Tutorial 8

1. Let X be a single observation from a distribution with density

$$p_X(x) = \theta x^{\theta-1} \qquad 0 \leq x \leq 1.$$

   (a) Let $Y = -\frac{1}{\log X}$. Evaluate the confidence level of the interval estimator $\left[\frac{Y}{2}, Y\right]$.

   (b) Find a pivotal quantity and use it to set up an interval estimator.

2. Let $X_1, \ldots, X_n$ be a sample from a $N(\mu, \sigma^2)$ population, both μ and σ^2 unknown.

   (a) Let $Z_1, \ldots, Z_n$ be i.i.d. $N(0,1)$ variables. The distribution of $W = Z_1^2 + \ldots + Z_n^2$ is χ_n^2. Let $\underline{Z} = (Z_1, \ldots, Z_n)^t$ where t denotes transpose. Let $Y_j = Z_j - \overline{Z}$ and let $\underline{Y} = (Y_1, \ldots, Y_n)^t$. Let $\underline{Y} = M_n \underline{Z}$ for a symmetric matrix M_n. What is M_n? Prove that $M_n^2 = M_n$. From this, what do you conclude about the eigenvalues of M_n? Use the fact that the sum of the eigenvalues is equal to the sum of the trace for symmetric matrices.

   Now consider the expression $M_n = PDP^t$ where D is diagonal and P is orthonormal. What is the distribution of $P^t \underline{Z}$? Hence what is the distribution of $\underline{Z}^t M_n^t M_n \underline{Z}$?

   Hence conclude that

$$\frac{\sum_{j=1}(X_j - \overline{X})^2}{\sigma^2} \sim \chi_{n-1}^2.$$

   (b) Show that

$$\overline{X} \perp X_1 - \overline{X}, \ldots, X_n - \overline{X}.$$

3. Let $Z \sim N(0,1)$ and let $V \sim \chi_m^2$. Let $Z \perp V$. Let $T = \frac{Z}{\sqrt{V/m}}$. Compute the density function for T.

4. Let $X_1, \ldots, X_n$ be a random sample from a $N(1, \sigma^2)$ population ($\mu = 1$ is known). Construct a symmetric $1 - \alpha$ interval estimator for σ, with as many degrees of freedom as possible.

5. Let $X_1, \ldots, X_n$ be independent $N(\mu, \sigma^2)$ random variables. Using a pivot based on $\sum_{j=1}^n (X_j - \overline{X})^2$, construct a symmetric confidence interval with confidence level $1 - \alpha$ for $\log \sigma^2$.

6. Suppose that $Y_1, \ldots, Y_n$ are independent and that

$$Y_i \sim N(x_i \beta, \sigma^2),$$

   where $x_1, \ldots, x_n$ are given, σ is known and β is an unknown parameter.

   (a) Compute the least squares estimator $\widehat{\beta}_{LS}$ of β.

   (b) Compute a confidence interval for β of the form $[\widehat{\beta}_{LS} - c, \widehat{\beta}_{LS} + c]$ with confidence level $1 - \alpha = 0.95$.

7. Let

$$X_i = \frac{\theta}{2} t_i^2 + \epsilon_i \qquad i = 1, \ldots, n$$

where ϵ_i are i.i.d. $N(0, \sigma^2)$ variables, where σ is known.

(a) Compute the MLE of θ.

(b) Using a pivot based on the MLE of θ, find a symmetric confidence interval for θ with confidence level $1 - \alpha$.

(c) Suppose the values for t_i may be chosen freely subject to the constraint that $0 \leq t_i \leq 1$ for each $i = 1, \ldots, n$. What values of t_i should be chosen to make the symmetric confidence interval as short as possible?

8. Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma_1^2)$. Suppose a lower confidence bound $\overline{X} - c$, intended to be of confidence level $1 - \alpha$ is computed under the assumption that $X_j \sim N(\mu, \sigma_0^2)$. What is the actual confidence level?

9. (a) Let $X_1, \ldots, X_n$ be a $N(\mu, \sigma^2)$ sample, where μ and σ^2 are both unknown. Show that the symmetric $1 - \alpha$ confidence interval is given by

$$\left[\overline{X} \pm \frac{S}{\sqrt{n}} t_{n-1;\alpha/2} \right]$$

where $S^2 = \frac{1}{n-1} \sum_{j=1}^{n} (X_j - \overline{X})^2$ and $t_{n,\alpha}$ denotes the number such that $\mathbb{P}(T > t_{n,\alpha}) = \alpha$ where $T \sim t_n$.

(b) Suppose we want to select a sample size N such that the interval in part (a) has length at most $l = 2d$ for some preassigned length. Stein's two stage procedure (1945) is the following: Begin by taking a fixed number $n_0 \geq 2$ of observations, calculate $\overline{X}_0 = \frac{1}{n_0} \sum_{j=1}^{n_0} X_j$ and $S_0^2 = \frac{1}{n_0 - 1} \sum_{j=1}^{n_0} (X_j - \overline{X}_0)^2$. Then take $N - n_0$ further observations where N is the smallest integer greater than or equal to n_0 and greater than or equal to $(S_0 t_{n_0-1;(\alpha/2)}/d)^2$.

Show that

$$\frac{\sqrt{N}(\overline{X} - \mu)}{S_0} \sim t_{n_0-1}$$

where $S_0^2 = \frac{1}{n_0-1} \sum_{j=1}^{n_0} (X_j - \overline{X}_0)^2$ and $\overline{X} = \frac{1}{N} \sum_{j=1}^{N} X_j$. It follows that $\left[\overline{X} \pm \frac{S_0}{\sqrt{N}} t_{n_0-1;\alpha/2} \right]$ is a confidence interval with confidence level $1 - \alpha$ for μ of length at most $2d$.

Hint recall that $\overline{X} \perp S^2$ where $\overline{X}$ is the estimator of μ and S^2 is the estimator of σ^2 based on a sample size n from a $N(\mu, \sigma^2)$ distribution. Consider the definition of a t distribution.

10. Let $X_1, \ldots, X_n$ be a random sample from a Rayleigh distribution

$$p(x, \sigma) = \begin{cases} \frac{x}{\sigma^2} \exp\left\{ -\frac{x^2}{2\sigma^2} \right\} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\sigma > 0$ is an unknown parameter.

(a) Compute the maximum likelihood estimator of σ^2.

(b) Compute a symmetric confidence interval, confidence level $1-\alpha$ of the form

$$\left[c_n \widehat{\sigma^2}_{ML}, d_n \widehat{\sigma^2}_{ML}\right]$$

for σ^2. Express your answer in terms of quantiles of an appropriate χ^2 distribution.

11. Let $X_1, \ldots, X_n$ be a $N(\mu, \sigma^2)$ random sample where σ is known. Show that the interval of shortest length of confidence $1-\alpha$ of the form

$$\left[\overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha_1}, \overline{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha_2}\right] \qquad \alpha_1 + \alpha_2 = \alpha$$

is obtained by taking $\alpha_1 = \alpha_2 = \frac{\alpha}{2}$.

Answers

1. (a) Coverage probability is:

$$\mathbb{P}_\theta\left(\frac{Y}{2} \le \theta \le Y\right) = \mathbb{P}\left(\theta \le Y \le 2\theta\right) = \mathbb{P}(e^{-1/\theta} \le X \le e^{-1/(2\theta)})$$
$$= \theta \int_{e^{-1/\theta}}^{e^{-1/(2\theta)}} x^{\theta-1} dx = e^{-1/2} - e^{-1}.$$

(b) X has c.d.f.

$$\mathbb{P}_\theta(X \le x) = \begin{cases} 0 & x < 0 \\ x^\theta & 0 \le x \le 1 \\ 1 & x \ge 1. \end{cases}$$

Let $Z = -\theta \log X$ then:

$$\mathbb{P}_\theta(Z \le z) = \mathbb{P}_\theta(-\theta \log X \le z) = \mathbb{P}(X \ge \exp\{-\frac{z}{\theta}\}) = \begin{cases} 1 - e^{-z} & z \ge 0 \\ 0 & z < 0. \end{cases}$$

Hence $Z = -\theta \log X \sim \text{Exp}(1)$ is a natural pivot variable (its distribution does not depend on θ). Then

$$\mathbb{P}(Z \le z_1) = \alpha_1 \Rightarrow 1 - e^{-z_1} = \alpha_1 \Rightarrow z_1 = \log \frac{1}{1-\alpha_1}$$
$$\mathbb{P}(Z \ge z_2) = \alpha_2 \Rightarrow e^{-z_2} = \alpha_2 \Rightarrow z_2 = \log \frac{1}{\alpha_2}$$

so that

$$1 - (\alpha_1 + \alpha_2) = \mathbb{P}_\theta\left(\log \frac{1}{1-\alpha_1} \le -\theta \log X \le \log \frac{1}{\alpha_2}\right)$$

giving

$$\mathbb{P}_\theta\left(\frac{\log \frac{1}{1-\alpha_1}}{-\log X} \le \theta \le \frac{\log \frac{1}{\alpha_2}}{-\log X}\right) = 1 - (\alpha_1 + \alpha_2)$$

An interval estimator with coverage probability $1 - (\alpha_1 + \alpha_2)$ is:

$$\left[\frac{\log(1-\alpha_1)}{\log X}, \frac{\log \alpha_2}{\log X}\right].$$

For a symmetric interval with coverage probabiltiy $1 - \alpha$, take $\alpha_1 = \alpha_2 = \frac{\alpha}{2}$.

Alternatively, from the c.d.f. for X, another pivot is: $W = X^\theta$ which has $U(0,1)$ distribution (which does not depend on θ).

Using $X^\theta \sim U(0,1)$, for any $0 \le a < b \le 1$,

$$\mathbb{P}(a \le X^\theta \le b) = b - a$$

$$b - a = \mathbb{P}(\log a \le \theta \log X \le \log b) = \mathbb{P}\left(\frac{\log b}{\log X} \le \theta \le \frac{\log a}{\log X}\right)$$

an interval estimator with confidence level $b - a$ is

$$\left[\frac{\log b}{\log X}, \frac{\log a}{\log X}\right].$$

If we want a *symmetric* confidence interval with confidence level $1 - \alpha$, then $a = \frac{\alpha}{2}$ and $b = 1 - \frac{\alpha}{2}$; interval is:

$$\left[\frac{\log(1 - \alpha/2)}{\log X}, \frac{\log(\alpha/2)}{\log X}\right].$$

2. (a) Let I_n denote the $n \times n$ identity matrix and let $\mathbf{1}_n$ denote a column vector, length n, with each entry 1. Then $\mathbf{1}_n \mathbf{1}_n^t$ is the $n \times n$ matrix with each ehtry 1. Then

$$M_n = I_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^t.$$

It follows that

$$M_n^2 = (I_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^t)^2 = I_n - \frac{2}{n} \mathbf{1}_n \mathbf{1}_n^1 t + \frac{1}{n^2} \mathbf{1}_n \mathbf{1}_n^t \mathbf{1}_n \mathbf{1}_n^t = M_n$$

using the fact that $\mathbf{1}_n^t \mathbf{1}_n = n$.

M_n is symmetric, hence has decomposition $M_n = PDP^t$ where D is diagonal and P is orthonormal. So $M_n^2 = M_n$ implies

$$M_n^2 = PDP^t PDP^t = PD^2 P^t = PDP^t = M_n \Rightarrow D^2 = D$$

hence all the eigenvalues are 0 or 1. Using the fact that the sum of eigenvalues is equal to the trace, the sum of eigenvalues is $n - 1$. Hence 1 has multiplicity $n - 1$ and 0 has multiplicity 1.

Use: If $\underline{X} \sim N(\underline{\mu}, \Sigma)$ then $A\underline{X} + \underline{b} \sim N(A\underline{\mu} + \underline{b}, A\Sigma A^t)$. Then

$$\underline{Y} := P^t \underline{Z} \sim N(P^t \underline{0}, \sigma^2 P^t I P) = N(0, \sigma^2 I)$$

since P is orthonormal (so that $P^t P = I$).

Take the columns of P such that $D = \text{diag}(1, \dots, 1, 0)$ (the 0 in the nth position). It follows that

$$\underline{Z}^t M_n^t M_n \underline{Z} = \underline{Y}^t D \underline{Y} = \sum_{j=1}^{n-1} Y_j^2 \sim \chi_{n-1}^2.$$

(b) Let $\Sigma^{(X)} = \sigma^2 I$ denote the covariance matrix of $\underline{X}$. Clearly, $\underline{Y} := (X_1 - \overline{X}, \dots, X_n - \overline{X}, \overline{X})^t = A\underline{X}$ is a normal random vector, since it is a linear transformation of $\underline{X}$. Let $\Sigma^{(Y)}$ denote the covariance matrix of Y. Here $A_{ij} = -\frac{1}{n}$ for $j \ne i$, $i = 1, \dots, n$, $A_{ii} = 1 - \frac{1}{n}$, $i = 1, \dots, n$, $A_{n+1,j} = \frac{1}{n}$, $j = 1, \dots, n$. Use:

$$\Sigma^{(Y)} = A\Sigma^{(X)}A^t = \sigma^2 AA^t$$

For $j \neq n+1$, $(AA^t)_{n+1,j} = \sum_{k=1}^n A_{n+1,k}A_{j,k} = \frac{1}{n}\sum_{k=1}^n A_{j,k} = 0$ hence $\mathbf{C}(\overline{X}, X_j - \overline{X}) = 0$.

3. Density of χ^2_m is:

$$p_V(v) = \frac{1}{2^{m/2}\Gamma(m/2)} v^{(m/2)-1} e^{-v/2} \mathbf{1}_{[0,+\infty)}(v)$$

so

$$\mathbb{P}(T \leq t) = \mathbb{P}(Z \leq t\sqrt{V/m}) = \frac{1}{2^{m/2}\Gamma(m/2)} \int_0^\infty dv \left(\int_{-\infty}^{t\sqrt{v/m}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \right) v^{(m/2)-1} e^{-v/2}$$

Density of T is

$$p_T(t) = \frac{d}{dt}\mathbb{P}(T \leq t) = \frac{1}{m^{1/2}\pi^{1/2}2^{(m+1)/2}\Gamma(m/2)} \int_0^\infty dv \left(v^{(m-1)/2} e^{-(1+(t^2/m))(v/2)} \right)$$

Now make the substitution $z = (1 + \frac{t^2}{m})(v/2)$ to get:

$$p_T(t) = \frac{1}{m^{1/2}\pi^{1/2}\Gamma(m/2)} \frac{1}{(1+t^2/m)^{(m+1)/2}} \int_0^\infty z^{(m-1)/2} e^{-z} dz = \frac{\Gamma(\frac{m+1}{2})}{m^{1/2}\pi^{1/2}\Gamma(\frac{m}{2})(1+t^2/m)^{(m+1)/2}}$$

4. Here $\frac{\sum_{j=1}^n (X_j - 1)^2}{\sigma^2} \sim \chi^2_n$. The interval estimator is

$$\left[\sqrt{\frac{\sum_{j=1}^n (X_j - 1)^2}{k_{n,(\alpha/2)}}}, \sqrt{\frac{\sum_{j=1}^n (X_j - 1)^2}{k_{n,1-(\alpha/2)}}} \right]$$

where $k_{n,\alpha}$ is the value such that $\mathbb{P}(V > k_{n,\alpha}) = \alpha$, $V \sim \chi^2_n$.

5.

$$\frac{\sum_{j=1}^n (X_j - \overline{X})^2}{\sigma^2} \sim \chi^2_{n-1}$$

and this is the pivot variable. Using tail probabilities $\mathbb{P}(V \geq k_{n,\alpha}) = \alpha$ for $V \sim \chi^2_n$,

$$\mathbb{P}\left(k_{n-1,1-(\alpha/2)} \leq \frac{\sum_{j=1}^n (X_j - \overline{X})^2}{\sigma^2} \leq k_{n-1,(\alpha/2)} \right) = 1 - \alpha$$

$$\mathbb{P}\left(\log\sum_{j=1}^n (X_j - \overline{X})^2 - \log k_{n-1,(\alpha/2)} \leq \log\sigma^2 \leq \log\sum_{j=1}^n (X_j - \overline{X})^2 - \log k_{n-1,1-(\alpha/2)} \right) = \alpha$$

A symmetric interval for σ with coverage probability $1 - \alpha$ is therefore:

$$\sigma \in \frac{1}{2}\left[\log\sum_{j=1}^n (X_j - \overline{X})^2 - \log k_{n-1,(\alpha/2)}, \log\sum_{j=1}^n (X_j - \overline{X})^2 - \log k_{n-1,1-(\alpha/2)} \right].$$

6. (a) Least squares estimator minimises $\frac{1}{\sigma^2}\sum_{j=1}^{n}(Y_i - x_i\beta)^2$ since $\mathbb{E}[Y_i] = x_i\beta$ and $\text{Var}(Y_i) = \sigma^2$ for each i. It follows that

$$\widehat{\beta}_{LS} = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}$$

(b) Since $\mathbb{E}\left[\frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}\right] = \frac{\sum_{i=1}^{n} x_i^2 \beta}{\sum_{i=1}^{n} x_i^2} = \beta$ and

$$\text{Var}\left(\widehat{\beta}_{LS}\right) = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$

$$\widehat{\beta}_{LS} \sim N\left(\beta, \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}\right)$$

giving a $1-\alpha$ confidence interval of

$$\beta \in \left[\widehat{\beta}_{LS} \pm \frac{\sigma}{\sqrt{\sum_{i=1}^{n} x_i^2}} z_{\alpha/2}\right]$$

where z_α denotes the value such that $\mathbb{P}(Z \geq z_\alpha) = \alpha$ for $Z \sim N(0,1)$. Here $\frac{\alpha}{2} = 0.025$ and $z_{0.025} = 1.96$ so interval is:

$$\beta \in \left[\widehat{\beta}_{LS} \pm 1.96 \frac{\sigma}{\sqrt{\sum_{i=1}^{n} x_i^2}} z_{\alpha/2}\right]$$

7. (a) $X_i \sim N(\frac{\theta}{2}t_i^2, \sigma^2)$.

$$L(\theta; x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{j=1}^{n}\left(x_j - \frac{\theta}{2}t_j^2\right)^2\right\}$$

Maximum

$$\sum_{j=1}^{n} \frac{t_j^2}{2}\left(x_j - \frac{\theta}{2}t_j^2\right) = 0 \Rightarrow \widehat{\theta}_{ML} = \frac{2\sum_{j=1}^{n} t_j^2 X_j}{\sum_{j=1}^{n} t_j^4}$$

(b)

$$\hat{\theta}_{ML} \sim N\left(\theta, \frac{4\sigma^2}{\sum_{j=1}^{n} t_j^4}\right).$$

$$Z = \frac{2\sum_{j=1}^{n} t_j^2 X_j - \theta\sum_{j=1}^{n} t_j^4}{2\sigma\sqrt{\sum_{j=1}^{n} t_j^4}} \sim N(0,1)$$

$$I = \left[\frac{2\sum_{j=1}^{n} t_j^2 X_j}{\sum_{j=1}^{n} t_j^4} \pm \frac{2\sigma}{\sqrt{\sum_{j=1}^{n} t_j^4}} z_{\alpha/2}\right]$$

(c) $t_i = 1$ $i = 1, \ldots, n$.

8. $c = \frac{\sigma}{\sqrt{n}} z_\alpha$. Let $1-\gamma$ denote the actual confidence level, σ_0 the assumed value of σ and σ_1 the true value. Then

$$\gamma = \mathbb{P}\left(\mu < \overline{X} - \frac{\sigma_0}{\sqrt{n}} z_\alpha\right) = \mathbb{P}\left(\frac{\sqrt{n}(\overline{X}-\mu)}{\sigma_1} > \frac{\sigma_0 z_\alpha}{\sigma_1}\right) = 1 - \Phi\left(\frac{\sigma_0 z_\alpha}{\sigma_1}\right)$$

9. (a) Follows directly from $\frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$.

(b) N depends only on $(X_1, \ldots, X_{n_0})$ and does so only through S_0. S_0 is independent of $\overline{X}_0 := \frac{1}{n_0}\sum_{j=1}^{n_0} X_j$ and also of $(X_{n_0+j})_{j\geq 1}$.

$$\overline{X} = \frac{n_0 \overline{X}_0 + \sum_{j=1}^{N-n_0} X_{n_0+j}}{N}$$

For $n \geq n_0 + 1$, let

$$Z_n = \frac{\sqrt{n}((\frac{1}{n}\sum_{j=1}^{n} X_j) - \mu)}{\sigma}.$$

Then for any borel set $A \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{P}\left(\frac{\sqrt{N}(\overline{X}-\mu)}{\sigma} \in A \mid N = n, S_0\right) = \mathbb{P}\left(Z_n \in A | N = n, S_0\right) = \mathbb{P}(Z_n \in A).$$

The conditioning may removed, because Z_n depends on $X_1, \ldots, X_{n_0}$ only through $\overline{X}_0$, hence is independent of S_0 hence is independent of N. From this, it follows that

$$\frac{\sqrt{N}(\overline{X}-\mu)}{\sigma} \sim N(0,1), \qquad \frac{\sqrt{N}(\overline{X}-\mu)}{\sigma} \perp S_0.$$

Since $\frac{(n_0-1)S_0^2}{\sigma^2} \sim \chi^2_{n_0-1}$ and independent of $\frac{\sqrt{N}(\overline{X}-\mu)}{\sigma}$, it follows that

$$\frac{\sqrt{N}(\overline{X}-\mu)}{S_0} \sim t_{n_0-1}.$$

10. (a)

$$\log L(\sigma; x_1, \ldots, x_n) = -2n\log\sigma + \sum_{j=1}^{n}\log x_j - \frac{1}{2\sigma^2}\sum_{j=1}^{n} x_j^2$$

Take derivative with respect to σ and set to 0 for ML (convexity gives that the result is the MLE)

$$-\frac{2n}{\sigma} + \frac{1}{\sigma^3}\sum_{j=1}^{n} x_j^2 = 0 \Rightarrow \widehat{\sigma^2}_{ML} = \frac{1}{2n}\sum_{j=1}^{n} X_j^2$$

If there is at least one $x_i > 0$, then log likelihood function is strictly concave, $\to -\infty$ for $\sigma \to 0$ and $\sigma \to +\infty$ hence maximum. If all $x_i = 0$, then likelihood maximised at $\sigma = 0$ as required.

(b) Distribution of $\widehat{\sigma^2}_{ML}$: let $Y = \frac{X^2}{\sigma^2}$ then

$$\begin{aligned}\mathbb{P}(Y > y) &= \mathbb{P}\left(\frac{X^2}{\sigma^2} > y\right) = \mathbb{P}(X > \sigma\sqrt{y}) \\ &= \int_{\sigma\sqrt{y}}^{\infty} \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} dx = \int_y^{\infty} \frac{1}{2} e^{-z/2} dz = \exp\{-\frac{1}{2}y\} \qquad Y \sim \text{Exp}(\frac{1}{2})\end{aligned}$$

so $\frac{X^2}{\sigma^2} \sim \text{Exp}(\frac{1}{2})$ and $\frac{1}{\sigma^2}\sum_{j=1}^n X_j^2 \sim \Gamma(n, \frac{1}{2}) = \chi^2_{2n}$. Let $W \sim \chi^2_{2n}$, then $\widehat{\sigma^2}_{ML} \overset{(d)}{=} \frac{\sigma^2}{2n} W$ so that

$$\frac{\alpha}{2} = \mathbb{P}(\sigma^2 < c_n \sigma^2 \frac{W}{2n}) = \mathbb{P}(W > \frac{2n}{c_n}) \Rightarrow \frac{2n}{c_n} = k_{2n,(\alpha/2)} \Rightarrow c_n = \frac{2n}{k_{2n,(\alpha/2)}}.$$

$$d_n = \frac{2n}{k_{2n,1-(\alpha/2)}}.$$

$k_{2n,\alpha}$ denotes the value such that $\mathbb{P}(W > k_{2n,\alpha}) = \alpha$ for $W \sim \chi^2_{2n}$.

11. length is $\frac{\sigma}{\sqrt{n}}(z_{\alpha_2} + z_{\alpha_1})$ so the aim is to find $\alpha_1 \in [0, \alpha]$ that maximises

$$z_{\alpha_1} + z_{\alpha - \alpha_1}.$$

$$\alpha = \int_{z_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \Rightarrow 1 = -\frac{dz_\alpha}{d\alpha} \frac{1}{\sqrt{2\pi}} e^{-z_\alpha^2/2} \Rightarrow \frac{dz_\alpha}{d\alpha} = \sqrt{2\pi} e^{z_\alpha^2/2}$$

$$\frac{d}{d\alpha_1}(z_{\alpha-\alpha_1} + z_{\alpha_1}) = \sqrt{2\pi}\left(e^{z_{\alpha_1}^2/2} - e^{z_{\alpha-\alpha_1}^2/2}\right) = 0 \Rightarrow z_{\alpha_1} = z_{\alpha-\alpha_1} \Rightarrow \alpha_1 = \alpha - \alpha_1 = \frac{\alpha}{2}.$$