Tutorial 10

1. Consider a situation where the parameter space has two elements, $\Theta = \{\theta_0, \theta_1\}$ Suppose we want to test $H_0 : \theta = \theta_0$ versus the alternative, $H_1 : \theta = \theta_1$. One way of doing this is to consider the test statistic

$$\nu(x) = \frac{L(\theta_1; x)}{L(\theta_0; x)},$$

the ratio of the likelihood functions. This is a different formulation, but gives the same test as the Likelihood Ratio statistic. We reject $H_0 : \theta = \theta_0$ in favour of $H_1 : \theta = \theta_1$ if $\nu(x)$ is large.

We have a single observation on a random variable X with distribution F, where F is either $U(0, 1)$ or $\text{Exp}(1)$. Construct the test described above, with significance level $\alpha = 0.05$ to test $H_0 : X \sim U(0, 1)$ versus the alternative $H_1 : X \sim \text{Exp}(1)$. Compute the rejection region for the test and compute its power when H_1 is true.

2. We have a single observation on the random variable X with density function

$$p(x, \theta) = \begin{cases} \theta e^{-x} + 2(1 - \theta)e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\theta \in [0, 1]$ is an unknown parameter.

   (a) Construct a test between the null hypothesis $H_0 : \theta = 0$ versus the alternative $H_1 : \theta > 0$ with significance level $\alpha = 0.05$. (Use LRT method).

   (b) Compute the power function of this test.

3. Let $(U_j)_{j\geq 1}$ be a sequence of i.i.d. $U(0, 1)$ random variables. Let X be a random variable. It is required to test

$$H_0 : X = \min\{U_1, \dots, U_k\} \qquad \text{versus} \qquad H_1 : X = \min\{U_1, \dots, U_l\} \qquad l < k.$$

   (a) Construct a test with significance level α based on the statistic $\nu(x) := \frac{L(H_1;x)}{L(H_0;x)}$ where $L(H_1; x)$ and $L(H_0; x)$ denote the likelihoods based on H_1 and H_0 respectively (each hypothesis corresponds to a single parameter value).

   (b) What is the largest value of the ratio $\frac{l}{k}$ so that a test with significance $\alpha = 0.05$ has power at least 0.95?

4. Consider a population with three types of individual, labelled 1, 2 and 3, which occur in the Hardy - Weinberg proportions

$$p_\theta(1) = \theta^2 \qquad p_\theta(2) = 2\theta(1 - \theta) \qquad p_\theta(3) = (1 - \theta)^2.$$

For a sample $X_1, \dots, X_n$ from this population, let $N_1 = \sum_{j=1}^n \mathbf{1}_1(X_j)$, $N_2 = \sum_{j=1}^n \mathbf{1}_2(X_j)$, $N_3 = \sum_{j=1}^n \mathbf{1}_3(X_j)$ denote the number of appearances of 1, 2, 3 respectively in the sample. Let $0 < \theta_0 < \theta_1 < 1$.

(a) Show that $\nu(\underline{x}; \theta_0, \theta_1) = \frac{L(\theta_1; \underline{x})}{L(\theta_0; \underline{x})}$ is an increasing function of $2N_1 + N_2$. (n is fixed).

(b) Show that if $c > 0$ and $\alpha \in (0,1)$ satisfy

$$\mathbb{P}_{\theta_0}(2N_1 + N_2 > c) = \alpha$$

then a test $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ with a given significance level α that rejects H_0 if and only if $2N_1 + N_2 > c$ corresponds to the test where $H_0 : \theta = \theta_0$ is rejected for large values of $\nu(\underline{x}; \theta_0, \theta_1)$, defined in the previous part.

5. Let $X_1, \dots, X_n$ be i.i.d. $U(0, \theta)$ variables and let $M_n = \max\{X_1, \dots, X_n\}$. Consider a test of $H_0 : \theta \le \theta_0$ versus the alternative $H_1 : \theta > \theta_0$ where H_0 is rejected if and only if $M_n > c$ for some value $c > 0$.

   (a) Compute the power function of this test and show that it is monotone increasing in θ.

   (b) For $\theta_0 = \frac{1}{2}$, compute the value of c which would give the test a size exactly 0.05.

   (c) Compute the value of n so that the test of size 0.05 for $\theta_0 = \frac{1}{2}$ has power 0.98 for $\theta = \frac{3}{4}$.

6. Consider a simple hypothesis test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$. Suppose that the test statistic T has a continuous distribution and the null hypothesis is rejected for $t \ge c$ where t is the observed value of T for some c and that, as a function of c, the size of the test is:

$$\alpha(c) = \mathbb{P}_{\theta_0}(T \ge c).$$

   Prove that, for $\theta = \theta_0$, $\alpha(T) \sim U(0,1)$.

7. Let $T_1, \dots, T_r$ be independent test statistics for the same simple $H_0 : \theta = \theta_0$ and that for each j, T_j has a continuous distribution. Let $\alpha_j(c) = \mathbb{P}_{\theta_0}(T_j \ge c)$. Show that, under H_0, $\tilde{T} = -2\sum_{j=1}^r \log \alpha_j(T_j) \sim \chi^2_{2r}$.

8. Let $F_0(y) = \mathbb{P}(Y < y)$ where Y is a non negative random variable representing a survival time. Assume that F_0 has a density f_0. Let $X_1, \dots, X_n$ be i.i.d. each with an alternative distribution, representing survival time under an alternative treatment. The new distribution is considered to take the form

$$G(y, \Delta) = 1 - (1 - F_0(y))^\Delta \qquad y > 0 \quad \Delta > 0.$$

   To test whether the new treatment is beneficial, test $H_0 : \Delta \le 1$ versus $H_1 : \Delta > 1$. Compute the Likelihood Ratio Test and compute the critical region for a test with significance level α in terms of n and an appropriate χ^2 distribution. (This is known as the *Lehmann alternative*).