Subclasses of Petri Nets

Questions and tools.

We focus on analysis of systems modelled with Petri nets.

Most important questions:

1. Place coverability,
2. Reachability,
3. Liveness,
4. Death of a transition,
5. Deadlock-freeness.

Most important tools:

1. Coverability: ExpSpace complete,
2. Boundedness: ExpSpace complete,
3. Reachability: at least Tower-Hard.

Two solutions:

Do not try to be precise (approximations).

1. Place invariant.
2. State equation.
3. Continuous reachability.
4. Traps and siphons.

Do not try to be general (sub-classes).

1. S-systems
2. T-systems
3. Free-choice Petri Nets.
4. Conflict free Petri nets.
5. One counter systems.
6. 2-dimensional VASS.

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Today Petri Nets do not have numbers on arcs

S-system

Definition

A net is an *S-net* if every transition consumes and produces exactly one token.

1. A number of tokens does not change.
2. Bunch of asynchronous automata.

S-systems

Lemma

If the graph of S-system $\mathcal{N}$ is strongly connected and the initial marking m_0 marks at least one place, then the system is live.

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Let m be a marking, by:

- $m(X)$ we denote the marking restricted to the subset of places X.
- In addition, by $|m(X)|$ we denote the number of tokens used by the marking m restricted to X i.e. $\sum_{q \in X} |m(q)|$.

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Lemma

Let a graph of the $S - net$ $\mathcal{N} = (\mathbb{P}, \mathbb{T})$ is strongly connected and let $|m_0(\mathbb{P})| = |m_0'(\mathbb{P})|$ where m_0, m_0' are two markings. Then $m_0' \in Reach[m_0\rangle$.

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The answer:

1. It is in NP as we may specify the run.
2. With epsilon transitions it is $NP - hard$.

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The answer:

1. It is in NP as we may specify the run.
2. With epsilon transitions it is $NP - hard$.
3. What without epsilons? (Homework).

T-systems

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- By a circuit we mean a simple cycle in the graph of the net. Let R be a set of places that are on the circuit γ. Then $m(\gamma) \stackrel{\text{def}}{=} m(R)$.
- We say that the circuit γ is marked if $|m(\gamma)| > 0$.
- A circuit of a system is initially marked if it is marked at the initial marking.

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Lemma

Let γ be a circuit of a $T-net$ $(\mathcal{N}, m_0)$. For every reachable marking m holds $|m(\gamma)| = |m_0(\gamma)|$.

T-systems

Theorem

A T-system is live iff every circuit is initially marked.

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Theorem

Let $(\mathcal{N}, m_0)$ be a strongly connected T-system. The following statements are equivalent:

1. $(\mathcal{N}, m_0)$ *is live.*
2. $(\mathcal{N}, m_0)$ *is deadlock-free.*
3. $(\mathcal{N}, m_0)$ *has an infinite run.*

T-systems

Definition

A $\mathcal{N}, m_0$ is b-bounded iff for every reachable marking m and every place p the number of tokens in $m(p) \leqslant b$.

Lemma

A live T-system $\mathcal{N}, m_0$ is b-bounded iff for every place s there is a circuit γ which contains s and satisfies $|m_0(\gamma)| \leqslant b$.

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Lemma

Let $(\mathcal{N}, m_0)$ be a live T-system.

1. *A place q of $(\mathcal{N}, m_0)$ is bounded iff it belongs to some circuit of $\mathcal{N}$.*
2. *If a place s of $(\mathcal{N}, m_0)$ is bounded, then its bound is equal to*

$$min\{|m_0(\gamma)| : \gamma \textit{ is a circuit of } \mathcal{N} \textit{ containing } s\}$$

3. *$(\mathcal{N}, m_0)$ is bounded iff $\mathcal{N}$ is strongly connected.*

T-systems

Genrich's Theorem

Let $\mathcal{N}$ be a strongly connected T-system having at least one place and one transition. There exists a marking m_0 of $\mathcal{N}$ such that $(\mathcal{N}, m_0)$ is a live and 1-bounded system.

Siphons and traps

We introduce a new notation:

1. ${}^\bullet t$ for $t \in \mathbb{T}$ is a set of places from which t consumes tokens.
2. $t^\bullet$ for $t \in \mathbb{T}$ is a set of places to which t produces tokens.
3. ${}^\bullet p$ for $p \in \mathbb{P}$ is a set of transition that produce tokens in p.
4. $p^\bullet$ for $p \in \mathbb{P}$ is a set of transitions that consumes tokens from p.

1. ${}^\bullet T$ for $T \subseteq \mathbb{T}$ is $\bigcup_{t \in T} {}^\bullet t$.
2. $T^\bullet$ for $T \subseteq \mathbb{T}$ is $\bigcup_{t \in T} t^\bullet$.
3. ${}^\bullet P$ for $P \subseteq \mathbb{P}$ is $\bigcup_{p \in P} {}^\bullet p$.
4. $P^\bullet$ for $P \subseteq \mathbb{P}$ is $\bigcup_{p \in P} p^\bullet$.

Siphons and traps

We introduce a new notation:

Definition

A set of places $\mathbb{S}$ is a *siphon* if ${}^\bullet\mathbb{S} \subseteq \mathbb{S}^\bullet$.

Definition

A set of places $\mathbb{S}$ is a *trap* if $\mathbb{S}^\bullet \subseteq {}^\bullet\mathbb{S}$.

Siphons and traps

Lemma

Every unmarked siphon remains unmarked.

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Every unmarked siphon remains unmarked.

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Every marked trap remains marked.

Lemma

Every nonempty siphon of a live system is initially marked.

Siphons and traps

Lemma

Let $(\mathcal{N}, m_0)$ *be a deadlocked system i.e., no transition can be fired from* m_0. *Then the set* R *of places of* $\mathcal{N}$ *unmarked at* m_0 *is a proper siphon.*

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Lemma

1. *The union of siphons (traps) is a siphon (trap).*
2. *Every siphon includes a unique maximal trap with respect to the set inclusion (which maybe empty).*
3. *A siphon includes a marked trap iff its maximal trap is marked.*

Free-choice

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A free-choice system is live if and only if every proper siphon includes an initially marked trap.

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Lemma

Let $\mathcal{N} = (\mathbb{P}, \mathbb{T})$ be a free-choice net. Suppose $R \subseteq \mathbb{P}$ and $Q \subseteq R$ is a maximal trap included in R. Then there is an order on $\mathcal{O}$ in places in $R \setminus Q$ such that for any marking m, $|m(Q)| = 0$ holds:

1. *If there is a transition in $R^\bullet$ which is fireable then there is $m' \in Reach[m\rangle$ such that $|m'(Q)| = 0$ and $m' < m$ in the lexicographic order induced by $\mathcal{O}$.*
2. *If R is a siphon such that $R^\bullet \not\subseteq Dead(m)$ then there is $m' \in Reach[m\rangle$ such that $|m'(Q)| = 0$ and $m' < m$ in the lexicographic order induced by $\mathcal{O}$.*

Lemma

Non-liveness in free-choice nets is in NP.

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1. guess a set of places $\mathbb{S}$;
2. check if $\mathbb{S}$ is a siphon;
3. if $\mathbb{S}$ is a siphon, then compute the maximal trap $Q \subseteq \mathbb{S}$;
4. if $|\mathfrak{i}(Q)| = 0$, then answer "non-live".

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The algorithm computing maximal trap included in a given set $\mathbb{S}$.

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Input: A net $\mathcal{N} = (\mathbb{P}, \mathbb{T})$ and a set $\mathbb{S} \subseteq \mathbb{P}$
Output: $\mathbb{X} \subseteq \mathbb{S}$ which is the maximal trap contained in $\mathbb{S}$.
Initialization: $\mathbb{X} = \mathbb{S}$.
begin
while there exists $s \in \mathbb{X}$ and $t \in s^{\bullet}$ such that $t \notin {}^{\bullet}X$ do
$\mathbb{X} := \mathbb{X} \setminus \{s\}$
endwhile
end

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Non-liveness in free-choice nets is NP-hard.

Proof

1. We encode the satisfiability of 3 CNF problem.

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Proof

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Proof

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3. One state for every literal in the formula.
4. One state to indicate that valuation is False.
5. For every variable two transitions to chose a valuation.
6. For every clause there is a transition which can be fired if the clause is valuated to false and it marks the state False
7. One transition which consumes from False and restarts the net.

A characterisation of minimal siphons

Definition (A cluster in a free-choice net $\mathcal{N}$.)

Let x be a node (place of transition) of a net. The cluster of x, denoted by $[x]$, is a minimal set of nodes such that

- $x \in [x]$
- if a place s belongs to $[x]$ then $s^\bullet \subseteq [x]$
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Lemma

Clusters form a partition of the nodes of the $\mathcal{N}$.

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Lemma

A nonempty set of places $\mathbb{S} \subset \mathbb{P}$ is a minimal siphon in the net $\mathcal{N} = (\mathbb{P}, \mathbb{T})$ if and only if

1. *every cluster C of the $\mathcal{N}$ contains at most one place of $\mathbb{S}$, and*
2. *the graph of a subnet generated by $\mathbb{S} \cup {}^\bullet\mathbb{S}$ is strongly connected.*

Proof ($\Longrightarrow$)

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Proof ($\Longrightarrow$)

1. a) Let $\mathbb{S}$ be a siphon and C a cluster such that $|C \cap \mathbb{S}| > 1$ then we can shrink the siphon.

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Proof

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2. b) $\mathbb{S}$ has to be strongly connected.
   - Let (x, y) be an arbitrary arc in $\mathcal{N}_{\mathbb{S}} = (\mathbb{S}, {}^{\bullet}\mathbb{S})$, we prove that there is a path from y to x in $\mathcal{N}_{\mathbb{S}}$. Define

$$\mathbb{X} = \{s \in \mathbb{S} | \text{ there exists a path from } s \text{ to } x \text{ in } \mathcal{N}_{\mathbb{S}}\}.$$

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   - $\mathbb{X}$ is a siphon.
   - So $\mathbb{X} = \mathbb{S}$.

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Proof ($\Longleftarrow$)

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2. $\mathbb{S}$ is a minimal siphon.

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Proof ($\Longleftarrow$)

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Proof ($\Longleftarrow$)

1. $\mathbb{S}$ is a siphon, due to strong connectivity.
2. $\mathbb{S}$ is a minimal siphon. Take a smaller siphon and prove that it is not a siphon.

Books

1. S-sytems, T-systems, siphons and traps:
   Free Choice Petri Nets, chapters 3 and 4
   `https://www7.in.tum.de/~esparza/fcbook-middle.pdf`
2. Commoner's Theorem:
   A concise proof of Commoner's theorem.
   `http://www.cs.vsb.cz/jancar/pnn95.pdf`