Tutorial 4: Written Exercises

1. Let $\{a_j\}$ be a sequence of numbers. Let

$$m(t) = c_0 + c_1 t + \ldots + c_k t^k$$

be a polynomial of degree k. Show that for any $(c_0, \ldots, c_k)$,

$$m(t) = \sum_j a_j m(t-j) \qquad \forall t$$

if and only if

$$\begin{cases} \sum_j a_j = 1 \\ \sum_j j^r a_j = 0 \quad \forall r = 1, \ldots, k. \end{cases}$$

2. Design a symmetric filter which eliminates seasonal components with period 3 and at the same time allows quadratic trend functions to pass without distortion.

3. Let

$$X_t = a + bt + Y_t$$

where $\{Y_t : t \in \mathbb{Z}\} \sim \text{IID}(0, \sigma^2)$ and where a and b are constants. Define

$$W_t = \frac{1}{1+2q} \sum_{j=-q}^{q} X_{t+j}.$$

Compute μ_W and $C_W(t, s) = \text{Cov}(W_t, W_{t+s})$.

4. Let $\{S_t : t = 0, 1, 2, \ldots\}$ be the random walk with constant drift μ, defined by

$$\begin{cases} S_0 = 0 \\ S_t = \mu + S_{t-1} + \epsilon_t \quad t = 1, 2, 3, \ldots \end{cases}$$

where $\{\epsilon_t\} \sim \text{IID}(0, \sigma^2)$. Show that ∇S is stationary and compute its mean and autocovariance function.

5. If $X_t = a + bt$ for $t = 1, \ldots, n$, let the sample autocorrelation function be defined by

$$\widehat{\rho}(k) = \frac{\sum_{t=1}^{n-k} (X_t - \overline{X})(X_{t+k} - \overline{X})}{\sum_{t=1}^{n} (X_t - \overline{X})^2}.$$

Show that $\lim_{n \to +\infty} \widehat{\rho}(k) = 1$ for each fixed k.

6. Let $\rho(k) = \text{Corr}(X_t, X_{t+k})$ denote the autocorrelation function (ACF) of a stationary process $\{X_t\}$. Suppose that $\{X_t\}$ is an MA(2) process

$$X_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} \qquad \{\epsilon_t\} \sim WN(0, \sigma^2).$$

Find $\rho(1)$ and $\rho(2)$ in terms of θ_1 and θ_2. Also, find the largest possible value of $\rho(1)$ for an MA(2) process and $\rho(2)$ for n MA(2) process.

7. Let $\{X_t\}$ be an AR(2) process with generating polynomial $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$. Suppose that (ϕ_1, ϕ_2) satisfy:

$$\begin{cases} \phi_1 + \phi_2 < 1 \\ \phi_2 - \phi_1 < 1 \\ |\phi_2| < 1. \end{cases}$$

Show that the process is causal.

Answers

1. For a polynomial $m(t) = c_0 + \ldots + c_k t^k$,

$$\begin{aligned} \sum_j a_j m_{t-j} &= c_0 \sum_j a_j + c_1 \sum_j a_j (t-j) + \ldots + c_k \sum_j a_j (t-j)^k \\ &= \sum_{i=0}^k t^i \sum_{m=i}^k c_m \binom{m}{i} \sum_j (-1)^{m-i} j^{m-i} a_j \\ &= \sum_{i=0}^k t^i \sum_{m=i}^k (-1)^{m-i} c_m \binom{m}{i} \sum_j j^{m-i} a_j \end{aligned}$$

It follows that for any collection $(c_0, \ldots, c_k)$ and each i,

$$c_i = \sum_{m=i} c_m (-1)^{m-i} \binom{m}{i} \sum_j j^{m-i} a_j$$

and hence that

$$\begin{cases} \sum_j a_j = 1 \\ \sum_j j^r a_j = 0 \quad r = 1, \ldots, k \end{cases}$$

as required.

2. Recall: a seasonal component with period d satisfies: $\sum_{j=1}^d s_{t+j} = 0$ for each t and $s_t = s_{t+d}$ for each t (it is taken to be mean 0 over the cycle).

For a *symmetric* filter, where $d = 1 + 2q$ (i.e. d is an odd number), this is achieved by taking $a_{-q} = a_{-q+1} = \ldots = a_{q-1} = a_q$; we need the a_i's to be equal over blocks of length $2q + 1$.

Let us compute the symmetric filter of smallest length which is symmetric, eliminates seasonal components of period 3 and which passes quadratic trends without distortion. Here $q = 1$. We need

$$\sum_j a_j = 1$$
$$a_j = a_{-j}$$

(which gives $\sum_j j a_j = 0$, so that we pass *linear* trends without distortion) and

$$\sum_j j^2 a_j = 0$$

to get quadratic trends without distortion. Let us consider the shortest possible, which is: $(a_{-4}, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, a_4)$ and these numbers satisfy:

- $a_j = a_{-j}$ (symmetry)
- $a_4 = a_3 = a_2 = a_{-2} = a_{-3} = a_{-4} = A_1$ and $a_{-1} = a_0 = a_1 = A_0$ (symmetry together with eliminating seasonal components of period 3). Such a filter satisfies $\sum_j j a_j = 0$.

- $A_0 + 2A_1 = \frac{1}{3}$ (comes from $\sum_j a_j = 1$)
- $(4^2+3^2+2^2)A_1 + A_0 = 0$ (comes from $\sum_j j^2 a_j = 0$)

A_0 and A_1 satisfy two equations

$$A_0 + 2A_1 = \frac{1}{3} \qquad 26A_1 + A_0 = 0$$

so that

$$A_1 = -\frac{1}{72} \qquad A_0 = \frac{13}{36}.$$

3.

$$\mu_W(t) = a + bt$$

(passes a linear trend without distortion). For $0 \le s \le 1+2q$,

$$\begin{aligned}
\mathrm{Cov}(W_t, W_{t+s}) &= \frac{1}{(1+2q)^2} \sum_{j_1=-q}^{q} \sum_{j_2=-q}^{q} \mathrm{Cov}(X_{t+j_1}, X_{t+s+j_2}) \\
&= \frac{\sigma^2}{(1+2q)^2} \sum_{j_1=-q}^{q} \sum_{j_2=-q}^{q} \mathbf{1}_s(j_1 - j_2). \\
&= \frac{\sigma^2(1+2q-s)}{(1+2q)^2}
\end{aligned}$$

Answer:

$$\begin{cases} \frac{\sigma^2(1+2q-|s|)}{(1+2q)^2} & s \in [-(1+2q), (1+2q)] \\ 0 & \text{otherwise} \end{cases}$$

4.

$$Y_t := \nabla S_t = S_t - S_{t-1} = \mu + S_{t-1} + \epsilon_t - S_{t-1} = \mu + \epsilon_t.$$

$$\mu_Y = \mu, \qquad \gamma_Y(h) = \begin{cases} 1 & h = 0 \\ 0 & h \neq 0. \end{cases}$$

5. $\overline{X} = a + b\frac{1}{n}\sum_{t=1}^{n} t = a + \frac{b}{2}(n+1)$ so that

$$\begin{aligned}
\sum_{t=1}^{n-k}(X_t - \overline{X})(X_{t+k} - \overline{X}) &= b^2 \sum_{t=1}^{n-k}(t - \frac{n+1}{2})(t + k - \frac{n+1}{2}) \\
&= b^2 \sum_{t=1}^{n}(t - \frac{n+1}{2})^2 + b^2 k \sum_{t=1}^{n-k}(t - \frac{n+1}{2}) - b^2 \sum_{t=n-k+1}^{n}(t - \frac{n+1}{2})^2
\end{aligned}$$

Now,

$$\lim_{n\to+\infty} \frac{1}{n^3} \sum_{t=1}^{n}(t - \frac{n+1}{2})^2 = \lim_{n\to+\infty} \frac{2}{n^3} \int_1^{n/2} x^2 dx = \frac{1}{4}$$

while $\frac{1}{n^2}|\sum_{t=1}^{n-k} k(t - \frac{n+1}{2})| \le k$ and $\frac{1}{n^2}|\sum_{t=n-k+1}^{n}(t - \frac{n+1}{2})^2| \le k$ so that, for each fixed k,

$$\widehat{\rho}(k) \overset{n\to+\infty}{\longrightarrow} 1.$$

6.

$$\gamma(0) = \sigma^2(1+\theta_1^2+\theta_2^2)$$

$$\gamma(1) = \sigma^2\theta_1(1+\theta_2)$$
$$\gamma(2) = \sigma^2\theta_2$$

hence

$$\rho(1) = \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} \qquad \rho(2) = \frac{\theta_2}{1+\theta_1^2+\theta_2^2}$$

Now consider maximising these:

$$\frac{d}{d\theta_1}\rho(1) = \frac{1+\theta_2}{1+\theta_1^2+\theta_2^2} - \frac{2\theta_1^2(1+\theta_2)}{(1+\theta_1^2+\theta_2^2)^2}$$
$$\frac{d}{d\theta_2}\rho(1) = \frac{\theta_1}{1+\theta_1^2+\theta_2^2} - \frac{2\theta_1\theta_2(1+\theta_2)}{(1+\theta_1^2+\theta_2^2)^2}$$

Set to 0 for critical points:

$$\begin{cases} 1-\theta_1^2+\theta_2^2 = 0 \\ 1+\theta_1^2-\theta_2^2-2\theta_2 = 0 \end{cases}$$

so that $\theta_1^2 = 1+\theta_2^2$ and

$$2-2\theta_2 = 0 \Rightarrow \theta_2 = 1, \qquad \theta_1^2 = 2 \Rightarrow \theta_1 = \pm\sqrt{2}$$

and the max value is $\rho(1) = \frac{1}{\sqrt{2}}$ (the + root gives the maximum).
$0 = \frac{1}{1+\theta_2} - 2\theta_2 \Rightarrow (\theta_2 - \frac{1}{2})^2 = \frac{1}{4} \Rightarrow \theta_2 = \frac{3}{4}$.
For $\rho(2)$,

$$\frac{d}{d\theta_1}\rho(2) = -\frac{2\theta_1\theta_2}{(1+\theta_1^2+\theta_2^2)^2}$$
$$\frac{d}{d\theta_2}\rho(2) = \frac{1}{1+\theta_1^2+\theta_2^2} - \frac{2\theta_2^2}{(1+\theta_1^2+\theta_2^2)^2}$$

setting equal to zero gives $\theta_1\theta_2 = 0$ and $1+\theta_1^2-\theta_2^2 = 0$. The maximiser is $\theta_1 = 0$, $\theta_2 = 1$ with $\rho(2) = \frac{1}{2}$.

7. Causal if and only if $1 - \phi_1 z - \phi_2 z^2 \neq 0$ for any $|z| \leq 1$.

   Roots are:

$$z = -\frac{1}{2\phi_2}(\phi_1 \pm \sqrt{(\phi_1^2 + 4\phi_2)})$$

   First suppose $\phi_1^2 + 4\phi_2 < 0$, then $\phi_2 < 0$ and

$$|z|^2 = \frac{\phi_1^2 - (4\phi_2 + \phi_1^2)}{4|\phi_2|^2} = \frac{1}{|\phi_2|} > 1$$

   since $|\phi_2| < 1$.

   Now consider $\phi_1^2 + 4\phi_2 \geq 0$ and consider $|z_-|$ where z_- denotes the root with lower modulus. Then

$$|z_-| = \begin{cases} \frac{1}{2|\phi_2|}\left(\sqrt{\phi_1^2 + 4\phi_2} - |\phi_1|\right) & \phi_2 > 0 \\ \frac{1}{2|\phi_2|}\left(|\phi_1| - \sqrt{\phi_1^2 + 4\phi_2}\right) & \phi_2 < 0 \end{cases}$$

   The conditions amount to $\phi_2 < 1 - |\phi_1|$, so that, for $\phi_2 > 0$, $|\phi_1| < 1$ and

$$\sqrt{\phi_1^2 + 4\phi_2} > \sqrt{\phi_1^2 + 4 - 4|\phi_1|} = ||\phi_1| - 2| = 2 - |\phi_1|$$

   hence $\sqrt{\phi_1^2 + 4\phi_2} - |\phi_1| > 2 - |\phi_1| - |\phi_1| = 2 - 2|\phi_1|$
   Hence $|z| \geq \frac{1-|\phi_1|}{|\phi_2|} > 1$. For $\phi_2 < 0$, $|\phi_1| < 1 + |\phi_2|$ so that $1 - |\phi_1| > -|\phi_2|$ and

$$\phi_1^2 - 4|\phi_2| < \phi_1^2 - 4|\phi_1| + 4 = (|\phi_1| - 2)^2$$

   and

$$|z_-| \geq \frac{1}{2|\phi_2|}(|\phi_1| - ||\phi_1| - 2|) = \begin{cases} \frac{1}{|\phi_2|} > 1 & |\phi_1| \geq 2 \\ \frac{2(1-|\phi_1|)}{2|\phi_2|} > 1 & |\phi_1| < 2 \end{cases}$$