Introduction to Combinatorics

Homework 9, due date: 2020-05-28

1. Let G be a d-regular simple undirected graph on n vertices. Let $\lambda_1 \geq \ldots \geq \lambda_n$ be eigenvalues of the adjacency matrix of G. Show that $\lambda_k = d$ if and only if G has at least k connected components.

2. Let $G = (V, E)$ be a d-regular simple undirected graph. For a set $S \subseteq V$ let $|\partial S|$ be the number of edges going from S to its complement. Show that for any $S \subseteq V$ we have $|\partial S| \leq \frac{d-\lambda_n}{2d}|E|$, where λ_n is the smallest eigenvalue of the adjacency matrix of G.

3. Let $G = (V, E)$ be an undirected simple graph. Let T denote the number of triangles in it. Prove that $9T^2 \leq 2|E|^3$.

4. We say that a d-regular simple undirected graph $G = (V, E)$ on n vertices is an (n, d, c)-vertex expander if the neighborhood $N(S) = \{v \in V \setminus S : \exists u \in S, uv \in E\}$ of every set S with $|S| \leq n/2$ satisfies $|N(S)| \geq c|S|$. Show that the diameter of G is at most $O(\log_{1+c}(n))$ (i.e. there exists an absolute constant B such that the diameter of G is at most $B \log_{1+c}(n)$).

5. Edges of K_{16} were colored in red, blue and green such that both red and blue edges form a Clebsch graph. Prove that green edges form a Clebsch graph as well. (Computer-assisted proofs will not be accepted!)
