Tutorial 13

1. Let $X_1, \ldots, X_n$ be i.i.d. $U(0, \theta)$; that is, the density is therefore:

$$p(x;\theta) = \frac{1}{\theta}\mathbf{1}_{[0,\theta]}(x)$$

   Let $l(\theta; x) = -\log p(x;\theta)$.

   (a) Show that $\frac{d}{d\theta}l(\theta, x) = \frac{1}{\theta}$ for $\theta > x$ and is undefined for $\theta \leq x$. If $X \sim U(0,\theta)$, conclude that $\frac{d}{d\theta}l(\theta, X)$ is defined with $\mathbb{P}_\theta$ probability 1, but that

$$\mathbb{E}_\theta\left[\frac{d}{d\theta}l(\theta; X)\right] = \frac{1}{\theta} \neq 0.$$

   (b) Recall that $\widehat{\theta}_{ML} = \max\{X_1, \ldots, X_n\}$. Show that: $n(\theta - \widehat{\theta}) \overset{n\to+\infty}{\longrightarrow}_{\mathcal{L}_\theta} \text{Exp}(1/\theta)$ ($\mathcal{L}_\theta$ denotes the law when the parameter value is θ).

2. Suppose $\lambda : \mathbb{R} \to \mathbb{R}$ satisfies $\lambda(0) = 0$, is bounded and has bounded second derivative λ''. Show that if $X_1, \ldots, X_n$ are i.i.d. with $\mathbb{E}[X_1] = \mu$ and $\mathbf{V}(X_1) = \sigma^2 < +\infty$, then

$$\left|\sqrt{n}\mathbb{E}\left[\lambda(|\overline{X} - \mu|)\right] - \lambda'(0)\sigma\sqrt{\frac{2}{\pi}}\right| \overset{n\to+\infty}{\longrightarrow} 0$$

3. Let $V_n \sim \chi^2_n$. Show that $(\sqrt{V_n} - \sqrt{n}) \overset{n\to+\infty}{\longrightarrow}_{\mathcal{L}} N(0, \frac{1}{2})$ ($\mathcal{L}$ denotes law).

4. Suppose that $X_1, \ldots, X_n$ are i.i.d. variables each with probability function

$$p_X(0) = \theta^2 \qquad p_X(1) = 2\theta(1-\theta) \qquad p_X(2) = (1-\theta)^2$$

   (a) Find a and b (in terms of n and θ) such that $Z_n = \frac{\overline{X}-a}{b} \overset{n\to+\infty}{\longrightarrow}_{\mathcal{L}} N(0,1)$.

   (b) Find c and d (in terms of n and θ) such that $Y_n = \frac{\sqrt{\overline{X}}-c}{d} \overset{n\to+\infty}{\longrightarrow}_{\mathcal{L}} N(0,1)$.

5. Let $X_1, \ldots, X_n$ be a sample from a population with mean μ and variance $\sigma^2 < +\infty$. Let h be a function and let $h^{(j)}$ denote its jth derivative. Suppose that h has a second derivative continuous at μ and that $h^{(1)}(\mu) = 0$.

   (a) Show that $\sqrt{n}(h(\overline{X}) - h(\mu)) \overset{n\to+\infty}{\longrightarrow} 0$, while $n\left(h(\overline{X}) - h(\mu)\right) \overset{n\to+\infty}{\longrightarrow} \frac{1}{2}h^{(2)}(\mu)\sigma^2 V$ where $V \sim \chi^2_1$.

   (b) Use part (a) to show that when $\mu = \frac{1}{2}$, then

$$n\left(\overline{X}(1-\overline{X}) - \mu(1-\mu)\right) \overset{n\to+\infty}{\longrightarrow}_{\mathcal{L}} -\sigma^2 V \qquad V \sim \chi^2_1$$

6. Show that if $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$ and $S^2 = \frac{1}{n-1}\sum_{j=1}^n (X_j - \overline{X})^2$, then

$$\sqrt{n}\begin{pmatrix} \overline{X} - \mu \\ S^2 - \sigma^2 \end{pmatrix} \overset{n\to+\infty}{\longrightarrow}_{\mathcal{L}} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix}\right)$$

7. Let $X_{ij} : i = 1, \ldots, p, j = 1, \ldots, k$ be independent with $X_{ij} \sim N(\mu_i, \sigma^2)$.

(a) Show that the MLEs of μ_i and σ^2 are:

$$\overline{\mu}_i = \frac{1}{k} \sum_{j=1}^{k} X_{ij} \qquad \widehat{\sigma^2} = \frac{1}{kp} \sum_{i=1}^{p} \sum_{j=1}^{k} (X_{ij} - \widehat{\mu}_i)^2$$

(b) Show that if k is fixed and $p \to +\infty$, then

$$\widehat{\sigma}^2 \xrightarrow[\mathcal{L}]{p \to +\infty} \left(1 - \frac{1}{k}\right) \sigma^2.$$

That is, the MLE $\widehat{\sigma}^2$ is not consistent.