$$\left\{ \boxed{\frac{2nm}{2n^2+m^2}} = a_{n,m} : n,m \in \mathbb{N} \right\} = A$$

KRES DOLNY $\qquad a_{n,m} > 0$

$\longrightarrow$ 0 jest ogr. dolnym.

Pytanie: czy $0 = \inf A$?

Pokażemy, że żadna liczba $\varepsilon > 0$ nie jest ogr. dolnym.

$\in A$

Ustalmy $\varepsilon > 0$. $\quad a_{1,m} = \frac{2m}{2+m^2} < \frac{1}{m} < \varepsilon$

$\Rightarrow$ ε nie jest ograniczeniem dolnym dla $m > \frac{1}{\varepsilon}$, $0 = \inf A$

To SAMO, INACZEJ:

$A \ni a_{n,1} =: b_n$, $\quad b_n \to 0$ dla $n \to \infty$

$\Rightarrow \quad 0 = \inf A$

FAKT: (1) a jest ogr. dolnym (górnym) A
(2) $A \supset (b_n)$, lim $b_n = a$ $\quad \Rightarrow \quad a = \inf A$ $\quad (= \sup A)$

$$\alpha\beta \leq \frac{\alpha^2}{2} + \frac{\beta^2}{2} \iff (\alpha - \beta)^2 \geq 0 \quad (*)$$

$= \iff \alpha = \beta$

$$a_{m,n} = \frac{2nm}{2n^2+m^2}$$

$$2nm = \sqrt{2} \cdot \sqrt{2}n \cdot m \leq \sqrt{2} \cdot \left(\frac{2n^2}{2} + \frac{m^2}{2} \right)$$

(to wynika z $(*)$ dla $\alpha = \sqrt{2} \cdot n$, $\beta = m$)

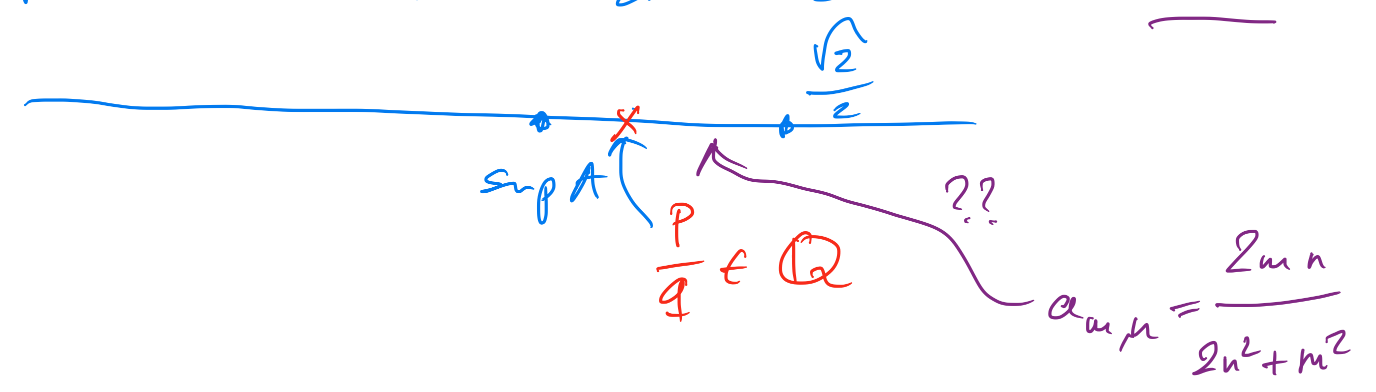
$$a_{m,n} = \frac{2nm}{2n^2+m^2} \underset{(**)}{\leq} \frac{\sqrt{2}}{2} \frac{(2n^2+m^2)}{2n^2+m^2} = \frac{\sqrt{2}}{2}$$

$\frac{\sqrt{2}}{2}$ jest ograniczeniem górnym A.

"$(**)$ jest prawie-równością dla $m \approx \sqrt{2}n$"

DOKOŃCZENIE DOWODU $\frac{\sqrt{2}}{2} = \sup A$

MOŻNA NP. PRZEZ ZAPRZECZENIE: PLAN

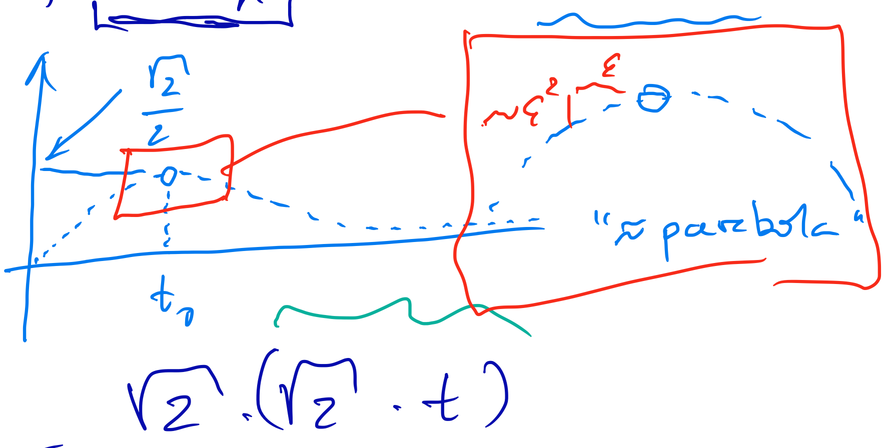


UWAGA (na boku)

$$a_{m,n} = \frac{2nm}{2n^2+m^2} = \frac{2 \cdot \frac{m}{n}}{2 + \frac{m^2}{n^2}}$$

$$= \frac{2t}{2+t^2}, \quad t = \frac{m}{n} \in \mathbb{Q} \cap (0, \infty)$$

$$f(t) = \frac{2t}{2+t^2}$$



$$f(t) = \frac{2t}{2+t^2} = \frac{\sqrt{2} \cdot (\sqrt{2} \cdot t)}{2+t^2}$$

$$\leq \frac{\sqrt{2}}{\cancel{2+t^2}} \left(\frac{\cancel{2+t^2}}{2} \right) = \frac{\sqrt{2}}{2}$$

$$f(t) - \frac{\sqrt{2}}{2} = \frac{2t}{2+t^2} - \frac{\sqrt{2}}{2} = \frac{4t - \sqrt{2}(2+t^2)}{2(2+t^2)}$$

$$\frac{\sqrt{2}}{2} - f(t) = \frac{\sqrt{2}}{2} \cdot \frac{1}{2+t^2} \left(t^2 + 2 - 2\sqrt{2} \cdot t \right)$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2+t^2} \left(t - \sqrt{2} \right)^2$$

$$\leq 1 \cdot \frac{1}{2} \cdot \left(t - \sqrt{2} \right)^2$$

$k \in \mathbb{N}$

$q > 1 \rightsquigarrow \exists c > 0 : q^n > c \cdot n^k$

WSKAZÓWKA

$q = 1 + \varepsilon, \quad \varepsilon > 0$

$$(1+\varepsilon)^n = 1 + \varepsilon \cdot \binom{n}{1} + \ldots + \varepsilon^k \binom{n}{k} + \varepsilon^{k+1} \binom{n}{k+1} + \ldots + \varepsilon^n$$

$= A$

WYSTARCZY DOWODZIĆ: $\exists c > 0$

$A > c \cdot n^k$

$a_n = nx - [nx] \qquad x \in \mathbb{R} \setminus \mathbb{Q}, \; x > 0$

1°. $a_n \in (0, 1)$

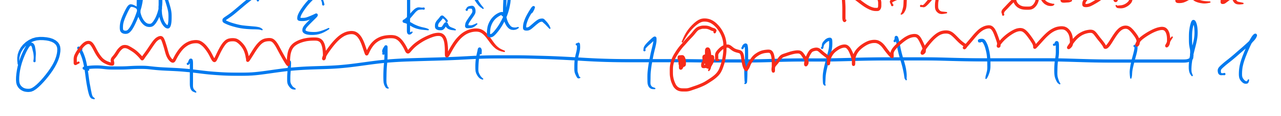
2°. $n \neq m \longrightarrow a_n \neq a_m$

$a_n = a_m \iff nx - mx = [nx] - [mx]$

$\iff x = \frac{[..] - [...]}{n - m} \in \mathbb{Q}$

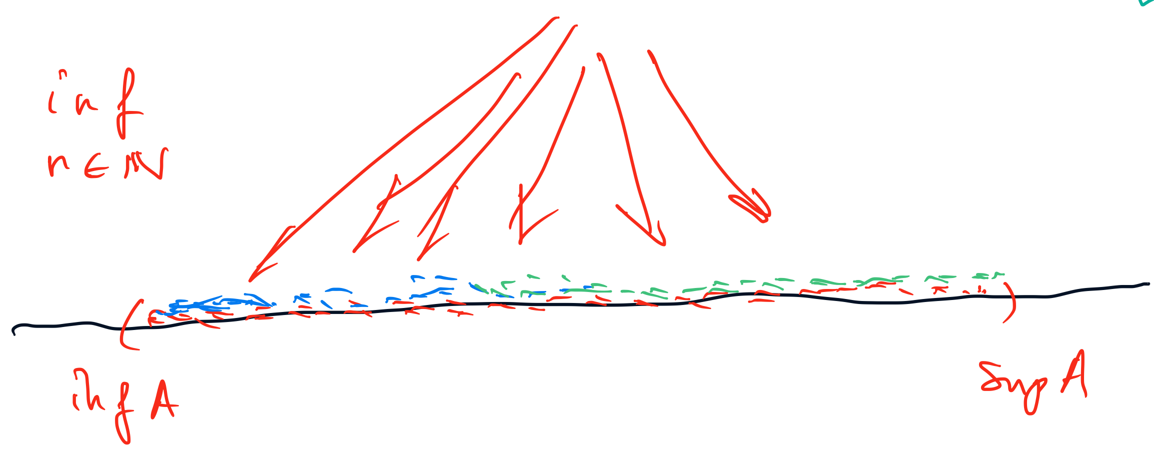
3°. $\varepsilon > 0$ małe, dzielę $(0, 1)$ na N części dł. $< \varepsilon$ każda

$N+1$ liczb a_n



$\boxed{\inf A}$ $\qquad A = \{a_{n,m} \mid n,m \in \mathbb{N}\}$ $\longleftarrow$ LEPIEJ

$\inf_{n \in \mathbb{N}}$



$A = B \cup C$

$\Rightarrow \inf A \leq \inf B, \; \leq \inf C$