

VASS reachability

Preparations

Solution of linear equations in \mathbb{N}

How to describe a set of solutions of a system of linear equations?

We say that a vector $\vec{x} \leq \vec{y}$ if $\vec{x}(i) \leq \vec{y}(i)$ for all i .

Definition

We say that $\vec{x} \in \mathbb{N}^d$ is a minimal solution of a system $A\vec{x} = 0$ if

- ① \vec{x} is a solution $A\vec{x} = 0$.
- ② for any nontrivial solution \vec{y} of $A\vec{y} = 0$ holds $\vec{y} \not\leq \vec{x}$.

A set of minimal solutions of a system of linear equations we called a Hilbert basis of the set of solutions.

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Lemma

Hilbert basis is finite.

Bound on the element of the basis.

Let r be a rank of a matrix A and $\|M\|_1 = \sup_{\vec{x} \in \text{Hilbert}(A\vec{x}=0)} \|\vec{x}\|_\infty$.
Let $\|A\|_{1,\infty} = \sup_i \{\sum_j |A(i,j)|\}$.

Theorem

$$\|M\|_1 \leq (1 + \|A\|_{1,\infty})^r$$

Proof.

Minimal solutions of linear Diophantine systems: bounds and algorithms
Loic Pottier. □

Lemma

Let $\mathcal{M} \in \mathbb{N}^d$ be a set of solutions of $A\vec{x} = \vec{b}$. There are two sets \mathcal{B} and \mathcal{P} such that any $\vec{x} \in \mathcal{M}$ can be expressed as $\vec{x}_1 + \vec{x}_2$ where $\vec{x}_1 \in \mathcal{B}$ and \vec{x}_2 is a sum of elements of \mathcal{P} .

and

$$\forall_{\vec{x} \in \mathcal{B} \cup \mathcal{P}} \|\vec{x}\|_1 \leq (2 + \|A\|_{1,\infty} + \|b\|_\infty)^m$$

where m is equal to the number of rows in the matrix A .

Proof.

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VASS reachability

INPUT: The VASS V and two configurations i and f .

QUESTION: If there is a run from i to f in the VASS V .

The general concept.

We have a set of conditions.

- 1 Check the conditions.
- 2 If the conditions are satisfied then the reachability holds.
- 3 If they fail in a certain way then the reachability does not hold.
- 4 otherwise, there is a procedure to simplify the net, and go to point 1.

The simplification process has to terminate.

Three steps.

- 1 First, we formulate the conditions in a simplified versions 1 and 2. Next, the final third version.
- 2 For each version we prove that the conditions implies the reachability.
- 3 For the third version we define the simplification procedures.
- 4 Next we define a well founded order on VASS-es such that the simplification procedure returns a net smaller with respect to the order.

\mathbb{Z} runs.

\mathbb{Z} semantics for VASS

Suppose VASS is d -dimensional. A set of configurations is equal $Q \times \mathbb{Z}^d$ and there is a step from p, \vec{v}_1 to q, \vec{v}_2 if there is a transition in the VASS from p to q labelled with $\vec{v}_2 - \vec{v}_1$.

\mathbb{Z} -reachability

There is a \mathbb{Z} -run from p, \vec{v}_1 to q, \vec{v}_2 if the pair of configurations (p, \vec{v}_1) , (q, \vec{v}_2) is in the transitive closure of the step relation.

We denote it by $p, \vec{v}_1 \Rightarrow_{\mathbb{Z}} q, \vec{v}_2$

Lemma

$p, \vec{v}_1 \Rightarrow_{\mathbb{Z}} q, \vec{v}_2$ iff there is a solution of the state equation for configurations $(p, \vec{v}_1), (q, \vec{v}_2)$.

The step one.

Let $i = p, \vec{v}$ and $f = q, \vec{v}'$.

Θ_1

For every $m \in \mathbb{N}$ there is a \mathbb{Z} run from p, \vec{v}_1 to q, \vec{v}_2 that uses every transition at least m times.

Θ_2

There are vectors Δ and Δ' strictly positive on all coordinates such that there are runs

$$p, \vec{v} \Rightarrow p, \vec{v} + \Delta$$

$$q, \vec{v}' + \Delta' \Rightarrow q, \vec{v}'$$

The step two.

split of the vector

Let D be the set of dimensions. We split it to two sets constrained and unconstrained C and \bar{C} . By $\vec{v} \oplus_C \vec{\bar{v}}$ we mean a sum of vectors \vec{v} and $\vec{\bar{v}}$ where \vec{v} is zero outside of C and $\vec{\bar{v}}$ is zero outside \bar{C} .

Partially unconstrained reachability

Input: a VASS (Q, E) , two subsets $C, C' \subseteq D$, $(p, \vec{v}) \in Q \times \mathbb{N}^C$ and $(q, \vec{v}') \in Q \times \mathbb{N}^{C'}$.

Question: does there exist a run from $(p, \vec{v} \oplus \vec{\bar{v}})$ to $(q, \vec{v}' \oplus \vec{\bar{v}}')$, for some $\vec{\bar{v}} \in \mathbb{N}^{\bar{C}}, \vec{\bar{v}}' \in \mathbb{N}^{\bar{C}'}$?

We do not assume $C = C'$.

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The step two.

Let $C, C' \subseteq D$, $\vec{v} \in \mathbb{N}^C$, and $\vec{v}' \in \mathbb{N}^{C'}$

By $\mathbb{N}_{\geq m}$ we mean the set of natural number greater than m .

By \Rightarrow_C we mean \mathbb{Z} -run such that intermediate configurations are non-negative on coordinates form the set C .

Θ_1

For every $m \in \mathbb{N}$ there is a \mathbb{Z} run from $p, \vec{v} \oplus \vec{\bar{v}}$ to $q, \vec{v}' \oplus \vec{\bar{v}'}$ such that

- $\vec{\bar{v}} \in \mathbb{N}_{\geq m}^{\bar{C}}$ and $\vec{\bar{v}'} \in \mathbb{N}_{\geq m}^{\bar{C}'}$,
- every transition at least m times.

Θ_2

There are vectors $\Delta \in \mathbb{N}_{\geq 1}^C$, $\bar{\Delta} \in \mathbb{Z}^{\bar{C}}$, $\Delta' \in \mathbb{N}_{\geq 1}^{C'}$, and $\bar{\Delta}' \in \mathbb{Z}^{\bar{C}'}$ such that

$$p, \vec{v} \oplus \vec{\bar{0}} \Rightarrow_{C_i} p, \vec{v} \oplus \vec{\bar{0}} + \Delta \oplus \bar{\Delta}$$

$$q, \vec{v}' \oplus \vec{\bar{0}} + \Delta' \oplus \bar{\Delta}' \Rightarrow_{C'_i} p, \vec{v}' \oplus \vec{\bar{0}}$$

The step three

By a component we mean a d -dimensional VASS (Q, E) together with the following data:

- 1 initial and final state $q, q' \in Q$,
- 2 a subset of rigid coordinates $D_{\bar{c}}$,
- 3 every transition is constant on rigid coordinates,
- 4 rigid vector $\vec{v} \in \mathbb{N}^{D_{\bar{c}}}$
- 5 two partitions of $D \setminus D_{\bar{c}}$ to $D_C, D_{\bar{c}}$ and $D_{C'}, D_{\bar{c}'}$,
- 6 initial and final vectors $\vec{v} \in \mathbb{N}^{D_C}$ and $\vec{v}' \in \mathbb{N}^{D_{C'}}$.

The generalized VASS

It is a finite sequence of d -dimensional components Com_i and a sequence of edges going from final state of Com_i to the initial state Com_{i+1} .

The step three.

generalized reachability

Input: a generalized VASS $Com_1 e_1 Com_2 e_2 Com_3 e_3 \dots Com_l$.

If there is a sequence of pairs of vectors $\vec{v}_i, \in \mathbb{N}^{D_{\vec{c}_i}}$, and $\vec{v}'_i, \in \mathbb{N}^{D_{\vec{c}'_i}}$. such that there is a run of the following form

$$q_1, \vec{v}_1 \oplus \vec{v}_1 \oplus \vec{v}_1 \Rightarrow q'_1, \vec{v}'_1 \oplus \vec{v}'_1 \oplus \vec{v}_1 \xrightarrow{e_1}$$

$$q_2, \vec{v}_2 \oplus \vec{v}_2 \oplus \vec{v}_2 \Rightarrow q'_2, \vec{v}'_2 \oplus \vec{v}'_2 \oplus \vec{v}_2 \xrightarrow{e_2}$$

...

$$q_l, \vec{v}_l \oplus \vec{v}_l \oplus \vec{v}_l \Rightarrow q'_l, \vec{v}'_l \oplus \vec{v}'_l \oplus \vec{v}_l$$

The step three.

Θ_1

For every $m \in \mathbb{N}$ there are configurations $\vec{v}_i \in \mathbb{N}_{\geq m}^{D_{\vec{c}_i}}$, and $\vec{v}'_i \in \mathbb{N}_{\geq m}^{D_{\vec{c}'_i}}$ such that there is a Z -run of a form of the generalized run, where transition in the components Com_i are used at least m times.

Θ_2

For every $i \in \{1 \dots l\}$ there are vectors $\bar{\Delta}_i \in \mathbb{N}^{D_{\vec{c}_i}}$ and $\bar{\Delta}'_i \in \mathbb{N}^{D_{\vec{c}'_i}}$ such that

$$q_i, \vec{v}_i \oplus \vec{0}_i \oplus \vec{v}_i \Rightarrow_{D_{C_i}} q_i, \vec{v}_i + \Delta_i \oplus \vec{0}_i + \bar{\Delta}_i \oplus \vec{v}_i.$$

$$q'_i, \vec{v}'_i + \Delta'_i \oplus \vec{0}_i + \bar{\Delta}'_i \oplus \vec{v}'_i \Rightarrow_{D_{C'_i}} q'_i, \vec{v}_i \oplus \vec{0}'_i \oplus \vec{v}_i.$$

The order on generalized nets.

component

For a component its rank is a triple $(d - |D_{\bar{c}}|, |E|, |D_{\bar{c}}| + |D_{\bar{c}'}|) \in \mathbb{N}^3$.
We order ranks lexicographically.

generalized VASSt

For a generalized VASS its rank is a multiset of ranks of its components, ordered lexicographically.

The order on generalized VASS is well founded.

Lemma

All refinement operations are reducing the rank of the generalized VASS.

Proof

Look to the notes and paper:

- ① https://www.mimuw.edu.pl/~sl/teaching/16_17/Kosaraju.pdf
- ② http://delivery.acm.org/10.1145/810000/802201/p267-kosaraju.pdf?ip=193.0.96.15&id=802201&acc=ACTIVE%20SERVICE&key=6AF5E6E07E3D4A13%2EF25B909119C68FF3%2E4D4702B0C3E38B35%2E4D4702B0C3E38B35&__acm__=1527858390_f66d35b467337d81ea2d7b79cbeeb15b