Tutorial 5 (Written Exercises)

1. For a linear stationary time series, define $\widehat{\gamma}$ (the estimated autocovariance) by

$$\widehat{\gamma}(h) = \frac{1}{n}\sum_{j=1}^{n-h}(X_j - \overline{X})(X_{j+h} - \overline{X}).$$

   Show that $\sum_h \widehat{\gamma}(h) = 0$ (summing over all possible lags h).

2. Construct a stationary process $\{X_t\}$ such that Γ_n, the covariance matrix of $(X_1, \ldots, X_n)$ is non-singular for each $n \geq 1$ and such that $\gamma(h) = \text{Cov}(X_t, X_{t+h}) \not\to 0$ as $h \to +\infty$.

3. Let $\{X_t\}$ be a causal AR(1) process:

$$X_t - \phi X_{t-1} = \epsilon_t \qquad \{\epsilon_t\} \sim WN(0, \sigma^2)$$

   and let $\widehat{X}_{n+1}$ be the best linear predictor of X_{n+1} (in the sense of minimising the mean squared error) based on $X_1, \ldots, X_n$. Define: $\theta_{n0} = 1$ and $\widehat{X}_1 := 0$. Find $\theta_{n1}, \ldots, \theta_{nn}$ such that

$$X_{n+1} = \sum_{j=0}^{n} \theta_{nj}\widehat{\epsilon}_{n+1-j}$$

   where $\widehat{\epsilon}_{n+1-j} = (X_{n+1-j} - \widehat{X}_{n+1-j})$.

4. Let $\{X_t\}$ be a causal invertible ARMA(p,q) process

$$\phi(B)X_t = \theta(B)\epsilon_t \qquad \{\epsilon_t\} \sim WN(0, \sigma^2)$$

   Given the sample $\{X_1, \ldots, X_n\}$, define:

$$\epsilon_t^* = \begin{cases} 0 & t \leq 0 \quad \text{or} \quad t \geq n+1 \\ \phi(B)X_t - \theta_1\epsilon_{t-1}^* - \ldots - \theta_q\epsilon_{t-q}^* & t = 1, \ldots, n \end{cases}$$

   where $X_t := 0$ for $t \leq 0$. Here, we are using ϵ_t^* to estimate ϵ_t. This is clearly of importance for estimating σ^2.

   (a) Show that $\phi(B)X_t = \theta(B)\epsilon_t^*$ for all $t \leq n$ and hence that $\epsilon_t^* = \pi(B)X_t$ where $\pi(z) = \frac{\phi(z)}{\theta(z)}$.

   (b) Writing $\pi(z) = 1 + \sum_{j=1}^{\infty}\pi_j z^j$, set

$$\widetilde{X}_{n+1}^T = -\sum_{j=1}^{n}\pi_j X_{n+1-j}.$$

   Show that

$$\widetilde{X}_{n+1}^T = \phi_1 X_n + \ldots + \phi_p X_{n+1-p} + \theta_1\epsilon_n^* + \ldots + \theta_q\epsilon_{n+1-q}^*.$$

5. Let $(X_1, \ldots, X_{p+1})^t$ be a random vector with mean $\underline{0}$ and non-singular covariance matrix Γ_{p+1} with entries $\Gamma_{p+1;i,j} = \gamma(i-j)$ (i.e. the process is stationary). Let $\widehat{X}_{p+1}$ be the one step predictor of X_{p+1} based on $X_1, \ldots, X_p$, so that

$$\widehat{X}_{p+1} = P_{\overline{\text{spa}}(X_1,\ldots,X_p)} X_{p+1} = \phi_1 X_p + \ldots + \phi_p X_1$$

where

$$\underline{\phi} = \Gamma_p^{-1} \underline{\gamma}_p.$$

Show that

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p \neq 0 \qquad \forall |z| \leq 1$$

Hint: if $\phi(z) = (1 - az)\xi(z)$, then consider $Y_j = \xi(B) X_j$.

6. Consider the process $W_t : t = 1, 2, \ldots, n$ defined by

$$W_t = \epsilon_t - \epsilon_{t-1} \qquad \{\epsilon_t\} \sim WN(0, \sigma^2).$$

Find the best linear predictor $\widehat{W}_{n+1} = P_{\overline{\text{spa}}\{W_1,\ldots,W_n\}} W_{n+1}$ of W_{n+1} and compute its mean squared error.

Answers

1. Let $Y_j = X_j - \overline{X}$ then $\sum_{j=1}^{n} Y_j = n\overline{X} - n\overline{X} = 0$.

   Note:
$$\widehat{\gamma}(h) = \frac{1}{n}\sum_{j=1}^{n-h} Y_j Y_{j+h} = Y' D^{(h)} Y$$

   where Y is the n-vector $(Y_1, \dots, Y_n)'$ and $D^{(h)}$ is the $n \times n$ matrix with entries $D_{i,i+h} = 1$ and $D_{i,j} = 0$ for $j \neq i + h$. Then
$$\sum_h \widehat{\gamma}(h) = \frac{1}{n} Y' (\sum_h D^{(h)}) Y$$

   and $\sum_h D^{(h)}$ is the $n \times n$ matrix M where $M_{ij} = 1$ for each $(i,j) \in \{1, \dots, n\}^2$ Hence $Y'M = 0$ and the result follows.

2. Example: $X_t = W + Z_t$ where $W \sim N(0,1)$ and $\{Z_t\} \sim \text{IID}(0,1)$. Then $\gamma_X(0) = 2$, $\gamma_X(h) = 1$ for $h \neq 0$ and for any $(a_j)_{j \geq 1}$, $\sum_{j=1}^{n} \sum_{k=1}^{n} a_j a_k \gamma_X(j-k) = (\sum_j a_j)^2 + \sum_j a_j^2 > 0$ if $\underline{a} \neq 0$. Hence Γ_n non singular. Also,
$$\text{Cov}(X_t, X_{t+h}) \overset{h \to \infty}{=\!\!\longrightarrow} 1.$$

3. Firstly, for an AR(1) process, predicting X_{t+1} based on $X_1, \dots, X_t$ for $t \geq 1$,
$$\begin{aligned}
&\mathbb{E}[|X_{t+1} - \sum_{j=1}^{t} a_j X_{t+1-j}|^2] \\
&\quad = \mathbb{E}[|\epsilon_{t+1} + (\phi - a_1) X_t + \sum_{j=2}^{t} a_j X_{t+1-j}|^2] \\
&\quad = \mathbb{E}[\epsilon_{t+1}^2] + 2\mathbb{E}[\epsilon_{t+1}((\phi - a_1)X_t + \sum_{j=2}^{t} a_j X_{t+1-j})] + \mathbb{E}[|(\phi - a_1)X_t + \sum_{j=2}^{t} a_j X_{t+1-j}|^2] \\
&\quad = \sigma^2 + \mathbb{E}[|(\phi - a_1)X_t + \sum_{j=2}^{t} a_j X_{t+1-j}|^2]
\end{aligned}$$

   This has minimum σ^2, which is achieved for $a_1 = \phi$ and $a_2 = \dots = a_n = 0$, so that
$$\widehat{X}_{t+1} = \phi X_t.$$

   Plugging into the expression $X_{n+1} = \sum_{j=0}^{n} \theta_{n,j} \widehat{\epsilon}_{n+1-j}$, with $\widehat{X}_1 = 0$ gives:
$$X_{n+1} = \sum_{j=0}^{n-1} \theta_{nj}(X_{n+1-j} - \phi X_{n-j}) + \theta_{nn} X_1 = \sum_{j=0}^{n-1} \theta_{nj} \epsilon_{n+1-j} + \theta_{nn} X_1.$$

Directly from the definition, $\theta_{00} = 1$. The expression for the AR(1) process may be expanded iteratively:

$$X_{n+1} = \phi X_n + \epsilon_{n+1} = \phi^2 X_{n-1} + \phi \epsilon_n + \epsilon_{n+1} = \phi^n X_1 + \sum_{j=1}^{n+1} \phi^{n+1-j} \epsilon_j$$

noindent and comparing the expressions gives:

$$\theta_{nn} = \phi^n \qquad \theta_{nj} = \phi^j$$

4. (a) The first part follows straight from the definition:

$$\epsilon_t^* + \theta_1 \epsilon_{t-1}^* + \ldots + \theta_q \epsilon_{t-q}^* := \theta(B)\epsilon_t^* = \phi(B) X_t.$$

Since From this, since the process is invertible, $\pi(B) := \theta^{-1}(B\phi(B)$ is well defined and:

$$\epsilon_t^* = \theta^{-1}(B)\phi(B) X_t = \pi(B) X_t.$$

(b) Recall that

$$\phi(z) = 1 - (\phi_1 z + \ldots + \phi_p z^p) \qquad \theta(z) = 1 + (\theta_1 z + \ldots + \theta_q z^q)$$

Note $\epsilon_t^* - X_t = -(\phi_1 X_{t-1} + \ldots + \phi_p X_{t-p}) - \left(\theta_1 Z_{t-1}^* + \ldots + \theta_q Z_{t-q}^*\right)$. With $X_t = 0$ for all $t \leq 0$,

$$\begin{aligned} \widetilde{X}_{n+1}^T &= -\sum_{j=1}^{n} \pi_j X_{n+1-j} = X_{n+1} - \pi(B) X_{n+1} = X_{n+1} - \epsilon_{n+1}^* \\ &= (\phi_1 X_n + \ldots + \phi_p X_{n+1-p}) + \left(\theta_1 \epsilon_n^* + \ldots + \theta_q \epsilon_{n+1-q}^*\right) \end{aligned}$$

as required.

5. The polynomial may be expressed as:

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p = \prod_{j=1}^{p} (1 - a_j z)$$

where $\frac{1}{a_1}, \ldots, \frac{1}{a_p}$ are the roots of the polynomial. Since $\widehat{X}_{p+1}$ is the one step predictor that minimises the mean square error, the coefficients $\phi_1, \ldots, \phi_p$ minimise

$$v_p := \mathbb{E}\left[\left|X_{p+1} - \widehat{X}_{p+1}\right|^2\right] = \mathbb{E}\left[|\phi(B) X_{p+1}|^2\right].$$

Now suppose that $a_1, \ldots, a_p$ are arranged so that $|a_1| \geq |a_2| \geq \ldots \geq |a_p|$. Let $\xi(B) = \prod_{j=2}^{p}(1 - a_j z)$, so that $\phi(z) = (1 - a_1 z)\xi(z)$. Then

$$v_p = \mathbb{E}\left[|\xi(B)X_{p+1} - a_1 B\xi(B)X_{p+1}|^2\right] = \mathbb{E}\left[|\xi(B)X_{p+1} - a_1\xi(B)X_p|^2\right].$$

Let $Y_p = \xi(B)X_p$, then using the fact that X_j are mean 0,

$$v_p = \mathbb{E}\left[|Y_{p+1} - a_1 Y_p|^2\right] = \mathrm{Var}(Y_{p+1}) + a_1^2\mathrm{Var}(Y_p) - 2a_1\mathrm{Cov}(Y_p, Y_{p+1}).$$

Since $a_1, \ldots, a_p$ minimise the mean squared error, it follows that once $a_2, \ldots, a_p$ are established,

$$a_1 = \frac{\mathrm{Cov}(Y_p, Y_{p+1})}{\mathrm{Var}(Y_p)} = \rho\sqrt{\frac{\mathrm{Var}(Y_p)}{\mathrm{Var}(Y_{p+1})}} = \rho$$

where $\rho = \rho(Y_p, Y_{p+1})$ and, by stationarity, $\mathrm{Var}(Y_p) = \mathrm{Var}(Y_{p+1})$. Hence the minimising a_1 is $a_1 = \rho$, so that $|a_1| \leq 1$. Now suppose that $a_1 = \rho = \pm 1$. Then $\mathbb{E}\left[|\phi(B)X_{p+1}|^2\right] = 0$ contradicting non-singularity of Γ_{p+1}.

6. $W_t = \epsilon_t - \epsilon_{t-1}$. We use

$$\widehat{W}_{n+1} = \sum_{j=1}^{n} a_j W_{n+1-j} = \sum_{j=1}^{n} a_j(\epsilon_{n+1-j} - \epsilon_j)$$

and the a_j's are chosen to minimise

$$v_n = \mathbb{E}[|\epsilon_{n+1} - \epsilon_n - \sum_{j=1}^{n} a_j(\epsilon_{n+1-j} - \epsilon_{n-j})|^2] = \sigma^2(1 + a_n^2 + (1+a_1)^2 + \sum_{j=1}^{n-1}(a_{j+1} - a_j)^2)$$

This is minimised by taking

$$a_{j+1} - 2a_j + a_{j-1} = 0 \quad j = 2, \ldots, n-1$$

$$a_n = \frac{1}{2}a_{n-1}$$

$$1 + 2a_1 = a_2$$

$$a_j = \alpha + \beta j \Rightarrow \alpha = -1, \beta = \frac{1}{n+1}.$$

$$a_{j,n} = -1 + \frac{j}{n+1} \quad j = 1, \ldots, n$$

Using $a_{j+1} - a_j = \beta = \frac{1}{n+1}$, $a_{n,n} = \frac{-1}{n+1}$, $1 + a_{1,n} = \frac{1}{n+1}$ we get

$$v_n = \sigma^2(1 + \frac{1}{n+1}).$$