume two equational theories E_1 and E_2 are given. If one has procedures for deciding these theories, then the natural question arises of whether they can be extended to a procedure deciding the union theory, $E_1 \cup E_2$. This question is practically important, since in principle it is equivalent to the question whether theorem provers for E_1 and E_2 can be joined into a theorem prover for $E_1 \cup E_2$.

The general answer is, quite obviously, in the negative. For a long time it has been known that if the signatures Σ_1 and Σ_2, over which E_1 and E_2 are built, are disjoint, then the answer is in the affirmative [Pig74]. The problem is usually only discussed in the case of E_1 and E_2 being conservative extensions of an "intersection theory" E over the signature $\Sigma = \Sigma_1 \cap \Sigma_2$. Recently it has been proven that in this case, if one additionally assumes that E_1 and E_2 are in a certain sense *effectively constructible* over Σ, then the answer is positive. This has been proven by Fiorentini and Ghilardi [FG03] and by Baader and

* This work has been partially supported by European Union project AGILE IST-2001-32747.

Tinelli [BT02]. Both papers are quite complex, which is to a large extent due to them dealing with signatures in which symbols of arbitrary arity may exist.

In this paper we only deal with *monadic theories*, i.e., theories in which all function symbols are unary. The category of such equational theories turns out to be equivalent to the category of monoids (Theorem 1). Showing this equivalence and transposing the considered problem from the level of theories to the level of monoids is the topic of Sections 1 and 2. It should be noted that the approach of Fiorentini and Ghilardi is, in fact, very similar — however, since they do not restrict themselves to monadic theories, they have to use the much more complex notion of category with products and product-preserving functors.

Moving to the category of monoids allows us to define the property of effective constructability is a concise way (see the definition of *base* in Section 3). The proof of the result of Fiorentini/Ghilardi and Baader/Tinelli for the monadic case also turns out very simple (Theorem 2). The main benefit of Section 3 is thus that known results are presented in such a way that they are easy to understand; this should facilitate future research on this topic. As an example, a simple extension of the known result to *cobases* is proposed (Corollary 1).

We then consider an idea stemming from [BT02], namely of dividing the considered problem into two stages. In the first stage, two properties are enforced: first, that the union theory conservatively extends E_1 and E_2 (conservativity property); second, that if a Σ_1-term t_1 is equivalent in the union theory to a Σ_2-term t_2, then there exists a Σ-term u such that $t_1 = u$ in E_1 and $u = t_2$ in E_2 (interpolation property). In the second stage, one would use the conservativity and interpolation properties to decide the union theory (we call this "lifting").

In Section 4 the above idea is formalized and it is shown that, in the monoid case, the conservativity and interpolation properties are very useful and natural. In particular, it is shown that the theories E_1 and E_2 conservatively extending E are really equivalent to a *monoid amalgam*, an object well known in semigroup theory [How96a,Hig92]. Moreover, the conservativity property is equivalent to the amalgam being *weakly embeddable*, while adding the interpolation property is equivalent to it being *embeddable*. These equivalences make it possible to use results from amalgamation theory concerning the embeddability of amalgams to guarantee or prove that conservativity and interpolation hold. In particular, we show that if the theory E, or rather the corresponding monoid, is unitary or quasiunitary in E_1 and E_2, then both properties hold. We also show that in certain cases the monoid corresponding to E can, by itself, guarantee that these poperties hold, in particular if it is a group or inverse semigroup.

Section 5 is concerned with stage two of the program presented above, i.e., with lifting the conservativity and interpolation to a solution to our main problem. The results of this section are negative, showing that conservativity and interpolation, even if augmented by one of the theories (monoids) having a base, may still fail to guarantee the decidability of the union theory (Theorem 3).

This negative result suggests that one should search for some additional condition which would guarantee that the lifting phase is indeed correct. The result is also interesting even if one abstracts away from the proposed two-stage program, since it shows that even a very slight weakening of the assumption of both theories having bases moves the problem into the realm of undecidability.

Finally, Section 6 contains conclusions and suggestions on directions for future work. In particular, the problem of extending the techniques and results of this paper to arbitrary arities and to multi-sorted logic is considered.

1 Monadic Theories vs Monoids

A *signature* Σ defines a set of *function symbols* together with their *arities*. By $T_\Sigma(X)$ the set of *terms* over Σ with variables X is denoted. A Σ-*equation* has the form $t_1 = t_2$, where $t_1, t_2 \in T_\Sigma(X)$. A Σ-*theory* is any set of Σ-equations; a *theory* is a pair (Σ, E), where E is a Σ-theory. A Σ-*model* consists of a non-empty set M and, for any function symbol f with arity n in Σ, of an interpretation of f, that is, of a function $M(f) : M^n \to M$. If $X = \{x_1, \dots, x_n\}$ is an n-element set and t is a Σ-term over X, then any Σ-model M naturally defines a function $M(t) : M^n \to M$, called the *interpretation* of t. An equation $t_1 = t_2$ *holds* in M if $M(t_1) = M(t_2)$. A Σ-model M is a *model* of a Σ-theory E if all equations in E hold in M. An equation is a *consequence* of E if it holds in all models of E.

A *signature morphism* $\sigma : \Sigma \to \Delta$ is a map taking any function symbol f of arity n in Σ to a Δ-term t over $\{x_1, \dots, x_n\}$. Signature morphisms compose naturally via term substitution. Moreover, the morphism $\sigma : \Sigma \to \Sigma$ taking any f with arity n to the term $f(x_1, \dots, x_n)$ is an identity. Thus, signatures and signature morphisms form a category. Note that any term and, hence, any equation can be translated via a signature morphism. A *theory morphism* $\sigma : (\Sigma, E) \to (\Delta, F)$ is a signature morphism $\sigma : \Sigma \to \Delta$ such that the translated equations in $\sigma(E)$ are all consequences of F. The composition of two theory morphisms is a theory morphism and identity morphisms are theory morphisms. Thus, theories and theory morphisms form a category.

The map taking any signature or any theory to the class of its models is, actually, functorial, since if $\sigma : (\Sigma, E) \to (\Delta, F)$ is a theory morphism and N is a Δ-model of F, then one can define the Σ-model $M = N|_\sigma$, called the σ-*reduct* of N, and M will be a model of E; this map is functorial, since the reduct along identity is identity and since reducts compose (contravariantly).

We focus on *monadic* theories. A signature is monadic if all its function symbols are unary. A term is monadic if it is built solely of unary symbols and uses only the variable "x". An equation is monadic if it is built of monadic terms. A theory is monadic if it is a theory over a monadic signature and consists only of monadic equations.

Let (Σ, E) be a monadic theory. Consider the set of all monadic Σ-terms and let $t_1 \equiv_E t_2$ hold iff $t_1 = t_2$ is a consequence of E. The set of equivalence classes w.r.t. $\equiv_E$ forms a monoid if one defines 1 to be the equivalence class of x and $[t_1]_{\equiv_E}[t_2]_{\equiv_E} = [t_2[t_1/x]]_{\equiv_E}$. This monoid is denoted by $\alpha(\Sigma, E)$. In fact, this operation is functorial, since if $\sigma : (\Sigma, E) \to (\Delta, F)$ is a theory morphism, then the map defined by $\alpha(\sigma)([t]_{\equiv_E}) = [\sigma(t)]_{\equiv_F}$ is a homomorphism and, moreover, this correspondence preserves identities and composition. Thus α is a functor from the category of monadic theories to the category of monoids.

Similarily, any monoid S defines a monadic theory $\beta(S) = (\Sigma, E)$, where function symbols in Σ are all elements of $S \setminus \{1\}$ and, for any $s_1, s_2 \in S \setminus \{1\}$ and $s \in S$ with $s_1 s_2 = s$ holding in S, E contains the equation $s_2(s_1(x)) = s(x)$ if $s \neq 1$, or the equation $s_2(s_1(x)) = x$ if $s = 1$. This map again extends to a functor, since any homomorphism $h : S \to T$ defines a theory morphism $\beta(h)$ taking any function symbol in $\beta(S)$, i.e., any element $s \in S \setminus \{1\}$, to the term x if $h(s) = 1$ or to the term consisting of a single function symbol $h(s)$ in $\beta(T)$ if $h(s) \neq 1$. Thus β is a functor from the category of monoids to the category of monadic theories.

We now have:

Theorem 1. *The functors α and β form an equivalence of the category of monadic theories and the category of monoids.*

Proof. We only define the natural isomorphisms $\gamma : \alpha;\beta \to \mathrm{id}$ and $\delta : \beta;\alpha \to \mathrm{id}$.

For any theory $T = (\Sigma, E)$, γ_T takes any function symbol K over $\beta(\alpha(T))$, where K is a term t over Σ, to the term t over Σ. The inverse transformation γ_T^{-1} takes any function symbol f in Σ to the term $K(x)$ built of a single function symbol K in $\beta(\alpha(T))$, where K is the term $f(x)$ over Σ.

Now take any monoid S and let $(\Sigma, E) = \beta(S)$. Then δ_S takes any element K of $\alpha(\beta(S))$, i.e., any term $K = s_n(\cdots s_1(x)\cdots)$, where $s_1, \ldots, s_n \in S \setminus \{1\}$, $n \geq 0$, to the element $s_1 \ldots s_n$ of S. The inverse transformation δ_S^{-1} takes any $s \in S$ to an element K of $\alpha(\beta(S))$, namely to the term $s(x)$ if $s \neq 1$ or to the term x if $s = 1$.

It is easy to check that γ and δ are indeed natural isomorphisms. $\square$

This equivalence allows one to translate notions from one category to notions from the other.

A signature morphism $\gamma : (\Sigma, E) \to (\Delta, F)$ is said to be *conservative*, if $\gamma(t_1) = \gamma(t_2)$ being a consequence of F implies that $t_1 = t_2$ is a consequence of E, for all Σ-terms t_1 and t_2; the equivalent notion is a homomorphism $g : S \to T$ which is one-one, i.e., a monomorphism.

The notion equivalent to checking that an equation $t_1 = t_2$ is a consequence of E is solving the *word problem* in S, i.e., checking whether $s_1 \ldots s_n = s'_1 \ldots s'_k$, for $s_1, \ldots, s_n, s'_1, \ldots, s'_k \in S$. Thus the decidability of E is equivalent to the word problem in S being decidable.

It should be noted that Fiorentini and Ghilardi also use an equivalence similar to the one defined in this section, but because they cover all, not only monadic, theories, instead of monoids they use categories with products and product-preserving functors.

2 Union of Monadic Theories vs Amalgam of Monoids

Let (Σ_1, E_1) and (Σ_2, E_2) be decidable monadic theories conservatively extending a theory (Σ, E), where $\Sigma = \Sigma_1 \cap \Sigma_2$. The equivalence constructed in the previous section gives us what, in monoid theory, is called an *amalgam*, namely monoids S and T with decidable word problems and monomorphisms $i : U \to S$ and $j : U \to T$; here S corresponds to E_1, T to E_2 and U to E. Because monomorphisms in the category of monoids are simply one-to-one homomorphisms, one can without loss of generality assume that i and j are inclusions, i.e., that $U = S \cap T$. In the sequel, an amalgam is written as $[U \subseteq_{i,j} S, T]$ or, if i and j are inclusions and $U = S \cap T$, just as $[U \subseteq S, T]$.

We are interested in deciding the theory $(\Sigma_1 \cup \Sigma_2, E_1 \cup E_2)$. Note that this theory is, in fact, the pushout of the inclusions of E into E_1 and E_2 in the category of monadic theories. Therefore, due to the equivalence of categories, our problem turns into the problem of solving the word problem in the pushout P of the span of monomorphisms $i : U \to S$ and $j : U \to T$; this pushout is given by certain homomorphisms $\mu : S \to P$ and $\nu : T \to P$ satisfying $i;\mu = j;\nu$. We call P, μ and ν the *pushout of the amalgam*.

The above considerations show that deciding the union of decidable monadic theories is equivalent to the following problem: given a monoid amalgam $[U \subseteq_{i,j} S, T]$ with S and T having decidable word problems, solve the word problem in the pushout P of the amalgam.

From now on we therefore assume that we have monoids S and T with decidable word problems. We write $U = S \cap T$, and $i : U \to S$ and $j : U \to T$ are the inclusions. Finally, $\mu : S \to P$ and $\nu : T \to P$ form the pushout; note that the pushout always exists (the category of monoids is cocomplete). We concentrate on solving the word problem in P. Note that, obviously, μ and ν are jointly epimorphic; more specifically, P may be seen as built of all sequences of elements of S and T, quotiented by an appropriate equivalence $\sim$. This equivalence is the least congruence satisfying $\langle 1 \rangle \sim \langle\rangle$ and, for all $s_1, s_2 \in S$ and $t_1, t_2 \in T$, $\langle s_1, s_2 \rangle \sim \langle s_1 s_2 \rangle$ and $\langle t_1, t_2 \rangle \sim \langle t_1 t_2 \rangle$.

3 The Solution via Bases

Solutions to the considered problem have been presented independently by Fiorentini/Ghilardi [FG03] and Baader/Tinelli [BT02]. In both solutions, virtually the same assumptions are made, cf. Prop. 7.10 of [BT02]. The main assumption is what Fiorentini and Ghilardi call *(effective) constructibility* of E_1 over E and of E_2 over E. An algebraic characterization of this notion may be found in Prop. 10.4 of [FG03]. The corresponding notion in the world of monoids is the notion of *(computable) base* (this terminology is borrowed from [BT02]).

Assume U is a submonoid of a monoid S. A *U-base* of S is a set $G \subseteq S$ such that $1 \in G$ and, for any $s \in S$, there exists a unique pair $(g, u) \in G \times U$ such that $s = gu$. The base is *computable* if the pair (g, u) may be computed by a recursive function.

It should be noted that, in fact, if S has a decidable word problem and G is a U-base of S, then the base G is computable; this is because one can simply enumerate all pairs, knowing that exactly one must have the needed property.

Observe that if G is U-base, then $G \cap U = \{1\}$. For if $g \in G \cap U$, then the pairs $(g, 1)$ and $(1, g)$ must be equal and so $g = 1$.

Note also that any monoid S has a computable $\{1\}$-base, where $\{1\}$ is the trivial monoid; one has to take $G = S$. If the original signatures Σ_1 and Σ_2 were disjoint, then U is indeed the trivial monoid. Hence, in this case both bases always exist.

For monadic theories, the results of Fiorentini/Ghilardi and Baader/Tinelli may thus, in algebraic terms, be expressed as follows:

Theorem 2. *Assume $[U \subseteq S, T]$ is an amalgam and the monoids S and T both have U-bases and decidable word problems. Then the pushout P of the amalgam has a decidable word problem.*

Proof. Assume G_S and G_T are the respective bases. Consider the set Q of all non-empty sequences of elements of G_S, G_T and U. Let $\to$ be a reduction relation on these sequences defined by:

- $x, g \to g', u'$ if $g \neq 1$, $xg = g'u'$ and $g' \neq 1$, where $x \in S$, $g, g' \in G_S$, $u' \in U$, or symmetrically for T,
- $x, g \to u'$ if $g \neq 1$, $xg = u'$, where $x \in S$, $g \in G_S$ and $u' \in U$, or symmetrically for T,
- $u, u' \to uu'$ if $u, u' \in U$,
- $g, 1 \to 1$ if $g \neq 1$, $g \in G_S$, or symmetrically for T.

It is obvious that $\to$ is terminating. It is also confluent: if two reductions overlap, then we must have elements x, x', x'' such that x, x' reduces and x', x'' reduces. The following cases thus arise:

1. x, x', x'' all come from S or all come from T; then both reductions may be performed in any order,
2. otherwise, assume $x' \neq 1$; since x, x' reduces and since it cannot be the case that $x, x' \in U$ (because we would then be in case 1), we must have that $x' \in G_S \setminus U$ or symmetrically for T, because if $x' \in U \setminus \{1\}$ and $x \notin U$, then x, x' is not a redex; but since x', x'' reduces, we must then have $x'' \in S$, or symmetrically for T, which would mean that we are actually in case 1; therefore $x' = 1$, so both reductions may be performed in any order.

Consider two sequences q, q' from Q to be equivalent, written $q \sim q'$, if they have the same normal formal form with respect to $\to$. Let P be the quotient $Q/_\sim$; then P with concatenation modulo $\sim$ and with $[1]_\sim$ as identity is a monoid. Define $\mu : S \to P$ by $\mu(s) = [g, u]_\sim$, where $gu = s$, $(g, u) \in G_S \times U$; define $\nu : T \to P$ analogically. We claim that μ must then be a homomorphism. Of course, $\mu(1) = [1]_\sim$. Also, if $s_1 = g_1 u_1$ and $s_2 = g_2 u_2$, with $g_1, g_2 \in G_S$, $u_1, u_2 \in U$, and if $s_1 s_2 = gu$ with $g \in G_S$, $u \in U$, then $\mu(s_1)\mu(s_2) = [g_1, u_1, g_2, u_2]_\sim = [g, u]_\sim = \mu(s_1 s_2)$. This is due to there existing $g_0 \in G_S$, $u_0 \in U$ such that $u_1 g_2 = g_0 u_0$; then $g_1, u_1, g_2, u_2 \to g_1, g_0, u_0, u_2$ (unless $g_0 = 1$); there further exist $g'_0 \in G_S$ and $u'_0 \in U$ such that $g_1 g_0 = g'_0 u'_0$; then $g_1, g_0, u_0, u_2 \to g'_0, u'_0, u_0, u_2 \to g'_0, u'_0 u_0, u_2 \to g'_0, u'_0 u_0 u_2$ (unless $g'_0 = 1$); but since $g'_0 u'_0 u_0 u_2 = g_1 u_1 g_2 u_2 = s_1 s_2 = gu$, we must have $g = g_0'$ and $u = u_0' u_0 u_2$, which completes the proof of the claim. An analogical argument shows that $\nu : T \to P$ is a homomorphism as well. Moreover, it is clear that $\mu(u) = [1, u]_\sim = \nu(u)$ and so $i; \mu = j; \nu$. Thus P, μ and ν form a cocone. Moreover, for any cocone $\mu' : S \to P'$, $\nu' : T \to P'$, there exists a unique factorizing $\eta : P \to P'$ given by $\eta([x_1, \dots, x_n]_\sim) = h(x_1) \cdot h(x_2) \cdots h(x_n)$, where $h(x_i) = \mu'(x_i)$ for $x_i \in S$ and $h(x_i) = \nu'(x_i)$ for $x_i \in T$ and where "$\cdot$" denotes multiplication in P'. This proves that P is a pushout.

What remains to be shown is that P has a decidable word problem. But this is obvious, since one may simply compute the normal forms of both sides of an equation and then check whether they are equal. $\square$

The algorithm for deciding the word problem in the pushout is thus straightforward: one simply has to repeatedly apply the relation $\to$, obtaining a normal form.

In the monadic case, a trivial generalization of this algorithm may be considered. If U is a submonoid of S, then a *U-cobase* of S is a set $G \subseteq S$ such that $1 \in G$ and, for any $s \in S$, there exists a unique pair $(u, g) \in U \times G$ such that $s = ug$. It is obvious that the following corollary holds:

Corollary 1. *Assume $[U \subseteq S, T]$ is an amalgam and the monoids S and T both have U-cobases and decidable word problems. Then the pushout P of the amalgam has a decidable word problem.* $\square$

It is an interesting question whether the notion of cobase can be generalized to non-monadic theories, or whether the use of cobases is excluded by function symbols with arities other than 1.

4 Conservativity and Interpolation vs Embeddability of Amalgams

In the paper [BT02], before the correctness of the whole proposed procedure is considered, two important lemmas are proved (Prop. 4.14 and Lemma 4.18). These lemmas show that under certain assumptions the union theory has the conservativity and interpolation properties.

The union theory $E_1 \cup E_2$ is said to have the *conservativity* property if it is a conservative extension of E_1 and a conservative extension of E_2. In other words, this property means that if t_1 and t_2 are terms over Σ_1, then the equality $t_1 = t_2$ is a consequence of $E_1 \cup E_2$ iff it is a consequence of E_1 alone, and that an analogous fact holds for Σ_2 and E_2.

The union theory is said to have the *interpolation property* if, for any term t_1 over Σ_1 and t_2 over Σ_2, the equation $t_1 = t_2$ is a consequence of $E_1 \cup E_2$ iff there exists a term t over $\Sigma_1 \cap \Sigma_2$, called the *interpolant*, such that $t_1 = t$ is a consequence of E_1 and $t = t_2$ is a consequence of E_2.

Together, the conservativity and interpolation properties mean that the equivalence relation defining the pushout P is trivial on sequences of length 1.

It is a natural idea to try to find a solution of the word problem in the pushout P in two stages: first to guarantee the conservativity and interpolation properties, and then, exploiting them, solve the problem. In this section the first stage is dealt with. In particular, we show that these properties correspond to classical notions of monoid amalgamation theory. This gives us many tools for checking whether these properties hold for a given union of theories. In the next section, the problem of "lifting" the solution for length 1 sequences to sequences of arbitrary length is considered, i.e., the problem of whether these properties can help us solve the word problem in P.

Let $\mu : S \to P$ and $\nu : T \to P$ be the pushout of the amalgam $[U \subseteq_{i,j} S, T]$. The word problem in P for sequences of length one is simply the problem of checking whether:

- $\mu(s) = \mu(s')$, for $s, s' \in S$,
- $\mu(s) = \nu(t)$, for $s \in S$, $t \in T$,
- $\nu(t) = \nu(t')$, for $t, t' \in T$.

It is clear that on the level of terms these problems are equivalent to checking whether the following equalities are consequences of the union theory $E_1 \cup E_2$:

- $s = s'$, for Σ_1-terms s and s',
- $s = t$, for a Σ_1-term s and a Σ_2-term t,
- $t = t'$, for Σ_2-terms t and t'.

In monoid theory, the problem of an amalgam $[U \subseteq_{i,j} S, T]$ being *embeddable* is considered, i.e., the question whether there exists a monoid Z such that $S, T \subseteq Z$, perhaps with the inclusions replaced by some homomorphisms. It can be easily shown that an amalgam is embeddable iff it is embeddable in its pushout P. This leads to the following definition:

- an amalgam is *weakly embeddable* if $\mu : S \to P$ and $\nu : T \to P$ are monomorphisms,
- an amalgam is *embeddable* if it is weakly embeddable and $\mu(S) \cap \nu(T) = \mu(i(U))$.

Note that, of course, $\mu(i(U)) = \nu(j(U))$.

Now, it is clear that an amalgam is weakly embeddable if and only if equalities of the form $\mu(s) = \mu(s')$ or $\nu(t) = \nu(t')$ hold in P iff $s = s'$ in S or $t = t'$ in T. On the level of terms this means that to check such an equality with respect to the theory $E_1 \cup E_2$ it suffices to check it with respect to the theories E_1 or E_2, respectively. In other words, the inclusions $E_1 \subseteq E_1 \cup E_2$ and $E_2 \subseteq E_1 \cup E_2$ are conservative. Thus:

Proposition 1. *The union theory $E_1 \cup E_2$ has the conservativity property iff the corresponding monoid amalgam is weakly embeddable.* □

It is also clear that an amalgam is embeddable if and only if it is weakly embeddable and, additionally, any equality $\mu(s) = \nu(t)$ holds in P iff $s, t \in U$ and $s = t$. On the level of terms embeddablity thus means that, in addition to weak embeddability, for all terms s over Σ_1 and t over Σ_2, the equation $s = t$ is a consequence of $E_1 \cup E_2$ iff there exists a term u over $\Sigma_1 \cap \Sigma_2$ such that $s = u$ is a consequence of E_1 and $u = t$ is a consequence of E_2. Thus:

Proposition 2. *The union theory $E_1 \cup E_2$ has the conservativity and interpolation properties iff the corresponding monoid amalgam is embeddable.* □

The following proposition simplifies checking that the equations considered at the beginning of this section hold. In particular, it may be used to prove the embeddability of an amalgam.

Proposition 3. *For any amalgam $[U \subseteq_{i,j} S, T]$, if $\sigma : S \to T$ is a homomorphism satisfying $i; \sigma = j$, then:*

- $\mu(s) = \nu(t)$ *implies* $\sigma(s) = t$,
- $\nu(t) = \nu(t')$ *iff* $t = t'$. □

In particular, if $\sigma : S \to T$ is an isomorphism, then the amalgam is weakly embeddable. Also, $\mu(s) = \nu(t)$ then implies that s and t are connected via the isomorphism, i.e., that $\sigma(s) = t$ and $\sigma^{-1}(t) = s$. It should be noted that in this case, that is if the pairs $U \subseteq S$ and $U \subseteq T$ are isomorphic, checking $\mu(s) = \nu(t)$ may be performed by considering, in the pushout P, only sequences of length 2; this is a consequence of Isbell's Zigzag Theorem (see [Isb66,How96b]). In this case, the object in which equality should be checked is called the *tensor* and denoted by $S \otimes_U T$.

One can try to guarantee the embeddability of an amalgam by enforcing certain restrictions on the pairs $U \subseteq S$ and $U \subseteq T$. For instance, $U \subseteq S$ is said to be *unitary* if, for all $u \in U$ and $s \in S$, $us \in U$ implies $s \in U$ and $su \in U$ implies $s \in U$. On the theory level, this means that composing a "new" term with an "old" one must give something provably equal to an "old" term. The connection with bases used in Section 3 is as follows:

Proposition 4. *If S has a U-base and a U-cobase, then $U \subseteq S$ is unitary.*

Proof. Assume S has a U-base G and $su \in U$, with $s \in S$, $u \in U$. Then $s = gu'$, for some $g \in G$, $u' \in U$. Thus $su = (gu')u = g(u'u)$ and $su = 1(su)$. Hence, by the property of the base, $g = 1$ and so $s = u' \in U$. The second part of the proof is analogous. □

It has been shown that if $U \subseteq S$ and $U \subseteq T$ are unitary, then the amalgam is embeddable (see [How96b]). This gives one method for guaranteeing that the amalgam is embeddable.

Actually, even more has been shown (we report these results after [How96b], where [Ren86a,Ren86b] are cited). The pair $U \subseteq S$ is said to be *quasiunitary* if there exists a map $\phi : S \to S$ such that:

- $\phi(\phi(s)) = \phi(s)$, for all $s \in S$,
- $\phi(us) = u\phi(s)$ and $\phi(su) = s\phi(u)$, for all $u \in U$, $s \in S$,
- $us \in U$ or $su \in U$ implies $\phi(s) \in U$, for all $u \in U$, $s \in S$.

It is obvious that if $U \subseteq S$ is unitary, then it is quasiunitary, since then as ϕ one may take the identity mapping. It turns out that if $U \subseteq S$ and $U \subseteq T$ are quasiunitary, then the amalgam is embeddable.

In some cases one can be sure that an amalgam is (weakly) embeddable by simply inspecting the monoid U (called the "core" of the amalgam). A monoid U is said to be a *(weak) amalgamation base* if any amalgam which has it as core is (weakly) embeddable. For example, if U is a group, then it is unitary in any S, since $us \in U$ implies that $s = u^{-1}us \in u^{-1}U \subseteq U$. Thus, any group is an amalgamation base. Note that a theory corresponds to a group if, for any term t, there is a term t^* such that $t(t^*(x)) = x$ and $t^*(t(x)) = x$ are consequences of the theory.

Actually, a much stronger result is know. A monoid U is said to be *inverse* if one can define an operation $(\_)^* : U \to U$ such that $(u^*)^* = u$, $uu^*u = u$ and $(u_1u_2)^* = u_2^*u_1^*$ for all $u, u_1, u_2 \in U$. It can be shown that any inverse monoid is an amalgamation base (again, see [Hal78]).

The above considerations give us a variety of methods for guaranteeing that the considered union theory has the conservativity and interpolation properties. A natural question would be whether it is decidable to check, given decidable theories E_1 and E_2 conservatively extending E, whether $E_1 \cup E_2$ has the conservativity and interpolation properties. It turns out that the answer is negative: this problem is undecidable. This is an easy consequence of the result of Sapir [Sap00], who proves (Theorem 1.2) that the problem of whether an amalgam of two finite semigroups is embeddable in a semigroup is undecidable.

5 Lifting

In the previous sections we have defined the conservativity and interpolation properties of the union theory $E_1 \cup E_2$, and we showed how to guarantee that these properties indeed hold. These properties allow us to solve the word problem in the pushout P for words of length 1 (i.e., on the term level, for so-called *pure* terms); what remains to be done is to lift this solution to words of arbitrary length.

Unfortunately, without additional assumptions this turns out to be impossible in general. That is, the word problem in the pushout P of an embeddable amalgam may be undecidable, even if the word problems in S and T are decidable. More precisely, we construct finitely presentable monoids S and T with decidable word problems and such that the amalgam $[U \subseteq S, T]$ is embeddable, but the word problem in its pushout P is undecidable.

Let MT be a deterministic, universal Turing machine. We may assume that there is exactly one accepting configuration and that in this configuration the machine is blocked. The problem of checking whether there is a computation in MT from a given configuration to the accepting configuration is undecidable. Now consider the problem of checking whether there is a two-way computation between those configurations, that is, a computation in which the machine may go in "reverse mode". Observe that since M is deterministic, if such a computation exists, then one may assume that it consists of a block of forward moves followed by a block of reverse moves. But since the last reverse move would have to end in the (blocked) accepting configuration, there cannot be any reverse moves. Therefore the two-way computation is actually a usual one-way computation in MT. Hence, the problem of checking whether a two-way computation exists is undecidable.

Let M be any Turing machine. Let Q be its set of states and Γ its set of tape symbols, with the symbol $0 \in \Gamma$ serving as blank; we assume that Q and Γ are disjoint. Also, let $\#$ be a special, distinct symbol which we will use to mark the end of the tape. Let U be the free monoid 0^* and let S be the monoid generated by 0 and $\#$ and satisfying the relation $\# = 0\#$. Note that U is a submonoid of S and that S has a decidable word problem. Also, the submonoid $G_S = \#^*$ is a U-base of S.

Let T be a monoid generated by $Q \cup \Gamma$ and satisfying the following relations: if in the state $q \in Q$ with the head over $a \in \Gamma$ the machine goes right and replaces a by $b \in \Gamma$ and changes its state to $q' \in Q$, then $aqx = bxq'$ for all $x \in \Gamma$; if it goes left then $aq = q'b$. Note that U is a submonoid of T, since if two elements are equal in T, then they must have the same length. Also, T has a decidable word problem, again because the relations do not change the length of elements of T.

Now let P be the pushout of the amalgam $[U \subseteq S, T]$. A *configuration* is a sequence of the form $a_1 \ldots a_n q b_1 \ldots b_k \#$, where $n > 0$, $k \geq 0$, $a_1, \ldots, a_n \in \Gamma$, $b_1, \ldots, b_k \in \Gamma \setminus \{0\}$, $q \in Q$. The problem of checking whether two configurations are equal in the pushout is equivalent to checking whether there is a two-way computation between the corresponding configurations of the Turing machine M. If M is the previously considered deterministic, universal Turing machine MT, then the latter problem is undecidable. Thus, if the machine is appropriately chosen, the word problem in P is undecidable.

It is easy to see that $U \subseteq S$ is unitary, since there is no way of removing the symbol $\#$ once it has been inserted, and an element of S not containing $\#$ at all must, in fact, be an element of U. Similarily, $U \subseteq T$ is unitary, since if $ut \in U$ for some $u \in U$ and $t \in T$, then t cannot possibly contain an element of Q, since this element could not be disposed of. But if t does not contain any element of Q, then no relation can be applied to ut, because all relations require at least one $q \in Q$ to appear. Hence, $ut \in U$ implies that t is built of 0s only, i.e., $t \in U$.

Thus, by the fact mentioned in the previous section, the amalgam $[U \subseteq S, T]$ is embeddable.

The above results may be summarized as follows:

Theorem 3. *There exists a monoid amalgam* $[U \subseteq S, T]$ *such that:*

- S *and* T *have decidable word problems,*
- *there exists a* U*-base of* S*,*
- *the pairs* $U \subseteq S$ *and* $U \subseteq T$ *are unitary and so the amalgam is embeddable,*

and such that the pushout P *has an undecidable word problem.* □

It may be noted that the monoid U is of course isomorphic to $\langle \mathbb{N}, + \rangle$, while S is isomorphic to the monoid $\langle \mathbb{N} \times \mathbb{N}, \cdot \rangle$ with multiplication defined by $(n, m) \cdot (n', m') = (n + n', m')$; finally, the inclusion of U in S takes any number n to the pair $(0, n)$.

The above theorem lets us state the following corollary for theories:

Corollary 2. *There exist decidable theories E_1 and E_2 conservatively extending a theory E and such that the union $E_1 \cup E_2$ has the conservativity and interpolation properties, but is undecidable.* □

This negative result should not be understood as meaning that the idea of exploiting conservativity and interpolation is flawed. However, what it means is that an appropriate property is still to be found that would facilitate the lifting of the conservativity and interpolation properties to a full solution of our problem. The fact that the monoid S has a U-base suggests that perhaps this property should not involve the existence of bases and that a new idea is needed here. We believe that, even from a purely algebraic point of view, the following question is natural and interesting: to what extent is the decidability of the word problem in the pushout of an *embeddable* amalgam a property stronger than the decidability of the word problem in S and T? That it is a stronger property at all has been proved above.

6 Conclusions

In the paper we investigate the problem of deciding the union of two decidable theories, but restricted to the monadic case. This problem is shown, via an equivalence of categories, to be equivalent to the decidability of the word problem in the pushout of an amalgam of monoids with decidable word problems.

Moving to the category of monoids leads to a simplified view of the problem, allowing us to produce a simple and easily comprehensible statement and proof of the known fact that the existence of bases guarantees decidability of the union theory. This simple proof should help in the search for yet more general theorems, in which assumptions are made that are weaker than the, admittedly rather strong, assumption on the existence of bases.

In the sections which follow, we make use of the equivalence of the considered problem to problems concerning monoid amalgams. We show that the conservativity and interpolation properties of the union theory, properties used, e.g., in [BT02], actually correspond to certain classical properties of amalgams, namely to weak embeddability and embeddability. This allows one to use the various ideas stemming from amalgamation theory to guarantee and prove that the union theory has the conservativity and interpolation properties. In particular, notions of unitary and quasiunitary extensions may be used here, as well as the property of the "core" theory E corresponding to an inverse monoid.

We then propose a program of proving decidability of the union theory by first guaranteeing conservativity and interpolation, and then exploiting these properties. While, as mentioned above, it turns out that the first part of the program may indeed be carried out, implementing the second part is shown to require some new approach. For, as stated in Corollary 2, the decidability of E_1 and E_2 plus conservativity and interpolation of the union theory $E_1 \cup E_2$ do not guarantee its decidability.

The latter result gives a clear suggestion on possible future work: one should look at properties of E_1 and E_2 that would guarantee that an implication of the above form holds. An additional argument in favor of this idea is that, in general, it is hardly reasonable to expect that a union theory without the conservativity and interpolation properties will be guaranteed decidable. Therefore we do not really lose much if we assume that these properties hold. The above task, in monoid terms, amounts to looking for such properties of embeddable amalgams of monoids with decidable word problems that would be strong enough to guarantee that the word problem in the pushout of the amalgam is decidable.

Another obvious direction of future work is a generalization to non-monadic theories. Of course, with respect to Theorem 2, this could simply lead back to the solution found in [FG03] and to categories with products and product-preserving functors. What is needed, however, is the development of an appropriate amalgamation theory, which would generalize known results for monoids to objects representing arbitrary equational theories. It remains to be seen whether categories with products and product-preserving functors turn out to be the ideal notion of such an object. It could also be fruitful to perform a limited generalization, e.g., to use operads instead of monoids: operads correspond to equational theories which are *strongly regular*, that is, defined by equations in which, on either side, the same sequence of variables appears (see [Lei04]).

Finally, a rather natural generalization, and one which should not involve too much difficulties, is switching from single-sorted to multi-sorted equational logic. For monadic theories this change simply means that, instead of monoids, one has to use categories. It seems that results concerning monoids transfer to categories, in particular, Theorem 2 holds for categories as well; from this an appropriate theorem on multi-sorted equational theories may be deduced. It may be noted that using multi-sorted logic and the operad approach proposed above would lead to the use of yet another categorical idea, namely of multicategories [Lei04], since it is precisely multicategories that may be used to represent strongly regular multi-sorted equational theories.

References

[BT02] Franz Baader and Cesare Tinelli. Deciding the word problem in the union of equational theories. *Information and Computation*, 178(2):346–390, 2002.

[FG03] Camillo Fiorentini and Silvio Ghilardi. Combining word problems through rewriting in categories with products. *Theoretical Computer Science*, 294:103–149, 2003.

[Hal78] Tom E. Hall. Amalgamation and inverse and regular semigroups. *Transactions of the AMS*, 246:395–406, 1978.

[Hig92] Peter M. Higgins. *Techniques of Semigroup Theory.* Oxford University Press, 1992.

[How96a] John M. Howie. *Fundamentals of Semigroup Theory.* London Mathematical Society Monographs, New Series, No. 12. Oxford University Press, 1996.

[How96b] John M. Howie. Isbell's zigzag theorem and its consequences. In K. H. Hofmann and M. W. Mislove, editors, *Semigroup Theory and its Applications,* *Proceedings of the 1994 Conference Commemorating the Work of Alfred H. Clifford, New Orleans, LA*, London Mathematical Society Lecture Notes Series Vol. 231, pages 81–91. Cambridge University Press, 1996.

[Isb66] John R. Isbell. Epimorphisms and dominions. In *Conference on Categorical Algebra, La Jolla, California, 1965, Proceedings*, pages 232–246. Springer, 1966.

[Lei04] Tom Leinster. *Higher Operads, Higher Categories.* London Mathematical Society Lecture Notes Series Vol. 298. Cambridge University Press, 2004.

[Pig74] Don Pigozzi. The join of equational theories. *Colloquium Mathematicum*, 30(1):15–25, 1974.

[Ren86a] James Renshaw. Extension and amalgamation in monoids and semigroups. *Proceedings of the London Mathematical Society (3)*, 52:119–141, 1986.

[Ren86b] James Renshaw. Flatness and amalgamation in monoids. *Journal of the London Mathematical Society (2)*, 33:73–88, 1986.

[Sap00] Mark V. Sapir. Algorithmic problems for amalgams of finite semigroups. *Journal of Algebra*, 229(2):514–531, 2000.