

AF1* – Problems from 26/11/2011

Problems discussed in class:

C1. $(X, \|\cdot\|)$ is a normed space, $A \subset X$ is a subset. Show that the following conditions are equivalent:

- (i) $\overline{\text{lin}A} = X$
- (ii) for every $x^* \in X^*$ if $x^*|_A = 0$ then $x^* = 0$.

Homework:

H1. Let $1 < p < \infty$ and $a = (a(n))_n$ be a scalar sequence. Define the subspace

$$G = \left\{ x \in \ell^p(\mathbb{N}) : \sum_{n=0}^{\infty} a(n)x(n) = 0 \right\}.$$

Show that G is a dense subspace of $\ell^p(\mathbb{N})$ if and only if $a \notin \ell^q(\mathbb{N})$, where $1/p + 1/q = 1$.

H2. Let $(X, \|\cdot\|_1)$ be a real normed space and $Y \subset X$ a linear subspace. Let $\|\cdot\|_2$ be a norm on X which is equivalent with the norm $\|\cdot\|_1$ on Y . Prove that there is a norm $\|\cdot\|_3$ on X which is equivalent to $\|\cdot\|_1$ on X and whose restriction to Y is $\|\cdot\|_2$.

H3. $(X, \|\cdot\|)$ is a real normed space, $A \subset X$ is open and convex, $L \subset X$ is a linear subspace such that $A \cap L = \emptyset$. Prove that there exists $x^* \in X^*$ such that $x^*|_L = 0$ and $x^*|_A > 0$.

H4. $(X, \|\cdot\|)$ is a real normed space, $A \subset X$ is open and convex, $B \subset X$ is convex, and $A \cap B = \emptyset$. Prove that there exist $x^* \in X^*$ and $\alpha \in \mathbb{R}$ such that $x^*|_A < \alpha$ and $x^*|_B \geq \alpha$.

H5. $(X, \|\cdot\|)$ is a real normed space, $A \subset X$ is closed and convex, $K \subset X$ is compact and convex, and $A \cap K = \emptyset$. Prove that there exists $x^* \in X^*$ and $\alpha \in \mathbb{R}$ such that $x^*|_A < \alpha$ and $x^*|_K > \alpha$.

H6. In the real space $\ell^2(\mathbb{N})$ consider the set

$$A = \left\{ \sum_{i=1}^n \alpha_i e_i : n \in \mathbb{N}, \alpha_i \in \mathbb{R} \text{ for } i = 1, \dots, n \text{ and } \alpha_n > 0 \right\}.$$

Let $B = -A$. Prove that A and B are disjoint convex sets and for any $x^* \in \ell^2(\mathbb{N})^*$, $x^* \neq 0$, we have $x^*(A) = x^*(B) = \mathbb{R}$.