Teoria współbieżności

Piotr Hofman

Theoretical aspects of concurrency

Lecture 3-4

HML and Bisimulation

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Lemma

If two Kripke structures S and S' have the same traces i.e. $\mathbb{TR}(S) = \mathbb{TR}(S')$ then for any LTL formula ϕ does not distinguish S and S' .

Concluding: some interesting properties can not be analysed if we look only into traces.

Transition Tree (derivation tree)

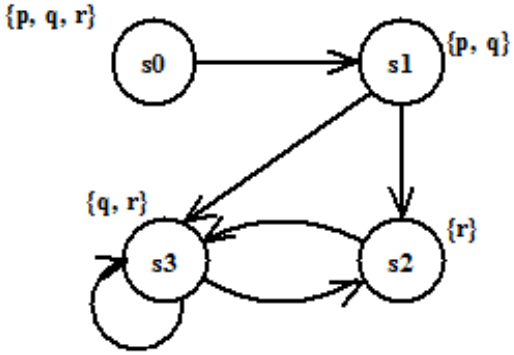

Figure 15. Example of a Kripke structure

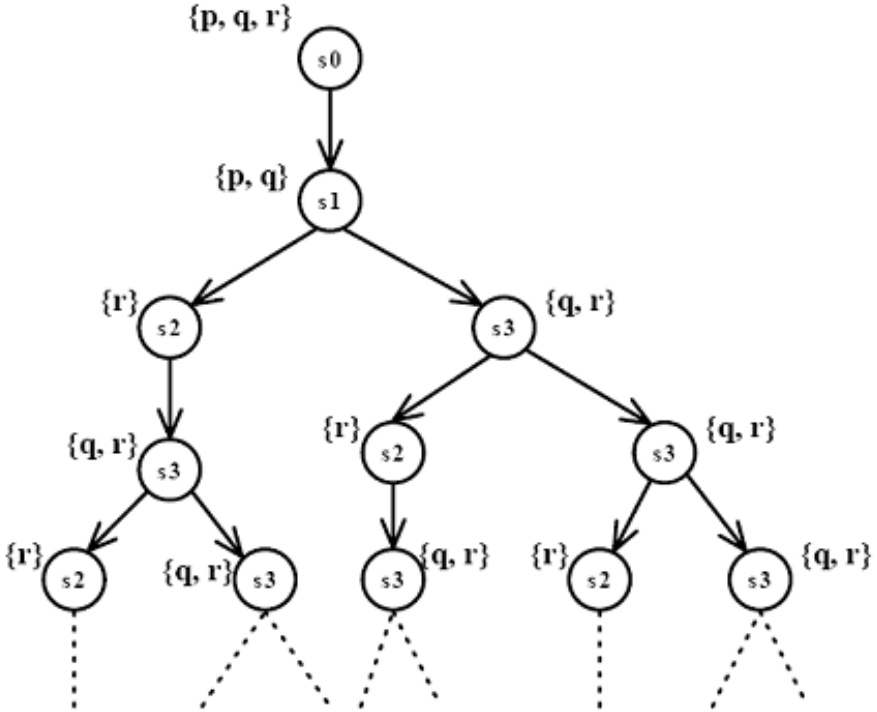

Figure 16. The unfolded tree of the Kripke structure of Figure

HMS Logic

Hennessy–Milner logic

A minimum to speak about tree, $\phi =$

- tt ($true$), ff ($false$),
- $\phi_1 \vee \phi_2$,
- $\phi_1 \wedge \phi_2$,
- $[]\phi$ or $\Box\phi$ or $AX\phi$,
- $<> \phi$ or $\Diamond\phi$ or $EX\phi$.

Semantic of HML

Formula is always evaluated in a node of the transition tree.

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Unlabelled case:

Definition

Bisimulation B is any relation on a set of configurations (nodes) that satisfies following conditions

1. if $(s, s') \in B$ then for every t such that $s \to t$ there is t' such that $s' \to t'$ and $(t, t') \in B$,
2. if $(s, s') \in B$ then for every t' such that $s' \to t'$ there is t such that $s \to t$ and $(t, t') \in B$.

We denote it by $s \sim_B s'$.

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Bisimulation relation

Theorem

A pair of configurations (s, s') can not be distinguished by HML formula if and only if there is a bisimulation relation B such that $s \sim_B s'$.

We need a few lemmas.

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Corollary

There is a biggest bisimulation.

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- The biggest bisimulation is called the bisimilarity relation and denoted by $\sim$.
- We say that two configurations (s, s') are bisimilar if $s \sim s'$.

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- Bisimilarity is an equivalence relation.

The proof of the theorem (idea).

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1. By negation, suppose there is a formula ϕ that distinguishes (s, s'), we prove that (s, s') is not en element of any bisimulation relation.

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1. Let X be a set of pairs of states that can not be distinguished by HML.
2. We prove that X is a bisimulation relation.

CTL and CTL*

An extension of HML - CTL

Definition

1. A state formula $\phi = tt|\neg\phi_1|\phi_1 \wedge \phi_2|p_i|A\alpha|E\alpha$
2. A path formula (restricted LTL) $\alpha = X\phi_1|\phi_1 U\phi_2|F\phi_1|G\phi_1$
3. where ϕ are state formulas and α are path formulas.

Semantics

- p_i means that the predicate p_i holds in the configuration in which the formula is evaluated (current configuration).
- $A\alpha$ for every run r starting at the current configuration the formula α holds for a sequence of states of r.
- $E\alpha$ there is a run r starting at the current configuration such that the formula α holds for the sequence of states of r.

Exercise

- $F\phi = trueU\phi$,
- $G\phi = \neg(F\neg\phi)$

Which of the following pairs of CTL formulas are equivalent? For those which are not, find a model of one of the pair which is not a model of the other:[a]

[a] exercise from https://www.win.tue.nl/ andova/education/2IF25/Ex2Solutions.pdf

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- Write a *CTL* formula which stays that there is always a possibility of braking.

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Evaluation of LTL in a state.

We consider all traces starting in a given state.

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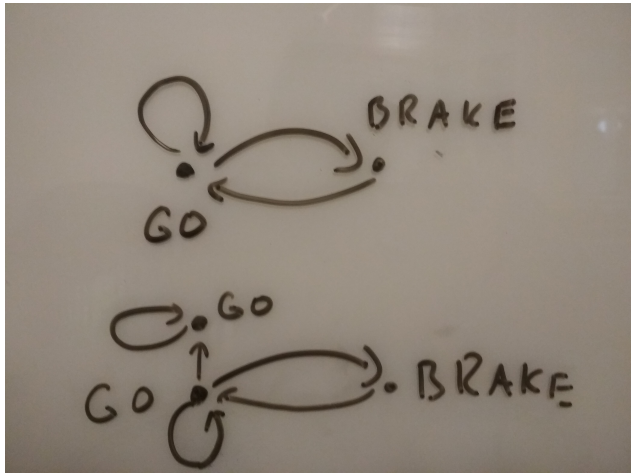

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1. $(T_i, s_0) \models AFG(black = true)$ but $(T_i', s_0) \not\models AFG(black = true)$.
2. (T_i, s_0) and (T_i', s_0') are not distinguished by any CTL formula of the modal depth i.

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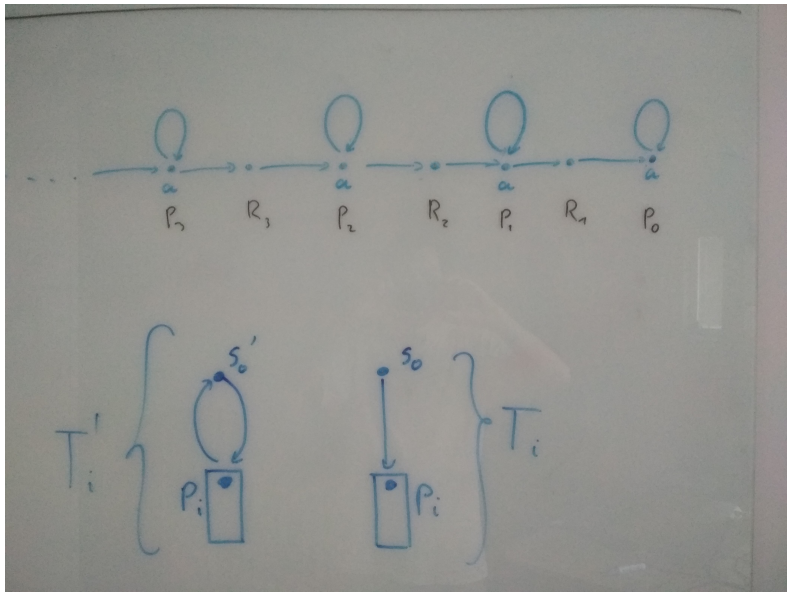
Look: `https://www.youtube.com/watch?v=0Af7q3X71-o` (min 58)

Proof.



Let $\sim_i$ not distinguishable by CTL formulas of depth i.

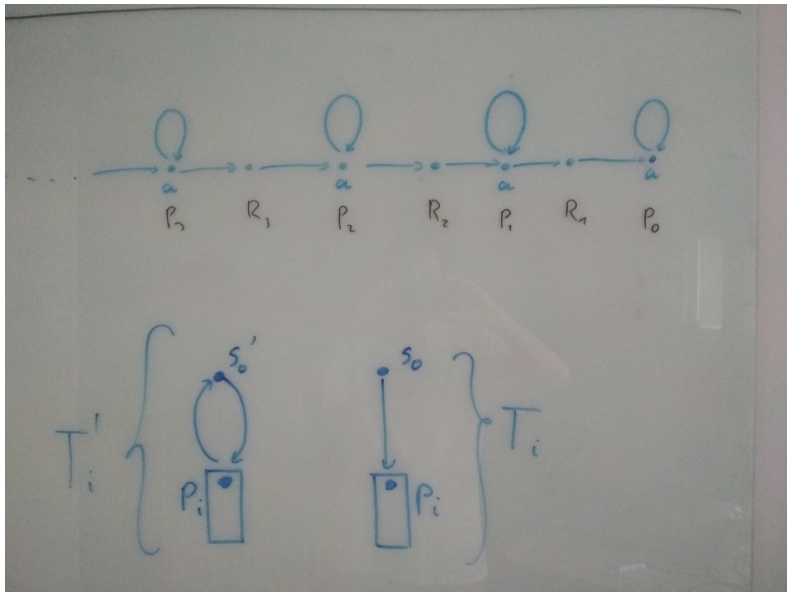
Lemma (Auxiliary)

$P_i \sim_i P_j$ for $i \leq j$. $R_i \sim_i R_j$ for $i \leq j$.

Proof.

Via induction on i, (the size of the formula).
Induction hypothesis:
$P_k \sim_i P_j$, $R_k \sim_i R_j$ for $i \leq k \leq j$. $\square$

Proof.



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$T_i \sim_i T_i'$.

Via induction on i, (the size of the formula).
Induction hypothesis:
$T_k \sim_i T_j'$ for $i \leq k \leq j$.

$\square$

Evaluation of CTL

Lemma

A CTL formula ϕ can be evaluated in time proportional to the length of the formula times size of the Kripke structure.

Proof.

By induction on the derivation tree of the formula. $\square$

Bisimulation and CTL

Lemma

Two configurations s and s' are bisimilar if and only if s and s' can not be distinguished by any CTL formula ϕ.

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- If they are not distinguishable by CTL, then they are not distinguishable by HML with predicates.

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- Indeed, HML with predicates is a fragment of CTL.
- We already proved that if (s, s') are not distinguished by HML then they are bisimilar.

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Proof $\rightarrow$

We need to extend our understanding of bisimulation first.

Game characterisation of Bisimilarity

Definition

A bisimulation game is played in rounds between two players Spoiler and Duplicator. Arena is a set of pairs of configurations of the Kripke structure. Suppose that current pair of configurations is (s, s').
Rules of a round are as follows:

- First Spoiler chooses one of configurations s or s'. Without lost of generality we may assume that it is s.
- Next he chooses a configuration t such that $s \to t$.
- Next Duplicator chooses a configuration t' such that $s' \to t'$ where s' is a configuration no chosen by Spoiler.
- The next round of the game will be plaid from (t, t').

Winning conditions:

- If $L(s) \neq L(s')$ then Spoiler wins.
- If any player can not make his part of the move then he looses.
- Infinite plays are won by Duplicator.

Lemma

Duplicator has a winning strategy in the bisimulation game starting from a pair of configurations (s, s') *if and only if* $s \sim s'$.

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Lemma

A winning strategy for Spoiler is a tree.

Bisimulation and CTL

Lemma

Two configurations s and s' are bisimilar if and only if s and s' are not distinguished by any CTL formula ϕ.

Proof $\rightarrow$ (bisimilar $\implies$ not distinguishable by any CTL formula).

- If they are distinguishable by CTL, then they are distinguishable by some formula ϕ.
- We construct a winning strategy for Spoiler via induction on the modal depth of the formula.

Extend even more - CTL^*

Definition

A state formula $\phi = tt|\neg\phi_1|\phi_1 \wedge \phi_2|p_i|A\alpha|F\alpha$
A path formula (restricted LTL) $\alpha = \phi|\neg\alpha_1|\alpha_1 \wedge \alpha_2|X\alpha_1|\alpha_1 U\alpha_2$

Semantics

- $A\alpha$ for every run r starting at the current configuration the formula α holds for $\mathbb{TR}(r)$.
- $F\alpha$ there is a run r starting at the current configuration such that the formula α holds for $\mathbb{TR}(r)$.

Fact

CTL^* subsumes CTL and LTL.

Bibliography

CTL, CTL*

https://www.youtube.com/channel/UCUXDMaaobCO1He1HBiFZnPQ

Unites: 9, 10, 11.

Bisimulation + CTL and more:

https://pdfs.semanticscholar.org/cb9f/325389bd6ee5894dcf435159d34f9e20da2d.pdf