Mazurkiewicz traces

Runs of the system.

We go back to the beginning, and take a look on the set of runs (traces) of a given system. We use them to characterise the behaviours of the system.

Example

2 independent automata Let A be an automaton that accepts words a^{10}, and B accepts b^{10}.

- What is the set of possible runs?
- How many of them there is?
- Is there a better way to describe the system?

Histories

Assumptions:

1. We have k processes running in parallel.
2. Each action of the process$_i$ has a unique name in Σ_i.
3. They communicate via rendezvous.
4. If a belongs to the alphabets of a few systems then all of them perform it synchronously.

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We consider a monoid $M = \Sigma_1^* \times \Sigma_2^* \times \ldots \Sigma_k^*$ with concatenation defined via coordinates
$(u_1, u_2, \ldots, u_k) + (w_1, w_2, \ldots, w_k) = (u_1 w_1, u_2 w_2, \ldots, u_k w_k)$.

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Let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \ldots \Sigma_k$. For $a \in \Sigma$ we define $u_a \in M$ as $(x_1, x_2, \ldots, x_k)$ where $x_i = \varepsilon$ if $a \notin \Sigma_i$ and $x_i = a$ if $a \in \Sigma_i$. We call the set of u_a elementary histories.

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Mathematical structure of histories

- Monoid structure.
- Projections goes form strings to the monoid of histories.

Algebraic approach

We consider strings over the alphabet Σ.

Dependency relation.

It is a relation on the letters from the alphabet Σ.

- reflexive,
- symmetric.

We denote by *Dep*.

Independency relation.

It is a a complement of the dependency relation. We denote it by *Ind*.

Mazurkiewicz traces

are the equivalence classes of the smallest congruence on the string monoid Σ^* that contains $ab \equiv ba$ if $(a, b) \in Ind$.

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Algebraic approach

We use:

- a, b, c for letters in Σ.
- x, y, u, v, w for strings in Σ^*.

The swap operation

Let $xaby$ we a string where $(a, b) \in Ind$ then $xbay$ is an effect of a swap operation.

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Lemma

Two strings u, v are in the same trace (equivalence class of the Mazurkiweicz traces relation) iff there is a sequence of swap operations that transforms u to v.

Algebraic approach

Our goal is to show that Mazurkiewicz traces and histories are isomorphic.

There is a theory similar to theory of strings developed for Mazurkiewicz traces.
There is an ongoing research around theory of traces but in recent years there are very few papers on top conferences about them.

By $u \div a$ we mean right cancellation of a from u. For u beeing a string it is defined as follows:

$$u \div a = \begin{cases} \varepsilon \ \mathit{for} \ \ u = \varepsilon, \\ v \ \mathit{for} \ \ u = va \\ (v \div a)b \ \mathit{for} \ \ u = vb \ \mathit{where} \ b \neq a \end{cases}$$

Lemma

$u \equiv w \implies u \div a \equiv w \div a$ *for any* $a \in \Sigma$.

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A dependency morphism w.r.t. *Dep* is any homomorphism from the monoid of strings over Σ on to another monoid such that

A1 $\phi(w) = \phi(\varepsilon) \implies w = \varepsilon$;

A2 $(a, b) \in Ind \implies \phi(ab) = \phi(ba)$;

A3 $\phi(ua) = \phi(v) \implies \phi(u) = \phi(v \div a)$;

A4 $\phi(ua) = \phi(vb) \wedge a \neq b \implies (a, b) \in Ind$

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Theorem

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Let ϕ be a dependency morphism, $u, v \in \Sigma^*$ $a, b \in \Sigma$. If $\phi(ua) = \phi(vb)$ and $a \neq b$ then there exists $w \in \Sigma^*$ such that $\phi(u) = \phi(wb)$ and $\phi(v) = \phi(wa)$.

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Theorem

Canonical homomorphism of string into histories is a dependency morphism.

Bibliography

Look to the notes and paper:

1. http://www.cas.mcmaster.ca/~cas724/2007/paper2.pdf