Tutorial 3

1. Let $X_1, \ldots, X_n$ be a random sample from a Poiss(θ) population where $\theta > 0$.

   (a) Show directly that $\sum_{j=1}^{n} X_j$ is sufficient for θ.

   (b) Establish the same result using the Factorisation Theorem.

2. Suppose that $X_1, \ldots, X_n$ is a random sample from a population with the following density:

$$p(x, \theta) = \begin{cases} \theta a x^{a-1} \exp\{-\theta x^a\} & x > 0, \quad \theta > 0, \quad a > 0 \\ 0 & \text{otherwise} \end{cases}$$

   where a is fixed. This is known as the *Weibull* density. Find a real-valued sufficient statistic for θ.

3. Let X be a random variable with state space $\mathcal{X} = \{v_1, \ldots, v_k\}$ and probability distribution $\mathbb{P}_\theta(X = v_i) = \theta_i$ for $i \in \{1, \ldots, k\}$ (so that $\sum_{i=1}^{k} \theta_i = 1$) and suppose that $\theta_i \in (0, 1)$ for each $i = 1, \ldots, k$. Let $X_1, \ldots, X_n$ be a random sample from X. Let

$$N_j = \sum_{i=1}^{n} \mathbf{1}_{\{v_j\}}(X_i).$$

   (the number of trials such that $X_i = v_j$).

   (a) What is the distribution of $(N_1, \ldots, N_k)$?

   (b) Show that $N = (N_1, \ldots, N_k)$ is sufficient for $\theta = (\theta_1, \ldots, \theta_k)$.

4. Let $X_1, \ldots, X_n$ be a random sample from a population with density $p(x, \theta)$ where:

$$p(x, \theta) = \frac{1}{\sigma} \exp\left\{ - \left(\frac{x - \mu}{\sigma} \right) \right\} \mathbf{1}_{[\mu, +\infty)}(x).$$

   Here $\theta = (\mu, \sigma)$, $\Theta = (-\infty, +\infty) \times (0, +\infty)$.

   (a) Show that $\min(X_1, \ldots, X_n)$ is sufficient for μ when σ is fixed.
   Note: you cannot use the factorisation theorem for this part since the support of the density depends on μ.

   (b) Find a one-dimensional sufficient statistic for σ when μ is fixed.

   (c) Find a two-dimensional sufficient statistic for $\theta = (\mu, \sigma)$.

5. Let $X_1, \ldots, X_n$ be a random sample from a distribution F. Treating F as a parameter, show that the order statistic $(X_{(1)}, \ldots, X_{(n)})$ is sufficient for F.

6. Let $X_1, \ldots, X_n$ be a random sample from $f(t - \theta), \theta \in \mathbb{R}$. Show that the order statistic is *minimal sufficient* for f when f is the Cauchy density

$$f(t) = \frac{1}{\pi(1 + t^2)} \qquad t \in \mathbb{R}.$$

7. Let $X_1, \ldots, X_m; Y_1, \ldots, Y_n$ be independent and distributed according to $N(\mu, \sigma^2)$ and $N(\eta, \tau^2)$ respectively. Find minimal sufficient statistics in the following three cases, where $(\mu, \eta, \sigma, \tau) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$:

   (a) μ, η, σ, τ arbitrary.

   (b) $\sigma = \tau$, μ, η, σ arbitrary.

   (c) $\mu = \eta$, μ, σ, τ arbitrary.

8. Let $Y = (Y_1, \ldots, Y_n)^t$ be a multivariate Gaussian random vector with distribution

$$Y \sim N(X\beta, \sigma^2 I)$$

   where X is an $n \times p$ design matrix (values are given) and $\beta = (\beta_1, \ldots, \beta_p)^t$ is a parameter vector. Compute a $p+1$ dimensional sufficient statistic for (β, σ^2).

9. Let $Y_1, \ldots, Y_n$ be independent Bernoulli trials, where

$$\mathbb{P}(Y_j = 1) = \frac{1}{1 + \exp\left\{-\sum_{k=1}^{p} X_{jk}\beta_k\right\}} \qquad j = 1, \ldots, n.$$

   Compute a p dimensional sufficient statistic for β.

Short Answers

1. (a) Let $T = \sum_{j=1}^n X_j$. Note: $T \sim \text{Poiss}(n\theta)$. For $x_1 + \ldots + x_n = t$

$$\begin{aligned} \mathbb{P}_\theta((X_1,\ldots,X_n) = (x_1,\ldots,x_n)|T=t) &= \frac{\mathbb{P}_\theta((X_1,\ldots,X_n) = (x_1,\ldots,x_n))}{\mathbb{P}(T=t)} \\ &= \frac{\theta^{\sum_{j=1}^n x_j}\prod_{j=1}^n \frac{1}{x_j!}e^{-n\theta}}{\theta^t \frac{1}{t!}e^{-n\theta}} = \frac{(\sum_j x_j)!}{\prod_{j=1}^n x_j!} \end{aligned}$$

which does not depend on θ.

(b)

$$\mathbb{P}_\theta((X_1,\ldots,X_n) = (x_1,\ldots,x_n)) = \frac{\theta^{\sum_{j=1}^n x_j}}{\prod_{j=1}^n x_j!}\exp\{-n\theta\}$$

This factorises as $g(\sum_{j=1}^n x_j, \theta)h(\underline{x})$ where $g(t,\theta) = \theta^t e^{-n\theta}$ and $h(\underline{x}) = \frac{1}{\prod_{j=1}^n x_j!}$.

2.

$$f(x_1,\ldots,x_n;\theta) = \begin{cases} \theta^n a^n \left(\prod_{j=1}^n x_j\right)^{a-1} \exp\left\{-\theta\sum_{j=1}^n x_j^a\right\} & x_1 > 0,\ldots,x_n > 0 \\ 0 & \text{other} \end{cases}$$

Set $t(x_1,\ldots,x_n) = \sum_{j=1}^n x_j^a$ then

$$f(x_1,\ldots,x_n;\theta) = g(t(x_1,\ldots,x_n),\theta)h(x_1,\ldots,x_n)$$

where

$$g(t,\theta) = \theta^n e^{-\theta t}, \qquad h(x_1,\ldots,x_n) = a^n \mathbf{1}_{\{x_1>0,\ldots,x_n>0\}}\left(\prod_{j=1}^n x_j\right)^{a-1}$$

Hence $t(X_1,\ldots,X_n)$ is sufficient for θ.

3. (a)

$$(N_1,\ldots,N_k) \sim \text{mult}(n;\theta_1,\ldots,\theta_k).$$

(b)

$$\mathbb{P}_\theta\left((X_1,\ldots,X_n) = (v_{a_1},\ldots,v_{a_n})\right) = \theta_1^{n_1}\ldots\theta_k^{n_k}$$

where $n_j = \sum_{i=1}^n \mathbf{1}(a_i = j)$. This is in the required form from the factorisation theorem.

4. (a) The joint density is:

$$p(x_1,\ldots,x_n,\theta) = \frac{1}{\sigma^n}\exp\left\{-\sum_{j=1}^n\left(\frac{x_j-\mu}{\sigma}\right)\right\}\mathbf{1}_{\{\min_j x_j \geq \mu\}}.$$

The factorisation theorem cannot be used, since the support of the density depends on μ. We therefore show that the conditional density $p(x_1,\ldots,x_n|\min_j X_j = y)$, for $y = \min_j x_j$, does not depend on μ.

Since $X - \mu \sim \text{Exp}(\frac{1}{\sigma})$, therefore $\left(\min_{j\in\{1,\dots,n\}} X_j\right) - \mu \sim \text{Exp}(\frac{n}{\sigma})$ so that the density of $Y := \min_j X_j$ is:

$$p_Y(y) = \frac{n}{\sigma}\exp\left\{-\frac{n(y-\mu)}{\sigma}\right\}\mathbf{1}_{\{y\geq\mu\}}$$

and therefore, for $\min_j x_j = y$,

$$p(x_1,\dots,x_n|\min_j X_j = y) = \frac{p(x_1,\dots,x_n)}{p_Y(y)} = \frac{1}{n\sigma^{n-1}}\exp\left\{-\frac{1}{\sigma}\sum_{j=1}^{n}(x_j - \min_i x_i)\right\}$$

which does not depend on μ, hence $\min_i X_i$ is sufficient for μ when σ is fixed.

(b) From the factorisation theorem, it follows that $\sum_{j=1}^n X_j$ is sufficient for σ when μ is fixed.

(c) From the factorisation theorem, applied to the conditional density $p_\sigma(x_1,\dots,x_n|\min_j X_j = y)$, $\sum_{j=1}^n X_j$ is sufficient for σ, conditioned on $\min_j X_j = y$ for any y.
Hence, for $\min_j x_j = y$, $p(x_1,\dots,x_n|\sum_j x_j = z, \min_j x_j = y)$ depends neither on μ nor on σ (definition of sufficiency applied to the one-parameter conditional distribution $p_\sigma(x_1,\dots,x_n|\min_j X_j = y)$), hence from the definition of sufficiency (applied to full distribution), $(\min_j X_j, \sum_j X_j)$ is sufficient for $\theta = (\mu,\sigma)$.

5. Once the order statistics $x_{(1)},\dots,x_{(n)}$ are given, the problem is then the random assignment (without replacement) of $x_1,\dots,x_n$ to $x_{(1)},\dots,x_{(n)}$. There are $n!$ permutations, each with equal probability. Suppose that there are m groups, group j contains n_j so that $n_1+\ldots+n_m = n$, and the order statistics are equal within each group. Then

$$\mathbb{P}((X_1,\dots,X_n) = (x_1,\dots,x_n)|x_{(1)},\dots,x_{(n)}) = \frac{\prod_{j=1}^m n_j!}{n!}$$

which does not depend on F.

6. We use the Dynkin Lehman Scheffe lemma; a statistic T is minimal sufficient if $\frac{L(\theta;x)}{L(\theta;y)}$ does not depend on θ for $T(x) = T(y)$ and does depend on θ for $T(x) \neq T(y)$.

$$L(\theta;x_1,\dots,x_n) = \frac{1}{\pi^n}\prod_{j=1}^n\frac{1}{(1+(x_j-\theta)^2)}$$

$$\frac{L(\theta;\underline{x})}{L(\theta;\underline{y})} = \prod_{j=1}^n\frac{(1+(y_j-\theta)^2)}{(1+(x_j-\theta)^2)}$$

Firstly, $L(x_1,\dots,x_n;\theta) = L(x_{(1)},\dots,x_{(n)};\theta)$ so that if $\underline{y}$ is a permutation of $\underline{x}$, then $\frac{L(\theta;\underline{x})}{L(\theta;\underline{y})} = 1$. To see that this is minimal, the function does not depend on θ only if the roots of the numerators and denominators are the same (considering as functions of θ), These are: $\theta = y_j \pm i$ for $j = 1,\dots,n$ (for the numerator) and $\theta = x_j \pm i$ for $j = 1,\dots,n$ for the denominator (where $i = \sqrt{-1}$). These are the same if and only if $(y_{(1)},\dots,y_{(n)}) = (x_{(1)},\dots,x_{(n)})$.

7. We use the Dynkin Lehman Scheffe lemma; a statistic T is minimal sufficient if $\frac{L(\theta;x)}{L(\theta;y)}$ does not depend on θ for $T(x) = T(y)$ and does depend on θ for $T(x) \neq T(y)$. This is equivalent to these properties holding for the log likelihood; $\log L(\theta; x) - \log L(\theta; y)$.

   The log likelihood function is:

$$\log L(\mu, \eta; \sigma, \tau; \underline{x}, \underline{y}) = -\frac{(n+m)}{2} \log(2\pi) - m \log \sigma - n \log \tau$$
$$-\frac{1}{2\sigma^2} \left(\sum_{j=1}^{m} x_j^2 - 2\mu \sum_{j=1}^{m} x_j + m\mu^2 \right) - \frac{1}{2\tau^2} \left(\sum_{j=1}^{n} y_j^2 - 2\eta \sum_{j=1}^{n} y_j + n\eta^2 \right).$$

   Write out

$$\begin{aligned} \log L(\theta; \underline{x}_1, \underline{y}_1) - \log L(\theta; \underline{x}_2, \underline{y}_2) &= -\frac{1}{2\sigma^2} \left(\sum_{j=1}^{m} x_{1j}^2 - \sum_{j=1}^{m} x_{2j}^2 \right) + \frac{\mu}{\sigma^2} \left(\sum_{j=1}^{m} x_{1j} - \sum_{j=1}^{m} x_{2j} \right) \\ &\quad -\frac{1}{2\tau^2} \left(\sum_{j=1}^{n} y_{1j}^2 - \sum_{j=1}^{n} y_{2j}^2 \right) + \frac{\eta}{\tau^2} \left(\sum_{j=1}^{n} y_{1j} - \sum_{j=1}^{n} y_{2j} \right) \end{aligned}$$

   and obtain:

   (a) $\sum_{j=1}^m X_j$, $\sum_{j=1}^m X_j^2$, $\sum_{j=1}^n Y_j$, $\sum_{j=1}^n Y_j^2$.
   (b) $\sum_{j=1}^m X_j$, $\sum_{j=1}^n Y_j$, $\sum_{j=1}^m X_j^2 + \sum_{j=1}^n Y_j^2$.
   (c) $\sum_{j=1}^m X_j$, $\sum_{j=1}^n Y_j$, $\sum_{j=1}^n X_j^2$, $\sum_{j=1}^n Y_j^2$. (same as part (a)).

8. Density is:

$$f(y_1, \dots, y_n) = \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left\{ -\frac{1}{2\sigma^2} (y^t y - 2y^t X\beta + \beta^t X^t X\beta) \right\}$$

   so, by the factorisation theorem, a $p+1$ dimensional sufficient statistic is $(y^t X, y^t y)$.

9. For an outcome $(Y_1, \dots, Y_n) = (y_1, \dots, y_n)$ where $(y_1, \dots, y_n)$ is a vector of 0's and 1's, we have

$$\mathbb{P}_{Y_1,\dots,Y_n}(y_1, \dots, y_n) = \frac{\prod_{j=1}^n \exp\{(1-y_j) \sum_{k=1}^p X_{jk}\beta_k\}}{\prod_{j=1}^n (1 + \exp\left\{ -\sum_{k=1}^p X_{jk}\beta_k \right\})}$$

   The sufficient statistic is therefore $(\sum_{j=1}^n y_j X_{jk} : k = 1, \dots, p)$.