Unification of discrete and continuous models of cellular dynamics

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Discrete and continuous dynamics of self-renewal and differentiation of cells can be described by a system of ODEs coupled with a system of transport equations. It is quite natural, however, to embed those two disparate dynamics into a purely continuous setting of the transport equation and its distributional solutions in measures:

$$\partial_t \mu(t) + \partial_x(g(v(t), x)\mu(t)) = p(v(t), x)\mu(t), \tag{1}$$

$$v(t) = \int_{\{x_N\}} d\mu(t). \tag{2}$$

The continuous part remains unaltered. The discrete part is represented by those structure parameter values x_i at which the differentiation speed g vanishes. This leads (in the interesting case when the drift time $\int \frac{dx}{g} < \infty$, compare Osgood's theorem) to the non-uniqueness of solutions. This usually undesirable effect is actually indispensable in modeling of self-renewing concentrated populations, since there is an additional constitutive parameter c_i describing what part of those cells differentiates in a unit of time:

$$g(v(t), \cdot)\frac{d\mu^{ac}(t)}{d\mathcal{L}^1}(x_i^+) = c_i(v(t)) \int_{\{x_i\}} d\mu(t), \quad i = 0, \ldots, N \tag{3}$$

This constitutive relation (or, in other words, transmission condition) allows to solve the problem uniquely. Nevertheless, equation (1) is satisfied in distributional sense in the whole of $\mathbb{R}_+ \times \mathbb{R}$. At points x_i we observe interesting regularization effects. The main technical issues stem from the fact that we are dealing with functions of insufficient regularity (e.g. BV, which is the maximum regularity of c_i, p, g) that are integrated with respect to singular measures what combined with the transmission conditions leads to remarkable difficulties.

This new approach can be easily extended to more complex signaling or branched settings, where node-cells differentiate into multiple-type progeny. It possesses the advantages of the discrete cell dynamics (e.g. semitrivial steady-states), while doing away with unphysical effects such as infinite-speed signal propagation in purely discrete models.

Moreover, it opens new horizons in natural modeling of very complex processes such as emergence or disappearance of "stem cell like" populations, dynamical variations in the number of branches, etc.